

$$\phi = V_{\infty} x + \frac{\Lambda}{4\pi} \ln((x-d)^2 + y^2)$$

$$a) \quad u = \frac{\partial \phi}{\partial x} = V_{\infty} + \frac{\Lambda}{2\pi} \frac{x-d}{(x-d)^2 + y^2}$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\Lambda}{2\pi} \frac{y}{(x-d)^2 + y^2}$$

at  $x, y = 0, 0$ , require  $u = 0$

$$\text{or } V_{\infty} - \frac{\Lambda}{2\pi d} = 0$$

$$2\pi V_{\infty} d = \Lambda \quad (1)$$

at  $x, y = d, \sqrt{Cd}$ , require  $\frac{v}{u} = \frac{dy}{dx}$ , where  $\frac{dy}{dx} = \frac{1}{2} \sqrt{\frac{C}{x}} \Big|_{x=d} = \frac{1}{2} \sqrt{\frac{C}{d}}$

$$\text{or } \frac{1}{V_{\infty}} \frac{\Lambda}{2\pi} \frac{\sqrt{Cd}}{Cd} = \frac{1}{2} \sqrt{\frac{C}{d}}$$

$$\Lambda = \pi V_{\infty} C \quad (2)$$

$$\text{Combine (1) \& (2)} \rightarrow C = 2d \rightarrow \boxed{d = C/2}$$

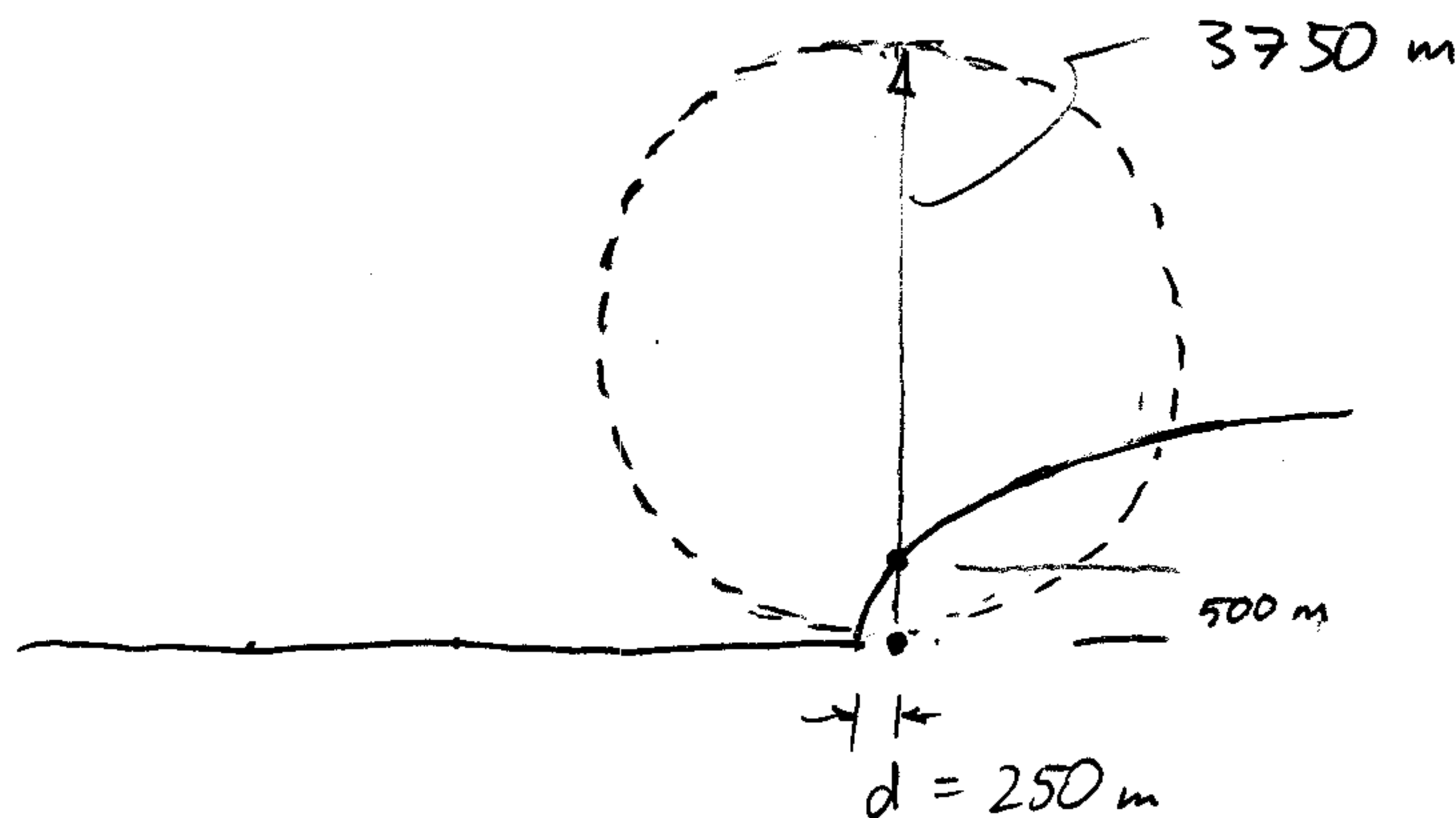
b) For  $C = 500 \text{ m}$ ,  $V_{\infty} = 15 \text{ m/s}$ ,  $\rightarrow d = 250 \text{ m}$ ,  $\Lambda = 7500\pi \text{ m}^2/\text{s}$

