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Unified Engineering
Fall 2004

Problem Set #11
Solutions

a) Pathline given by $x_A(t)$, $y_A(t)$

Defining equations:

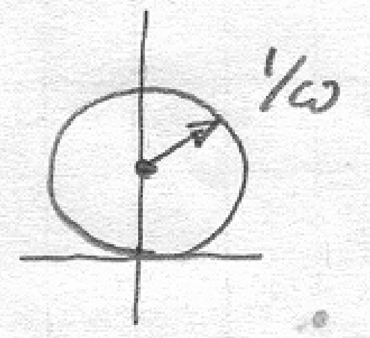
$$x_A(t) = \cancel{x_0}^{\,0} + \int_0^t u(x_A(\tau), y_A(\tau), \tau)\, d\tau$$
$$y_A(t) = \cancel{y_0}^{\,0} + \int_0^t v(x_A(\tau), y_A(\tau), \tau)\, d\tau$$

Velocity components depend only on τ (spatially uniform),

$$u(\tau) = \cos\omega\tau \qquad v = \sin\omega\tau$$

$$\rightarrow \left[\begin{array}{l} x_A(t) = 0 + \int_0^t \cos\omega\tau\, d\tau = \frac{1}{\omega}\sin\omega t \\ y_A(t) = 0 + \int_0^t \sin\omega\tau\, d\tau = \frac{1}{\omega}\left[1-\cos\omega t\right] \end{array}\right.$$

parametric definition of a circle of radius $1/\omega$



Note: Can write $\cos\omega t = \pm\sqrt{1-\sin^2\omega t} = \pm\sqrt{1-(\omega x_A)^2}$

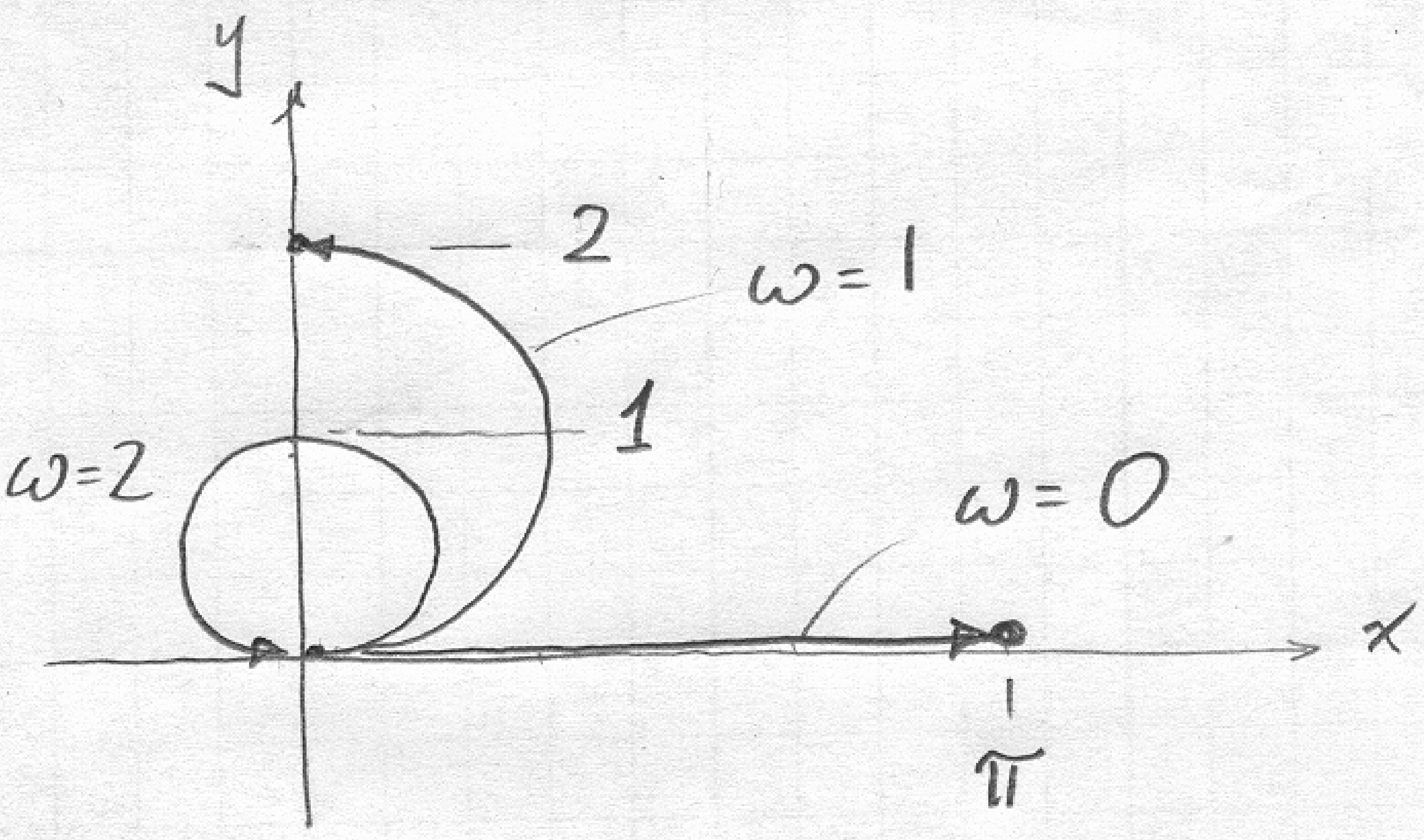
So that $y_A = \frac{1}{\omega} \pm \sqrt{\frac{1}{\omega} - x_A^2}$

