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Unified Engineering
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Problem Set #10
Solutions

UNIFIED ENGINEERING

Problem Set #10 -- SOLUTIONS

10(M).1 Begin by writing out the stress equilibrium equations as we have them in tensorial notation:

$$\frac{\partial \sigma_{mn}}{\partial x_n} + f_n = 0$$

expand this:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

Recall the symmetry of the stress tensor: $\sigma_{mn} = \sigma_{nm}$

(a) To go from tensorial to engineering notation recall:

$$x_1 \rightarrow x$$
$$x_2 \rightarrow y$$
$$x_3 \rightarrow z$$

And a similar conversion on subscripts on the stresses. So:

$$\boxed{\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0 \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0 \\ &\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{aligned}}$$

Notes:
- expression of stress with subscripts still remains symmetric
- τ can be used in place of σ for <u>shear</u> stresses (2 different subscripts: $\tau_{xy}, \tau_{xz}, \tau_{yz}$)

(b) A state of plane stress has
- no out-of-plane components
  $\Rightarrow \sigma_z = \sigma_{yz} = \sigma_{xz} = 0$
- no out-of-plane gradient
  $\Rightarrow \frac{\partial}{\partial z} = 0$

And, since there are no forces in the out-of-plane (z) direction, the body force in that direction (f_z) must be zero.

We thus end up with two equations:

$$\boxed{\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0 \\ &\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \end{aligned}}$$

This could also be done in tensorial notation:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

10 (M). 2

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{bmatrix} B_{111} & B_{122} & 2B_{112} \\ B_{211} & B_{222} & 2B_{212} \\ B_{311} & B_{322} & 2B_{312} \end{bmatrix} \begin{Bmatrix} C_{11} \\ C_{22} \\ C_{12} \end{Bmatrix}$$

First write this out in full (as it may help):

$$A_1 = B_{111} C_{11} + B_{122} C_{22} + 2 B_{112} C_{12}$$

$$A_2 = B_{211} C_{11} + B_{222} C_{22} + 2 B_{212} C_{12}$$

$$A_3 = B_{311} C_{11} + B_{322} C_{22} + 2 B_{312} C_{12}$$

Look at this piece by piece:

① the subscript on A must be a free index because it changes with equation and represents separate equations. It must be latin since it takes on the values 1, 2, 3

$(= A_m)$

(2) The subscripts on C take on the values 1 and 2 and therefore must be greek. The change independently and thus must be different

($C_{\alpha\beta}$)

(3) The first subscript on B matches the subscript on A

($A_m = B_{m??} C_{\alpha\beta}$)

(4) The second and third subscripts match those on C. By making them the same, they are also summed on (as occurs in the equations).

$$\Rightarrow \quad \boxed{A_m = B_{m\alpha\beta} C_{\alpha\beta}}$$

But, one must also make the assumption that $C_{\alpha\beta}$ is symmetric ($C_{\alpha\beta} = C_{\beta\alpha}$) and $B_{m\alpha\beta}$ is symmetric in the last two indices ($B_{m\alpha\beta} = B_{m\beta\alpha}$) to get the factor of 2 in the final equations on the C_{12} terms with B_{m12} as multipliers.

10(M).3 We have a two-dimensional field of displacement, thus all out-of-plane strains are zero:

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$$

Define the in-plane strain-displacement relations

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

and $\underline{u} = u_1 \underline{i}_1 + u_2 \underline{i}_2$

(a) $\underline{u} = (0.015 x_1)\underline{i}_1 - (0.030 x_2)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.015$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.030$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0+0) = 0$$

$$\boxed{\begin{aligned} \epsilon_{11} &= 0.015 \\ \epsilon_{22} &= -0.030 \\ \epsilon_{12} &= 0 \end{aligned}}$$