Maximum radius where  $v = 1 \text{ m/s}$

$$\text{or } v = \frac{\Lambda}{2\pi} \frac{y}{r^2} = \frac{\Lambda}{2\pi} \frac{\sin \theta}{r} = 1 \text{ m/s}$$

$$\rightarrow \boxed{r_{\max}(\theta) = \frac{\Lambda}{2\pi \cdot 1 \text{ m/s}} \sin \theta = 3750 \text{ m} \cdot \sin \theta}$$

circle of diameter 3750 m above source.





$$C_p = 1 - \frac{V^2}{V_\infty^2} = 1 - \left(\frac{u}{V_\infty}\right)^2 - \left(\frac{v}{V_\infty}\right)^2$$

a) Source:  $u = V_\infty + \frac{\Lambda}{2\pi} \frac{x}{x^2+y^2}$   
 $v = \frac{\Lambda}{2\pi} \frac{y}{x^2+y^2}$

Along  $-x, y=0$ :  $C_p = 1 - \frac{u^2}{V_\infty^2} = 1 - \left(\frac{V_\infty}{V_\infty} - \frac{\Lambda}{2\pi V_\infty} \frac{1}{x}\right)^2 = \left[\frac{\Lambda}{\pi V_\infty} \frac{1}{x} - \left(\frac{\Lambda}{2\pi V_\infty}\right)^2 \frac{1}{x^2}\right]$  (dominant term circled, note: much smaller for large x)

Along  $y, x=0$ :  $C_p = 1 - \left(\frac{V_\infty}{V_\infty}\right)^2 - \left(\frac{\Lambda}{2\pi V_\infty}\right)^2 \frac{1}{y^2} = -\left(\frac{\Lambda}{2\pi V_\infty}\right)^2 \frac{1}{y^2}$

b) Vortex:  $u = V_\infty + \frac{\Gamma}{2\pi} \frac{y}{x^2+y^2}$   
 $v = \frac{\Gamma}{2\pi} \frac{-x}{x^2+y^2}$

Along  $-x, y=0$ :  $C_p = 1 - \left(\frac{V_\infty}{V_\infty}\right)^2 - \left(\frac{\Gamma}{2\pi V_\infty}\right)^2 \frac{1}{x^2} = -\left(\frac{\Gamma}{2\pi V_\infty}\right)^2 \frac{1}{x^2}$  (dominant term circled)