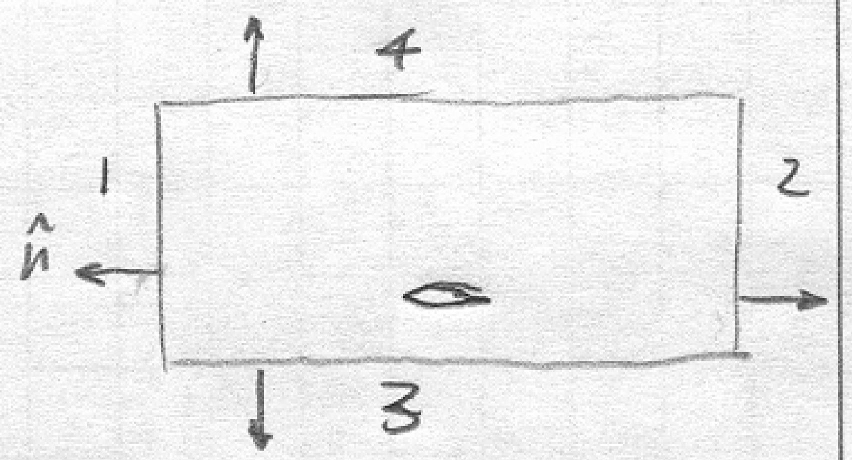
b) Must first evaluate limiting case of $\omega = 0$

$$x_A(t) = \lim_{\omega\to 0} \frac{\sin\omega t}{\omega} = \frac{t\cos\omega t}{1} = t$$
$$y_A(t) = \lim_{\omega\to 0} \frac{1-\cos\omega t}{\omega} = \frac{t\sin\omega t}{1} = 0$$

Could also have noted that $u=1$, $v=0$, so $x_A = ut = t$, $y_A = vt = 0$

Plots:





a) On 1: $\hat{n} = -\hat{\imath}$, $\vec{V} = V_\infty \hat{\imath}$, $v = 0$

so that $\oint_1 \rho(\vec{V}\cdot\hat{n}) v\, dA = 0$ $\oint_1 p\, \hat{n}\cdot\hat{\jmath}\, dA = 0$

On 2: $\hat{n} = \hat{\imath}$, $\vec{V} = V_\infty \hat{\imath}$, $v = 0$

so that $\oint_2 \rho(\vec{V}\cdot\hat{n}) v\, dA = 0$, $\oint_2 p\, \hat{n}\cdot\hat{\jmath}\, dA = 0$