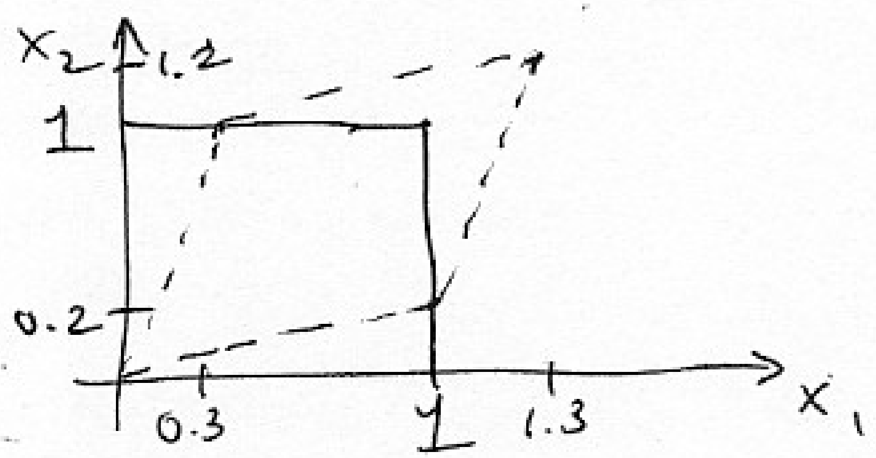
This is pure elongation in two directions (one being positive, one being negative)