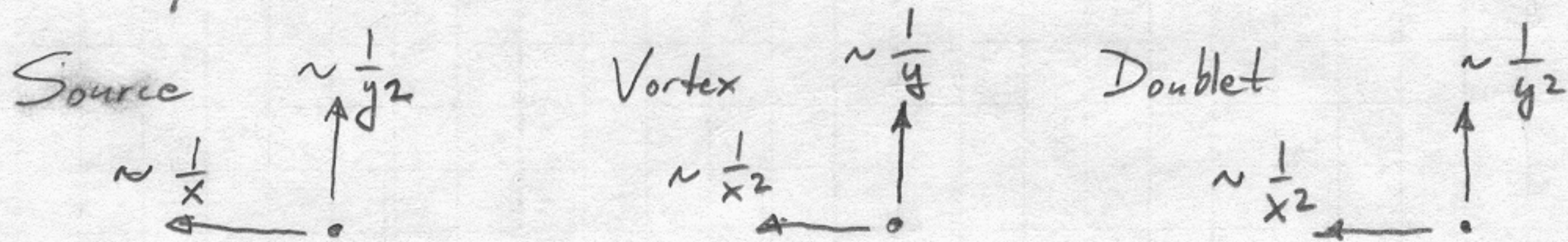
Along  $y, x=0$ :  $C_p = 1 - \left(\frac{u}{V_\infty}\right)^2 = 1 - \left(\frac{V_\infty}{V_\infty} + \frac{\Gamma}{2\pi V_\infty} \frac{1}{y}\right)^2 = \left[-\frac{\Gamma}{\pi V_\infty} \frac{1}{y} - \left(\frac{\Gamma}{2\pi V_\infty}\right)^2 \frac{1}{y^2}\right]$

c) Doublet:  $\phi = V_\infty x + \frac{\kappa}{2\pi} \frac{x}{x^2+y^2}$   $\left\{ \begin{aligned} u &= \frac{\partial \phi}{\partial x} = V_\infty + \frac{\kappa}{2\pi} \frac{y^2-x^2}{(x^2+y^2)^2} \\ v &= \frac{\partial \phi}{\partial y} = \frac{\kappa}{2\pi} \frac{-2xy}{(x^2+y^2)^2} \end{aligned} \right.$

Along  $-x, y=0$ :  $C_p = 1 - \left(\frac{u}{V_\infty}\right)^2 = 1 - \left(\frac{V_\infty}{V_\infty} - \frac{\kappa}{2\pi V_\infty} \frac{1}{x^2}\right)^2 = \frac{\kappa}{\pi V_\infty} \frac{1}{x^2} - \left(\frac{\kappa}{2\pi V_\infty}\right)^2 \frac{1}{x^4}$

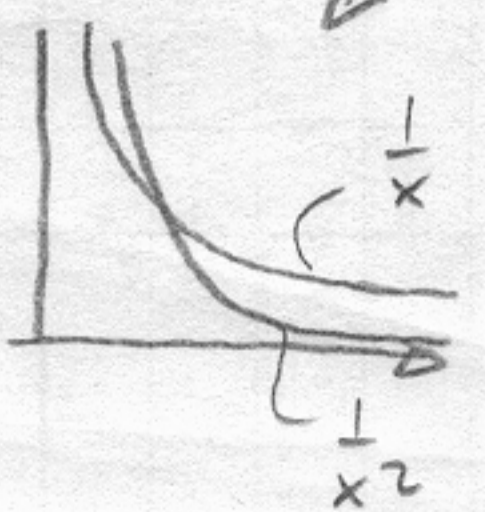
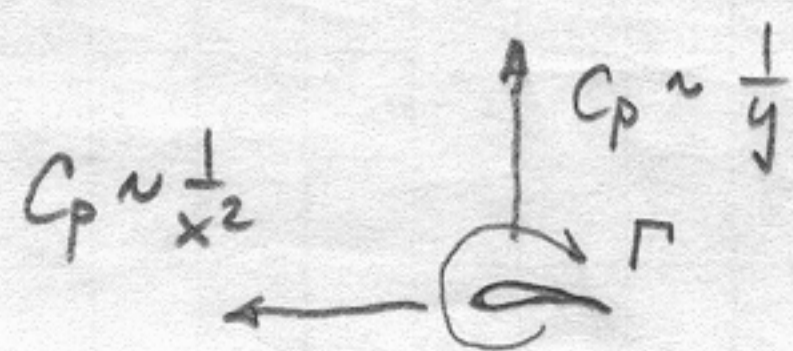
Along  $y, x=0$ :  $C_p = 1 - \left(\frac{u}{V_\infty}\right)^2 = 1 - \left(\frac{V_\infty}{V_\infty} + \frac{\kappa}{2\pi V_\infty} \frac{1}{y^2}\right)^2 = -\frac{\kappa}{\pi V_\infty} \frac{1}{y^2} - \left(\frac{\kappa}{2\pi V_\infty}\right)^2 \frac{1}{y^4}$

The  $C_p$  fields decrease with distance as follows:



Far away the  $\frac{1}{x}$  and  $\frac{1}{y}$  terms dominate ( $\frac{1}{x^2}$  and  $\frac{1}{y^2}$  die off much faster)

A lifting airfoil has a nonzero  $\Gamma$ , so it looks mostly like a vortex far away. Largest  $C_p$  is above & below.



