10(m).1 Begin by writing out the stress equilibrium equations as we have them in tensorial notation:

\[ \frac{\partial \tau_{mn}}{\partial x_n} + f_m = 0 \]

Expand this:

\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0 \]

\[ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0 \]

\[ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0 \]

Recall the symmetry of the stress tensor: \( \tau_{mn} = \tau_{nm} \)

(a) To go from tensorial to engineering notation, recall:

\[ x_1 \rightarrow x \]
\[ x_2 \rightarrow y \]
\[ x_3 \rightarrow z \]

And a similar conversion on subscripts on the stresses, so:
\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z &= 0
\end{align*}
\]

Notes:
- Expression of stress with subscript still remains symmetric.
- It can be used in place of \( \tau \) for shear stresses (2 different subscripts: \( \tau_{xy}, \tau_{xz}, \tau_{yz} \))

(5) A state of plane stress has:
- no out-of-plane components
  \[ \sigma_z = \tau_{yz} = \tau_{xz} = 0 \]
- no out-of-plane gradient
  \[ \frac{\partial \sigma}{\partial z} = 0 \]

And since there are no forces in the out-of-plane (z) direction, the body force in that direction (\( f_z \)) must be zero.

We thus end up with two equations:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0
\end{align*}
\]
This could also be done in tensorial notation:
\[
\begin{align*}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 &= 0 \\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 &= 0
\end{align*}
\]

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\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} =
\begin{bmatrix}
B_{111} & B_{122} & 2B_{112} \\
B_{211} & B_{222} & 2B_{212} \\
B_{311} & B_{322} & 2B_{312}
\end{bmatrix}
\begin{bmatrix}
C_{11} \\
C_{22} \\
C_{12}
\end{bmatrix}
\]

First write this out in full (as it may help):
\[
\begin{align*}
A_1 &= B_{111} C_{11} + B_{122} C_{22} + 2B_{112} C_{12} \\
A_2 &= B_{211} C_{11} + B_{222} C_{22} + 2B_{212} C_{12} \\
A_3 &= B_{311} C_{11} + B_{322} C_{22} + 2B_{312} C_{12}
\end{align*}
\]

Look at this piece by piece:
1. The subscript on \( A \) must be a free index because it changes with equation and represents separate equations. It must be latin since it takes on the values 1, 2, 3
   \( (= \text{A}_{1m}) \)
(2) The subscripts on C take on the values 1 and 2 and therefore must be Greek. The change independently and thus must be different

\( (C_{\alpha\beta}) \)

(3) The first subscript on B matches the subscript on A

\( (A_m = B_m?C_{\alpha\beta}) \)

(4) The second and third subscripts match those on C. By making them the same, they are also summed on C as occurs in the equation.

\[ \Rightarrow \quad A_m = B_{m\alpha\beta}C_{\alpha\beta} \]

But, one must also make the assumptions that \( C_{\alpha\beta} \) is symmetric \( (C_{\alpha\beta} = C_{\beta\alpha}) \) and \( B_{m\alpha\beta} \) is symmetric in the last two indices \( (B_{m\alpha\beta} = B_{m\beta\alpha}) \) to fit the factor of 2 in the final equations on the C, z terms with \( B_{m\beta\alpha} \) as multipliers.
We have a two-dimensional field of displacement, thus all out-of-plane strains are zero:

\[ \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0 \]

Define the in-plane strain-displacement relations:

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \]

and

\[ u = u_1 \hat{i}_1 + u_2 \hat{i}_2 \]

(a) \[ u = (0.015x_1) \hat{i}_1 - (0.030x_2) \hat{i}_2 \]

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.015 \]

\[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.030 \]

\[ \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (0 + 0) = 0 \]

--- undehomed
--- deformed

This is pure elongation in two directions (one being positive, one being negative).
(b) \( \mathbf{u} = (0.030x_2) \mathbf{i}_1 + (0.020x_1) \mathbf{i}_2 \)

\[
\begin{align*}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = 0 \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = 0 \\
\varepsilon_{12} &= \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (0.030 + 0.020) = 0.025
\end{align*}
\]