(b) $\underline{u} = (0.030 x_2)\underline{i}_1 + (0.020 x_1)\underline{i}_2$

$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$

$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$

$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0.030 + 0.020) = 0.025$



$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0.025 \end{aligned}}$

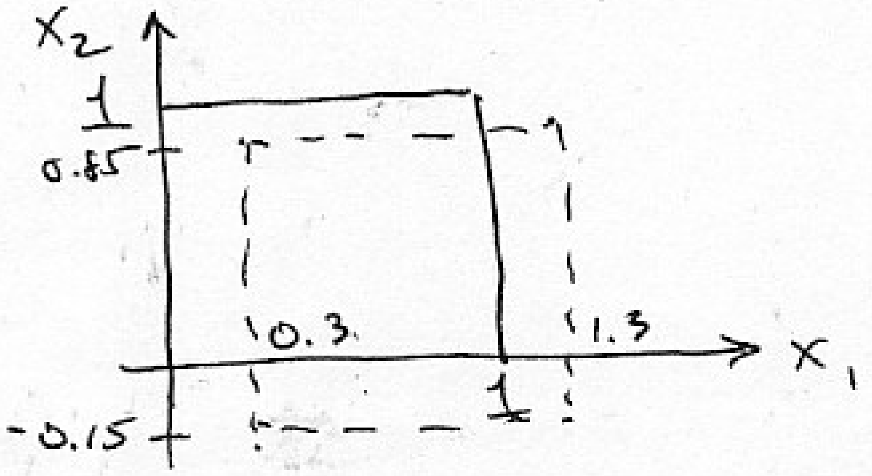
This is <u>pure</u> <u>shear</u>

(c) $\underline{u} = (0.030)\underline{i}_1 - (0.015)\underline{i}_2$

$\epsilon_{11} = \partial u_1/\partial x_1 = 0$

$\epsilon_{22} = \partial u_2/\partial x_2 = 0$

$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0 + 0) = 0$



$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$

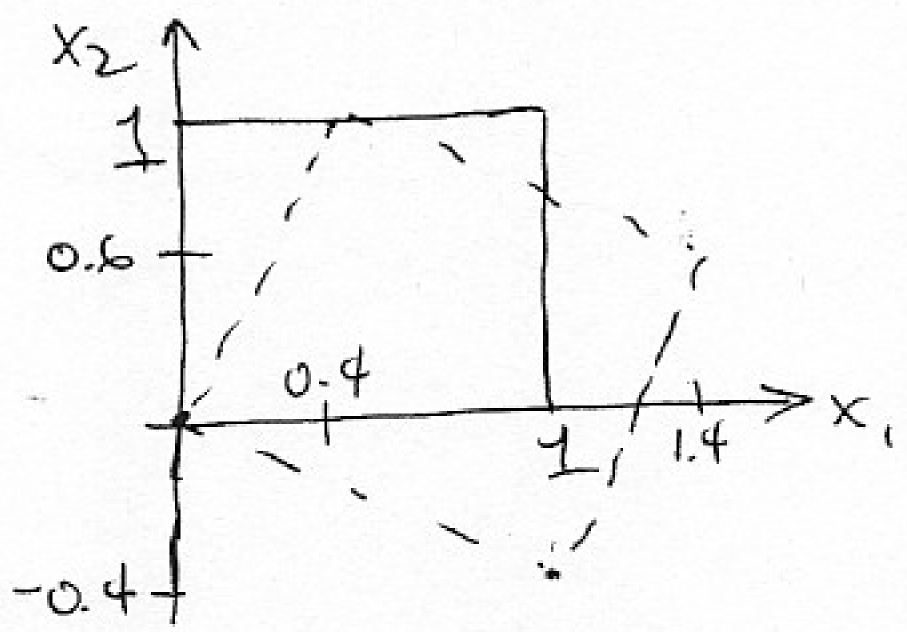
This is <u>pure</u> <u>translation</u> in x_1 and x_2

(d) $\underline{u} = (0.040 x_2)\underline{i}_1 - (0.040 x_1)\underline{i}_2$

$\epsilon_{11} = \partial u_1/\partial x_1 = 0$

$\epsilon_{22} = \partial u_2/\partial x_2 = 0$

$\epsilon_{12} = \frac{1}{2}\left(\partial u_1/\partial x_2 + \partial u_2/\partial x_1\right) = \frac{1}{2}(0.040 - 0.040) = 0$



$$\boxed{\begin{aligned}\epsilon_{11} &= 0\\ \epsilon_{22} &= 0\\ \epsilon_{12} &= 0\end{aligned}}$$

This is pure rotation

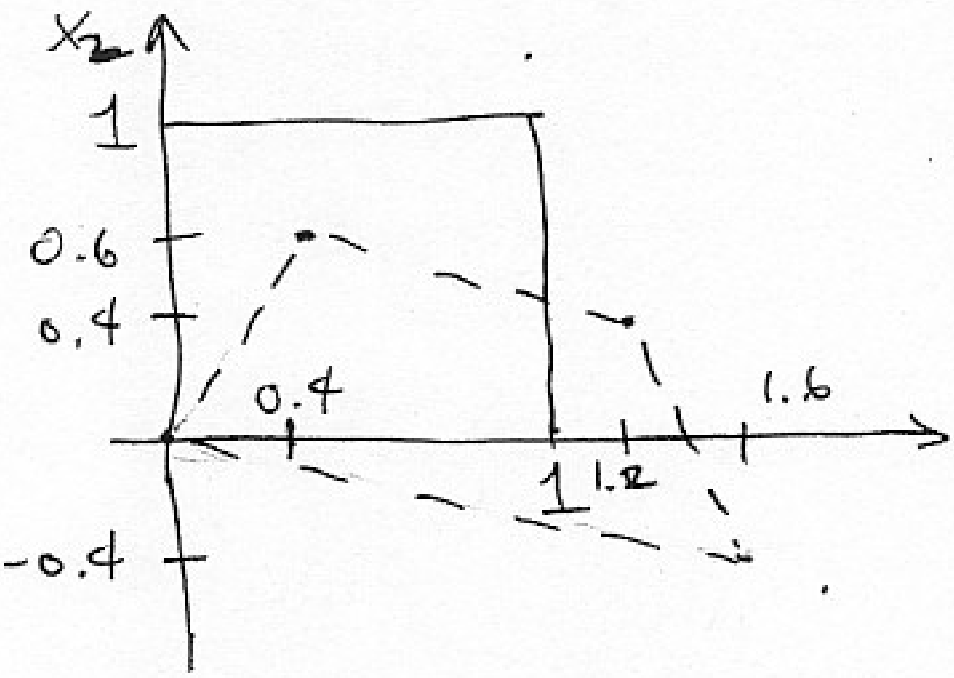
(e) $\underline{u} = (0.060 x_1 - 0.040 x_2)\underline{i}_1 + (0.040 x_1 - 0.020 x_2)\underline{i}_2$