500 SHEETS, FILLER, 5 SQUARE  
 50 SHEETS EYE-EASE, 5 SQUARE  
 100 SHEETS EYE-EASE, 5 SQUARE  
 200 SHEETS EYE-EASE, 5 SQUARE  
 400 RECYCLED WHITE, 5 SQUARE  
 200 RECYCLED WHITE, 5 SQUARE  
 13-782  
 42-381  
 42-382  
 42-389  
 42-392  
 42-399  
 Made in U.S.A.

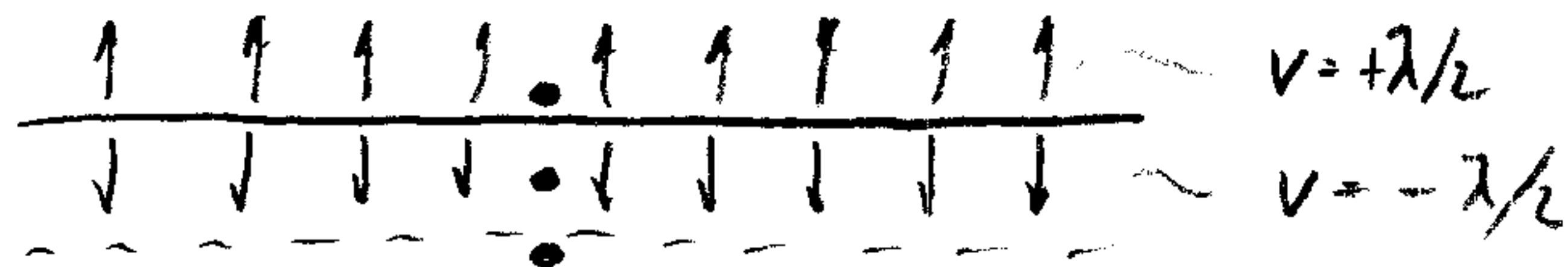




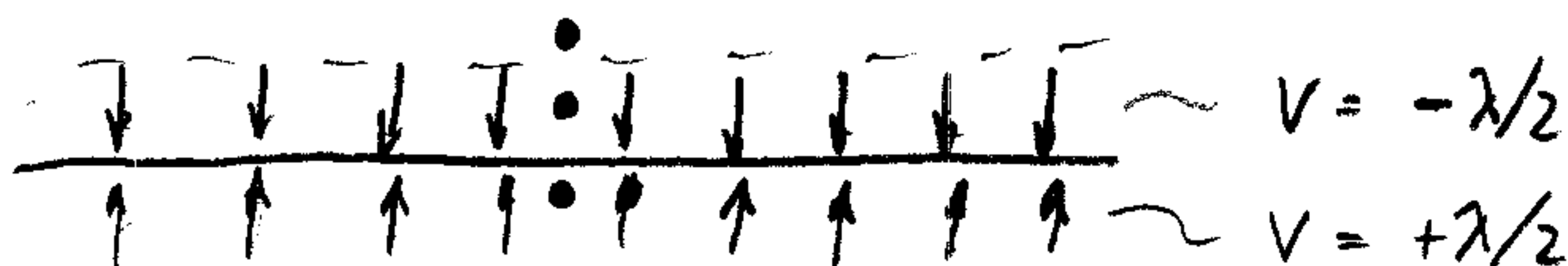
Since the panels are very long, the limiting case  $y \rightarrow 0$  applies for all 3 points A, B, C. (i.e.  $h$  is irrelevant.)

Only vertical velocities are nonzero.