On 3: $\hat{n} = -\hat{\jmath}$, $\vec{V} = V\hat{\imath}$, $v = 0$, $p \neq p_\infty$

so that $\oint_3 \rho(\vec{V}\cdot\hat{n}) v\, dA = 0$, $\oint_3 p\, \hat{n}\cdot\hat{\jmath}\, dA = \oint_3 -p\, dA \neq 0$

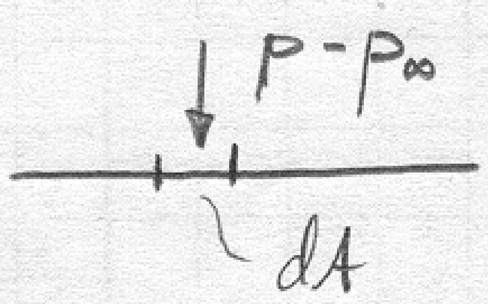
On 4: $\hat{n} = \hat{\jmath}$, $\vec{V} = V_\infty \hat{\imath}$, $v = 0$, $p = p_\infty$

so that $\oint_4 \rho(\vec{V}\cdot\hat{n}) v\, dA = 0$, $\oint_4 p\, \hat{n}\cdot\hat{\jmath}\, dA = \oint_4 p_\infty\, dA \neq 0$

Integral equation is $\boxed{\oint_3 -p\, dA + \oint_4 p_\infty\, dA = -L'}$

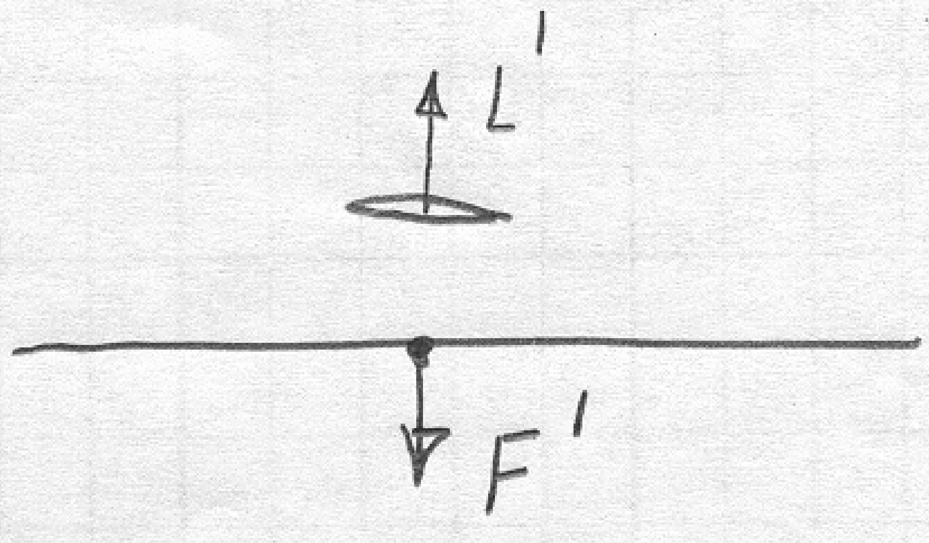
b) Force on the ground due to overpressure is