This is pure shear

(c) \( \mathbf{u} = (0.030) \mathbf{i}_1 - (0.05) \mathbf{i}_2 \)

\[
\begin{align*}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = 0 \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = 0 \\
\varepsilon_{12} &= \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (0 + 0) = 0
\end{align*}
\]

This is pure translation in \( x_1 \) and \( x_2 \)
(d) \[ u = (0.040x_2)i_1 - (0.040x_1)i_2 \]

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0 \]
\[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0 \]
\[ \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (0.040 - 0.040) = 0 \]

This is pure rotation

(e) \[ u = (0.060x_1 - 0.040x_2)i_1 + (0.040x_1 - 0.020x_2)i_2 \]

\[ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.060 \]
\[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.020 \]
\[ \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \frac{1}{2} (-0.040 - 0.040) = -0.040 \]
$\varepsilon_{11} = 0.060$

$\varepsilon_{22} = -0.020$

$\varepsilon_{12} = -0.040$

This is combined elongation and shear.
a) Using given formulas: \( a_1 = 20 \frac{T_1}{y} = 20(223)^{\frac{1}{6}} = 299 \text{ m/s} \)
\[
P_1 = \rho_1 RT_1 = 0.414 \cdot 287 \cdot 223 = 2.65 \times 10^5 \text{ Pa}
\]
also \( M_1 = \frac{V_1}{a_1} = \frac{250}{299} = 0.836 \)

b) To match the two flows, must have \( R_1 = R_2 \), \( M_1 = M_2 \)

\[
Re: \quad \frac{\rho_1 V_1 l}{\mu_1} = \frac{\rho_2 V_2 l}{\mu_2}
\]
where \( l \) is some length on object: (e.g. chord)

For \( \frac{1}{5} \) scale model, \( l_2 = \frac{1}{5} l_1 \), also \( \mu_1 = 10^{-6} t_1 \) and \( \mu_2 = 10^{-6} T_2 \)

so \( \frac{\rho_1 V_1 l}{10^{-6} T_1 l} = \frac{1}{5} \frac{\rho_2 V_2 l}{10^{-6} T_2 l} \) or \( \frac{\rho_1 V_1}{T_1 l} = \frac{1}{5} \frac{\rho_2 V_2}{T_2 l} \)

\[
M: \quad \frac{V_1}{a_1} = \frac{V_2}{a_2} \quad \text{or} \quad \frac{V_1}{20 T_1 l} = \frac{V_2}{20 T_2 l} \quad \text{or} \quad \frac{V_1}{T_1 l} = \frac{V_2}{T_2 l}
\]

\[
\rho_2 = 5 \rho_1 = 2.07 \text{ kg/m}^3
\]
\( \rho_2 = 10^5 \text{ Pa} \) is given

\[
T_2 = \frac{P_2}{\rho_2 R} = \frac{10^5}{2.07 \cdot 287} = 168 \text{ K} \quad \text{(cold!)}
\]
Also, \( a_2 = 20 (168)^{\frac{1}{6}} = 259 \text{ m/s} \)

\[
V_2 = M_2 a_2 = 0.836 \cdot 259 = 216 \text{ m/s}
\]
a) Segment 1: \( u = x = 0 \), so \( \mathbf{V} = x \mathbf{\hat{u}} - y \mathbf{\hat{v}} \)
\[ \mathbf{\hat{u}} = -\mathbf{\hat{i}} , \quad \rightarrow \mathbf{V} \cdot \mathbf{\hat{u}} = -y \mathbf{\hat{u}} \cdot \mathbf{\hat{i}} = 0 \]
\[ I_1 = 0 \]

Segment 2: \( v = -y = 0 \), so \( \mathbf{V} = u \mathbf{\hat{u}} = x \mathbf{\hat{u}} \)
\[ \mathbf{\hat{v}} = -\mathbf{\hat{j}} \quad \rightarrow \mathbf{V} \cdot \mathbf{\hat{v}} = 0 \]
\[ I_2 = 0 \]