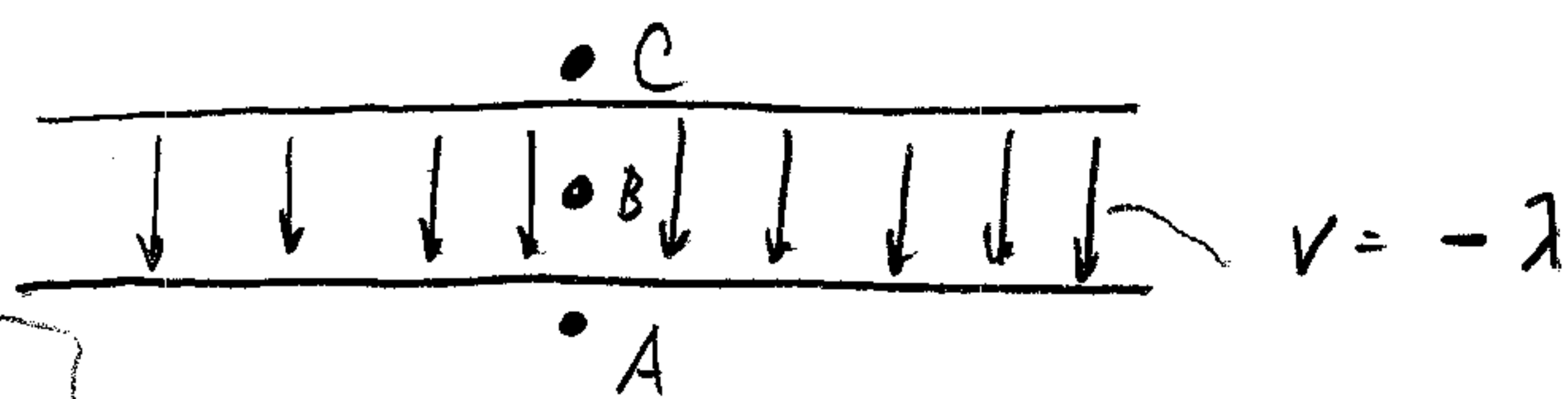
Top panel alone:



Bottom panel alone:



Superimpose:



$$\text{Point A: } v = -\lambda/2 + \lambda/2 = 0$$

$$\text{B: } v = -\lambda/2 - \lambda/2 = -\lambda$$

$$\text{C: } v = \lambda/2 - \lambda/2 = 0$$

The velocity field is analogous to the electric field of a capacitor.

M19.

a) This is a transversely isotropic material  
So it will require 5 elastic constants

b) Specimen (1) is loaded in the longitudinal fiber  
direction  $\epsilon_L =$

$$E_L = \frac{\sigma_L}{\epsilon_L} = \frac{14 \times 10^3}{(10 \times 10^{-3})^2} \times \frac{1}{500 \times 10^{-6}} = 2.8 \times 10^{11} = 280 \text{ GPa}$$

$$\nu_{LT} = -\frac{\epsilon_T}{\epsilon_L} = -\frac{(-120)}{500} = 0.24$$

$$E_T = \frac{\sigma_T}{\epsilon_T} = \frac{14 \times 10^3}{(10 \times 10^{-3})^2} \times \frac{1}{700 \times 10^{-6}} = 2.0 \times 10^{11} = 200 \text{ GPa} \Leftarrow$$

$$\nu_{TL} = \frac{\epsilon_T}{\epsilon_L} = \frac{1426 - (-125)}{700 \times 10^{-6}} = 0.18 \Leftarrow$$

*\*gauge L*  
*gauge d*

$$\nu_{TT} = -\frac{\epsilon_d}{\epsilon_d} = -\frac{210}{700} = 0.3$$

$$\text{Hence } G_{TT} = \frac{E_{TT}}{2(1 + \nu_{TT})} = 77 \text{ GPa} \Leftarrow$$

c) from longitudinal modulus:

$$E_L = V_f E_f + (1 - V_f) E_m$$

$$E_L V_f = V_f (E_f - E_m) + E_m$$

$$\frac{E_L - E_m}{E_f - E_m} = V_f = \frac{280 - 110}{450 - 110} = 0.5 \leftarrow$$

for transverse modulus, lower bound estimate

$$E_T = \frac{1}{\frac{V_f}{E_f} + \frac{1 - V_f}{E_m}}$$

$$\frac{E_T V_f}{E_f} + \frac{E_T (1 - V_f)}{E_m} = 1$$

$$E_T E_m V_f + E_T E_f (1 - V_f) = 1$$

$$V_f (E_T E_m - E_T E_f) = -E_T E_f$$

$$V_f = \frac{-E_T E_f}{E_T E_m - E_T E_f} = \frac{-200 \times 450}{200 \times 110 - 200 \times 450}$$

$$= \frac{E_f}{E_f - E_m} = \frac{450}{450 - 110} = \frac{450}{340} = 1.32$$

This is greater than 0.5 so not inconsistent!