$$F' = \iint_3 (p - p_\infty)\, dA \qquad \text{positive down}$$



$$= \oint_3 p\, dA - \oint_4 p_\infty\, dA = L'$$

Airfoil pushes down on the ground with force $F' = L'$

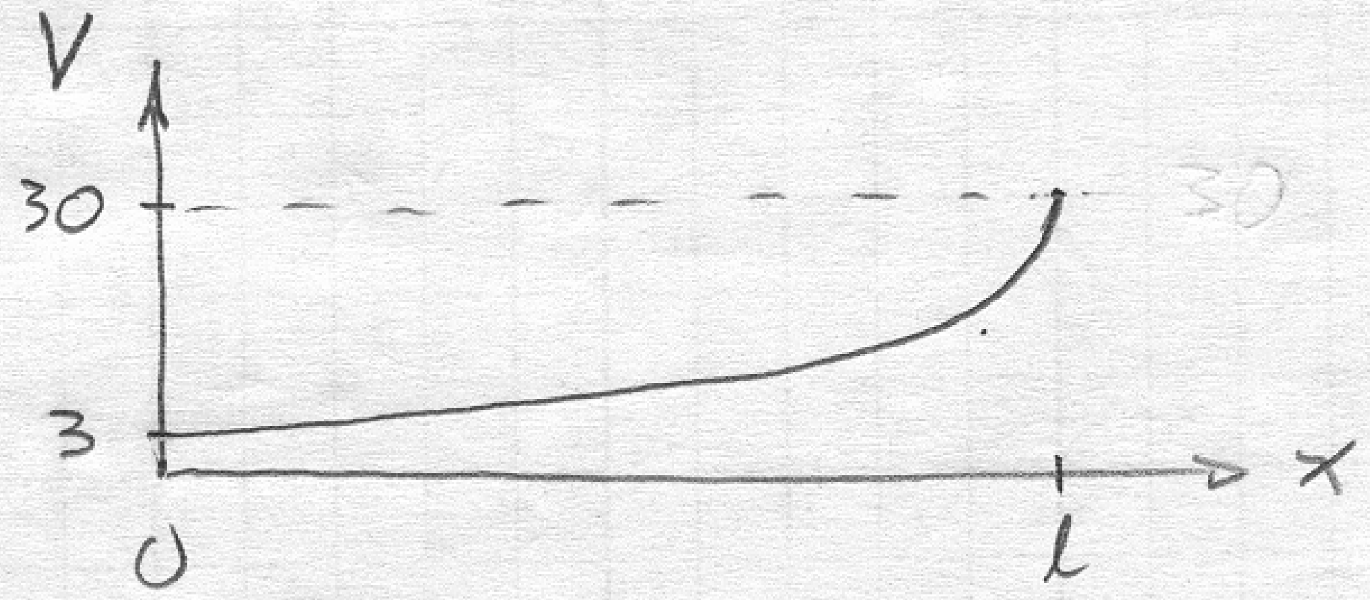


a) Continuity equation: $\rho VA = \text{const}$ or $VA = \text{const}.$

$$V(x)\,A(x) = V(l)\,A(l) = V_1 A_1 \quad ; \; A(x) = A_0 + (A_1 - A_0)\frac{x}{l}$$

$$V(x) = V_1 \frac{A_1}{A(x)} = 30\,\text{m/s}\,\frac{A_1}{A_0 + (A_1 - A_0)\frac{x}{l}} = 30\,\frac{4}{40 - 36\,x/l}$$

(m/s)