$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.060$

$\epsilon_{22} = \partial u_2/\partial x_2 = -0.020$

$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(-0.040 - 0.040)$

$= -0.040$



$$\boxed{\begin{aligned} \epsilon_{11} &= 0.060 \\ \epsilon_{22} &= -0.020 \\ \epsilon_{12} &= -0.040 \end{aligned}}$$

This is combined elongation and shear

a) Using given formulas: $a_1 = 20\,T_1^{1/2} = 20(223)^{1/2} = 299$ m/s

$$\boxed{P_1 = \rho_1 R T_1 = 0.414 \cdot 287 \cdot 223 = 2.65 \times 10^4 \text{ Pa}}$$

also $\boxed{M_1 = \frac{V_1}{a_1} = \frac{250}{299} = 0.836}$

b) To match the two flows, must have $Re_1 = Re_2$, $M_1 = M_2$

Re: $\frac{\rho_1 V_1 l_1}{\mu_1} = \frac{\rho_2 V_2 l_2}{\mu_2}$ where l is some length on object. (e.g. chord)

For $\frac{1}{5}$ scale model, $l_2 = \frac{1}{5} l_1$, also $\mu_1 = 10^{-6} T_1^{1/2}$ and $\mu_2 = 10^{-6} T_2^{1/2}$

so $\frac{\rho_1 V_1 \cancel{l_1}}{\cancel{10^{-6}} T_1^{1/2}} = \frac{1}{5} \frac{\rho_2 V_2 \cancel{l_1}}{\cancel{10^{-6}} T_2^{1/2}}$ or $\frac{\rho_1 V_1}{T_1^{1/2}} = \frac{1}{5} \frac{\rho_2 V_2}{T_2^{1/2}}$

M: $\frac{V_1}{a_1} = \frac{V_2}{a_2}$ or $\frac{V_1}{20\,T_1^{1/2}} = \frac{V_2}{20\,T_2^{1/2}}$ or $\frac{V_1}{T_1^{1/2}} = \frac{V_2}{T_2^{1/2}}$

combine

$$\boxed{\rho_2 = 5 \rho_1 = 2.07 \text{ kg/m}^3}$$

$p_2 = 10^5$ Pa is given

$$\rightarrow \boxed{T_2 = \frac{p_2}{\rho_2 R} = \frac{10^5}{2.07 \cdot 287} = 168 \text{ K}} \quad \text{(cold!)}$$

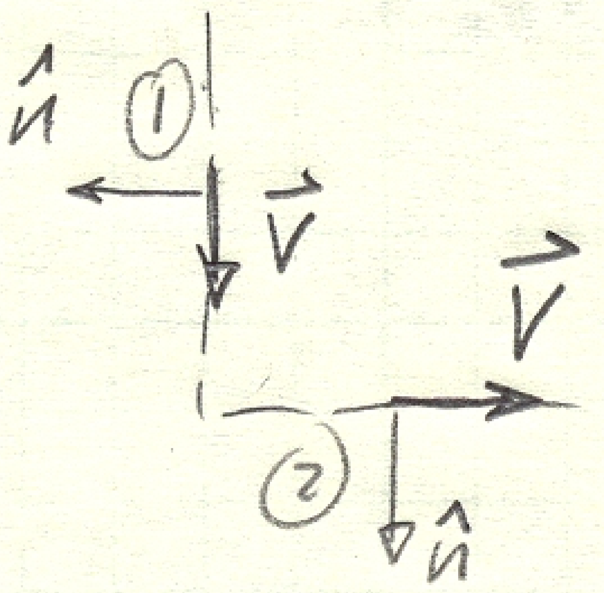
Also, $a_2 = 20\,(168)^{1/2} = 259$ m/s

$$\boxed{V_2 = M_2 a_2 = 0.836 \cdot 259 = 216 \text{ m/s}}$$

a) Segment 1: $u = x = 0$, so $\vec{V} = v\hat{j} = -y\hat{j}$

$\hat{n} = -\hat{i}$, $\rightarrow \vec{V}\cdot\hat{n} = -y\hat{j}\cdot\hat{i} = 0$