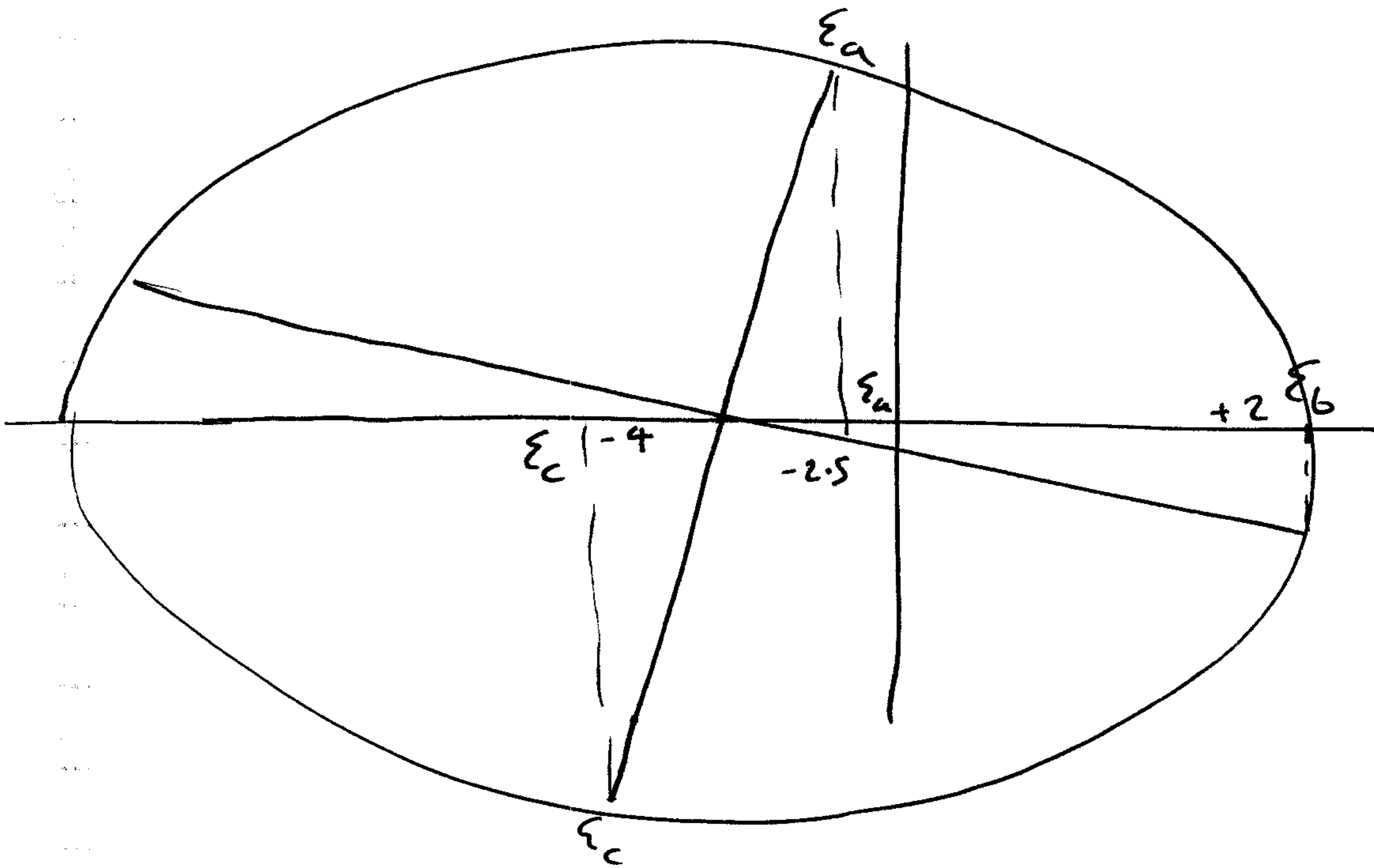
M20

45° Rosette

$$\epsilon_a = -2.5 \text{ m}\epsilon = 2500 \mu\epsilon$$

$$\epsilon_b = +2.0 \text{ m}\epsilon = 2000 \mu\epsilon$$

$$\epsilon_c = -4.0 \text{ m}\epsilon = 4000 \mu\epsilon$$



center of circle @  $-3.25 \text{ m}\epsilon$

$$\text{Radius} = \sqrt{(2 - (-3.25))^2 + (3.25 - 2.5)^2} = 5.3 \text{ m}\epsilon$$