Segment 3: \( u = x = R \cos \theta \), \( v = -y = -R \sin \theta \)
\[ \mathbf{V} = R \cos \theta \mathbf{\hat{u}} - R \sin \theta \mathbf{\hat{v}} \]
\[ \mathbf{\hat{u}} = \cos \theta \mathbf{\hat{i}} + \sin \theta \mathbf{\hat{j}} \]
\[ \mathbf{V} \cdot \mathbf{\hat{u}} = R (\cos^2 \theta - \sin^2 \theta) = R \cos 2\theta \]
Also, \( dA = R \, d\theta \)
\[ I_3 = \int_{\theta_1}^{\theta_2} \rho \mathbf{V} \cdot \mathbf{\hat{n}} \, R \, d\theta = \int_{\theta_1}^{\theta_2} \rho R^2 \cos 2\theta \, d\theta \]
\[ I_1 = I_2 = 0 \quad \text{as shown above} \]
\[ I_3 = \rho R^2 \int_{\theta_1}^{\theta_2} \cos 2\theta \, d\theta = \rho R^2 \left( \frac{1}{2} \sin 2\theta \right)_{\theta_1}^{\theta_2} = 0 \]
\[ I = I_1 + I_2 + I_3 = 0 \]

b) This flow satisfies mass conservation, which requires \( \oint p \mathbf{V} \cdot \mathbf{n} \, dA = 0 \) for any control volume. We have \( I = 0 \) for any \( R \).
Alternatively, using Gauss' Theorem, we have \( \oint p \mathbf{V} \cdot \mathbf{n} \, dA = \iiint \nabla \cdot (p \mathbf{V}) \, dV \)
and for this flow, \( \nabla \cdot (p \mathbf{V}) = p \nabla \cdot \mathbf{V} = p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \)
so \( \iiint \nabla \cdot (p \mathbf{V}) \, dV = \iiint 0 \, dV = 0 \)
a) \[ \oint p \nabla \cdot \hat{n} dA = \oint_1 + \oint_2 + \oint_3 + \oint_4 = 0 \]

On \(\oint_3, \oint_4\): \(\nabla \cdot \hat{n} = 0\), so \(\oint_3 + \oint_4 = 0\)

On \(\oint_1\): \(\nabla \cdot \hat{n} = -u_1 \frac{dy}{h} = -\frac{\bar{u}_1}{h}\)

On \(\oint_2\): \(\nabla \cdot \hat{n} = u_2\)

\[ \oint p \nabla \cdot \hat{n} dA = \int_0^h \left(-p \bar{u}_1 \frac{dy}{h}\right) + \int_0^h \rho u_2 dy = 0 \]

\[ -p \bar{u}_1 \frac{1}{2} h + \rho u_2 h = 0 \]

\[ u_2 = \frac{1}{2} \bar{u}_1 \]

b) Take x-component of momentum integral:

\[ \oint \hat{x} \cdot [p (\nabla \cdot \hat{n}) \hat{V} + \rho \hat{n}] dA = 0 \]

On \(\oint_3, \oint_4\): \(\nabla \cdot \hat{n} = 0\), \(\oint_3 + \oint_4 = 0\)

On \(\oint_1\): \(\hat{x} \cdot \left[p (\nabla \cdot \hat{n}) \hat{V} + \rho \hat{n}\right] = -p \bar{u}_1 u_1 - p_1 = -\frac{1}{2} \rho \bar{u}_1^2 \frac{y^2}{h} - p_1\)

On \(\oint_2\): \(\hat{x} \cdot \left[p (\nabla \cdot \hat{n}) \hat{V} + \rho \hat{n}\right] = \rho u_2 \cdot u_2 + p_2 = \rho u_2^2 + p_2\)

\[ \oint \hat{x} \cdot [p (\nabla \cdot \hat{n}) \hat{V} + \rho \hat{n}] dA = \int_0^h \left(-\frac{1}{2} p \bar{u}_1^2 \frac{y^2}{h} - p_1\right) dy + \int_0^h \left[\rho u_2^2 + p_2\right] dy = 0 \]

\[ -\frac{1}{2} \rho \bar{u}_1^2 \frac{1}{2} h - p_1 h + \rho u_2^2 h + p_2 h = 0 \]

but since \(u_2 = \frac{1}{2} \bar{u}_1\):

\[ -\frac{1}{6} \rho \bar{u}_1^2 - p_1 + \frac{1}{4} \rho \bar{u}_1^2 + p_2 = 0 \]

\[ p_2 - p_1 = \frac{1}{12} \rho \bar{u}_1^2 > 0 \]

pressure increases during the mixing