$\therefore \; I_1 = 0$

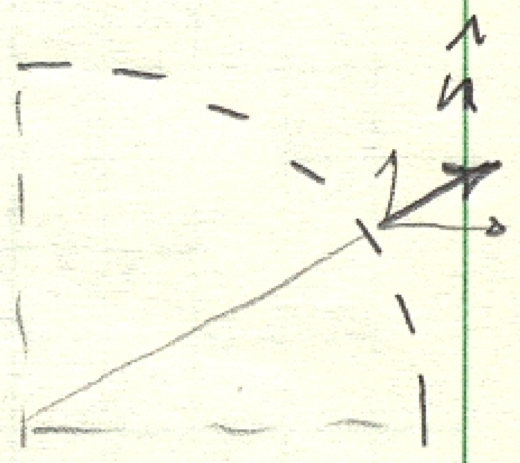


Segment 2: $v = -y = 0$, so $\vec{V} = u\hat{i} = x\hat{i}$

$\hat{n} = -\hat{j} \rightarrow \vec{V}\cdot\hat{n} = 0$

$\therefore \; I_2 = 0$

Segment 3: $u = x = R\cos\theta$, $v = -y = -R\sin\theta$



$\vec{V} = \hat{i}\,R\cos\theta - \hat{j}\,R\sin\theta$

$\hat{n} = \hat{i}\cos\theta + \hat{j}\sin\theta$

$\vec{V}\cdot\hat{n} = R(\cos^2\theta - \sin^2\theta) = R\cos 2\theta$

Also, $dA = R\,d\theta$

$$\therefore \; I_3 = \int_0^{\pi/2} \rho\vec{V}\cdot\hat{n}\,R\,d\theta = \int_0^{\pi/2} \rho R^2 \cos 2\theta\,d\theta$$

$I_1 = I_2 = 0$ as shown above

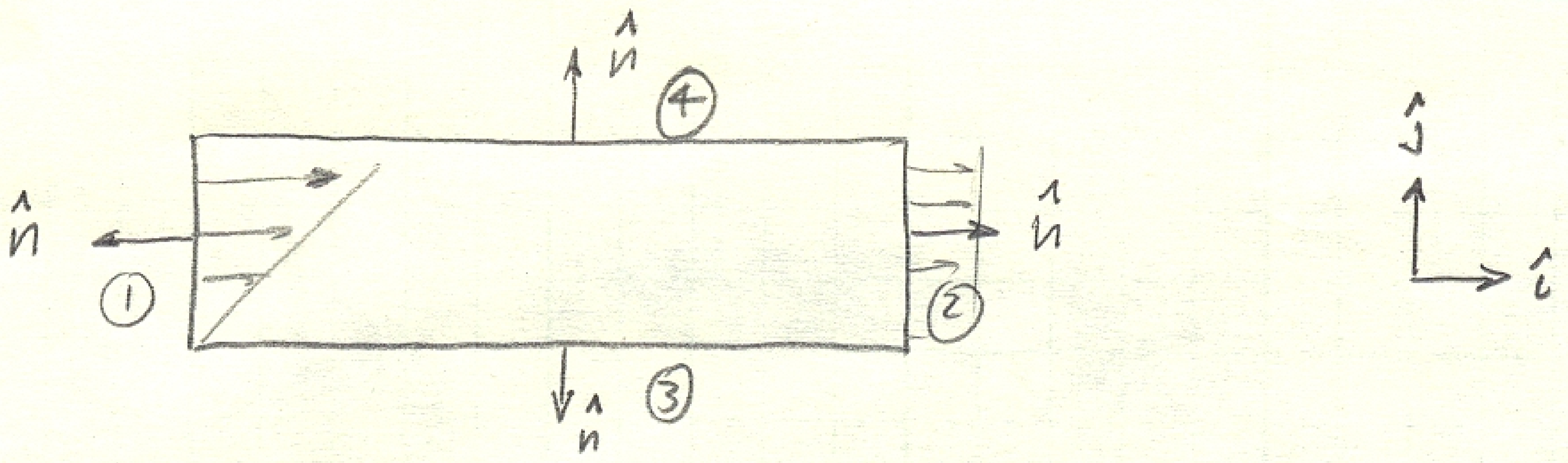
$$I_3 = \rho R^2 \int_0^{\pi/2} \cos 2\theta\,d\theta = \rho R^2 \left(\tfrac{1}{2}\sin 2\theta\Big|_0^{\pi/2}\right) = 0$$