Principal strains =  $-3.25 \text{ m}\epsilon \pm 5.3 \text{ m}\epsilon$

$$\epsilon_{\text{I}} = +2053 \mu\epsilon$$

$$\epsilon_{\text{II}} = -8553 \mu\epsilon$$

a) State of strain

$$\epsilon_{11} = \epsilon_a = -2500 \mu\epsilon, \quad \epsilon_{22} = -4000 \mu\epsilon, \quad \epsilon_{12} = \frac{1}{2}(20 - (-32.5)) = 2625 \mu\epsilon$$

$$\epsilon_{11} = \epsilon_b = +2000 \mu\epsilon, \quad \epsilon_{22} = -32.5 - (5.25) = -8500 \mu\epsilon, \quad \epsilon_{12} = \frac{1}{2}(7.5) = 3750 \mu\epsilon$$

Linear elasticity

Plane stress

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \nu \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E} = 0 \quad (1)$$

$$\epsilon_{22} = -\nu \frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \nu \frac{\sigma_{33}}{E} = 0 \quad (2)$$

Multiply (1) by  $\nu$  and add to (2)

$$\nu \epsilon_{11} + \epsilon_{22} = \frac{\sigma_{22}}{E} (1 - \nu^2)$$

$$\sigma_{22} = \frac{E (\nu \epsilon_{11} + \epsilon_{22})}{(1 - \nu^2)} = \frac{70 \times 10^9 (0.33 \times (-2500) + (-4000)) \times 10^{-6}}{(1 - (0.33)^2)}$$

$$\sigma_{\epsilon} = \sigma_{22} = -3800 \text{ MPa} \Leftarrow$$

$$\sigma_a = \sigma_{11} = -300 \text{ MPa} \Leftarrow \left( \frac{E (\nu \epsilon_{22} + \epsilon_{11})}{(1 - \nu^2)} \right)$$

$$\text{Similarly for } \sigma_b = \sigma_{33} = \frac{E (\nu \epsilon_{22} + \epsilon_{33})}{(1 - \nu^2)} = -63 \text{ MPa} \Leftarrow$$

Principal stresses for principal strains

$$\sigma_I = \frac{E (\epsilon_{II} + \nu \epsilon_{II})}{(1 - \nu^2)} = -60.4 \text{ MPa} \Leftarrow$$

$$\sigma_{II} = \frac{E (\epsilon_{II} + \nu \epsilon_I)}{(1 - \nu^2)} = -618.7 \text{ MPa} \Leftarrow$$

$$\sigma_{III} = 0$$



M21 a) uniaxial loading

$$\epsilon_z = \frac{\sigma_z}{E} = \frac{100 \times 10^6}{3 \times 10^9} = 0.033 \text{ } \Leftarrow$$

$$\epsilon_y = \epsilon_x = -\nu \epsilon_z = -0.3 \times (0.033) = -0.01 \text{ } \Leftarrow$$

b) Assume  $\epsilon_x = \epsilon_y = 0$  ( $E_{\text{SiC}} \gg E_{\text{epoxy}}$ )

$$\begin{pmatrix} 0 \\ 0 \\ \epsilon_z \end{pmatrix} = \begin{pmatrix} \frac{\sigma_x}{E} & -\frac{\nu \sigma_y}{E} & -\frac{\nu \sigma_z}{E} \\ -\frac{\nu \sigma_x}{E} & \frac{\sigma_y}{E} & -\frac{\nu \sigma_z}{E} \\ -\frac{\nu \sigma_x}{E} & -\frac{\nu \sigma_y}{E} & \frac{\sigma_z}{E} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

by symmetry  $\sigma_x = \sigma_y = \sigma_T$   $\sigma_z = 100 \text{ MPa}$

$$0 = \frac{\sigma_T}{E} (1 - \nu) - \frac{\nu \sigma_z}{E} \quad \textcircled{1}$$

$$\epsilon_z = \frac{-2\nu \sigma_T}{E} + \frac{\sigma_z}{E} \quad \textcircled{2}$$