$$I = I_1 + I_2 + I_3 = 0$$

b) This flow satisfies mass conservation, which requires $\oint \rho\vec{V}\cdot\hat{n}\,dA = 0$ for any control volume. We have $I=0$ for any R.

Alternatively, using Gauss' Theorem, we have $\oint \rho\vec{V}\cdot\hat{n}\,dA = \iint \nabla\cdot(\rho\vec{V})\,dx\,dy$ and for this flow, $\nabla\cdot(\rho\vec{V}) = \rho\,\nabla\cdot\vec{V} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

so $\iint \nabla\cdot(\rho\vec{V})\,dx\,dy = \iint 0\,dx\,dy = 0$



a) $\oint \rho \vec{V}\cdot\hat{n}\, dA = \int_{①} + \int_{②} + \int_{③} + \int_{④} = 0$

On ③, ④, $\vec{V}\cdot\hat{n} = 0$ so $\int_{③} + \int_{④} = 0$

On ①: $\vec{V}\cdot\hat{n} = -u_1(y) = -\bar{u}_1 \frac{y}{h}$

On ②; $\vec{V}\cdot\hat{n} = u_2$

$\therefore \oint \rho \vec{V}\cdot\hat{n}\, dA = \int_0^h -\rho \bar{u}_1 \frac{y}{h}\, dy + \int_0^h \rho u_2\, dy = 0$

$-\rho \bar{u}_1 \frac{1}{2} h + \rho u_2 h = 0$

$$\boxed{u_2 = \frac{1}{2}\bar{u}_1}$$

b) Take x-component of momentum integral: $\hat{i}\cdot\oint\left[\rho(\vec{V}\cdot\hat{n})\vec{V} + p\hat{n}\right] dA = 0$

On ③, ④: $\vec{V}\cdot\hat{n} = 0$, $\hat{i}\cdot\hat{n} = 0$, so $\int_{③} + \int_{④} = 0$

On ①: $\hat{i}\cdot\left[\rho(\vec{V}\cdot\hat{n})\vec{V} + p\hat{n}\right] = -\rho u_1 u_1 - p_1 = -\rho \bar{u}_1^2 \left(\frac{y}{h}\right)^2 - p_1$

On ②: $\hat{i}\cdot\left[\rho(\vec{V}\cdot\hat{n})\vec{V} + p\hat{n}\right] = \rho u_2 \cdot u_2 + p_2 = \rho u_2^2 + p_2$

$\therefore \oint \hat{i}\cdot\left[\rho(\vec{V}\cdot\hat{n})\vec{V} + p\hat{n}\right] dA = \int_0^h \left[-\rho \bar{u}_1^2 \left(\frac{y}{h}\right)^2 - p_1\right] dy + \int_0^h \left[\rho u_2^2 + p_2\right] dy = 0$

$-\rho \bar{u}_1^2 \frac{1}{3} h - p_1 h + \rho u_2^2 h + p_2 h = 0$

but since $u_2 = \frac{1}{2}\bar{u}_1$: $-\frac{1}{3}\rho \bar{u}_1^2 - p_1 + \frac{1}{4}\rho \bar{u}_1^2 + p_2 = 0$

$$\boxed{p_2 - p_1 = \frac{1}{12}\rho \bar{u}_1^2} > 0$$

pressure increases during the mixing