$$\text{from } \textcircled{1} \quad \sigma_T = \frac{\nu \sigma_z}{(1 - \nu)} = \frac{0.3 \times 100 \times 10^6}{(1 - 0.3)}$$

$$\sigma_T = 42.9 \text{ MPa}$$



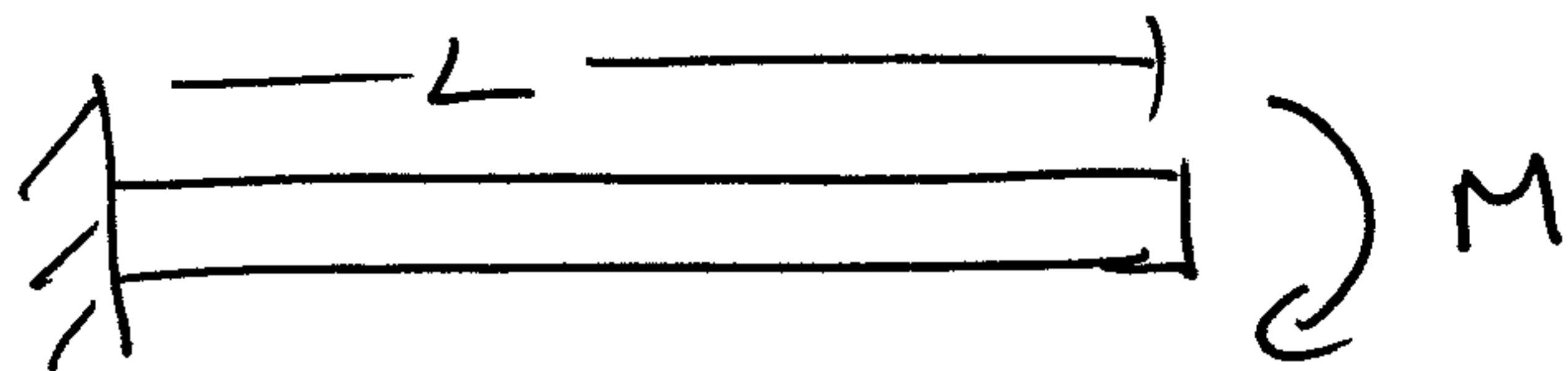
$$\epsilon_z = \frac{1}{E} (-2\nu\sigma_T + \sigma_z)$$

$$= \frac{1}{2 \times 10^9} (-2 \times 0.3 \times 42.9 + 100) \times 10^6$$

$$= 0.0371 \quad \Leftarrow$$

(Note  $\frac{\epsilon_z \sigma_z}{E} = 0.05$  so laminate restraint makes epoxy appear stiffer)





M22

$$\delta = \frac{2ML^2}{\pi R^4 E}$$

$$\text{mass, } m = \rho \pi R^2 L$$

R is the free variable,  $\therefore$  eliminate

$$R = \sqrt{\frac{m}{\rho \pi L}}$$

$$\Rightarrow \delta = \frac{2ML^2}{\pi E} \left( \frac{\rho \pi L}{m} \right)^2$$

$$\text{Mass} = \pi \left( \frac{2M}{\pi \delta} \right)^{\frac{1}{2}} \cdot (L^3) \left( \frac{\rho}{E^{\frac{1}{2}}} \right)$$

F                      G                      M

minimize  $\left( \frac{\rho}{E^{\frac{1}{2}}} \right)$ , maximize  $\frac{E^{\frac{1}{2}}}{\rho}$   $\Leftarrow$

	$\rho$ (kg/m <sup>3</sup> )	E (GPa)	$E^{\frac{1}{2}}/\rho$
6) Steel	7.9	203	1.8
Al	2.8	71	3.0
Ti	4.5	120	2.4
CFRP	1.5	230	10.11 $\Leftarrow$
HDPE	0.96	1.1	1.1
Wood	0.6	12	5.8
SiC	3.0	410	6.8

Choose CFRP



c) choose material comparable in bending stiffness to bone - match  $E^{1/2}/\rho$

18 GPa,  $\rho = 1.55$

Draw line of constant  $E^{1/2}/\rho$  on graph

possibilities

- low modulus GFRP, CFRP, KFRP laminates

Cement

Rock, stone

Ti alloys  $\Leftarrow$

ZrO<sub>2</sub>

Choose Ti - these are used!