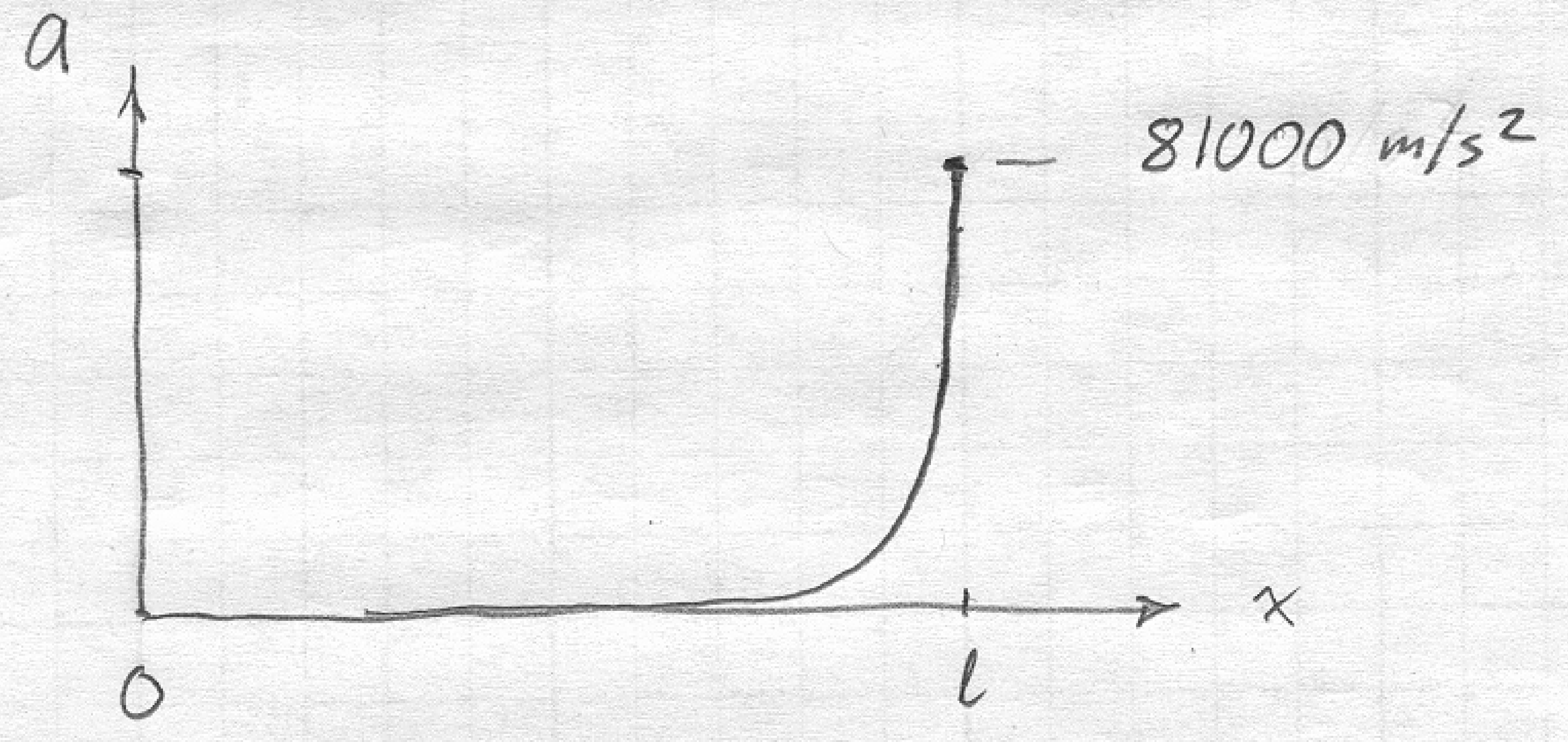


b) $$a(x) = \frac{DV}{Dt} = \cancelto{0}{\frac{\partial V}{\partial t}} + V\frac{\partial V}{\partial x} = V(x)\frac{dV}{dx}$$

$$\frac{dV}{dx} = -\frac{V}{A}\frac{dA}{dx}$$

$$a = V\frac{dV}{dx} = -\frac{V^2}{A}\frac{dA}{dx} = \left[30\,\text{m/s}\,\frac{4}{40 - 36\frac{x}{l}}\right]^2 \frac{36/0.1\,\text{m}}{40 - 36\frac{x}{l}}$$

(m/s^2)

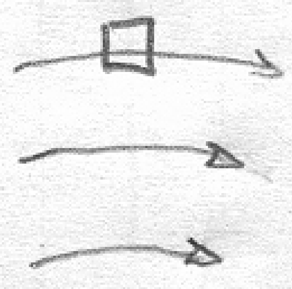


c) Gravitational acceleration is $g = 9.8\ \text{m/s}^2$

Compare: $$\frac{g}{a_{max}} = \frac{9.8}{81000} = 0.00012 = 0.012\,\%$$

<u>negligible</u>

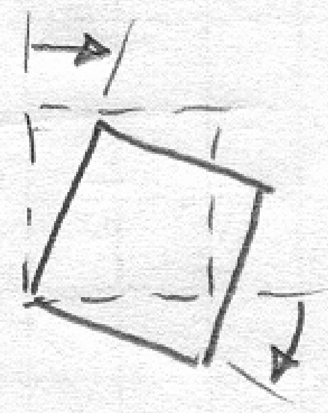
a) From F11 notes: $\dfrac{d\Delta\theta_1}{dt} = -\dfrac{\partial u}{\partial y} = -1$



$$\frac{d\Delta\theta_2}{dt} = \frac{\partial v}{\partial x} = -1$$

For small Δt: $\Delta\theta_1 \simeq \dfrac{d\Delta\theta_1}{dt}\Delta t = -\Delta t$

$$\Delta\theta_2 \simeq \frac{d\Delta\theta_2}{dt}\Delta t = -\Delta t$$



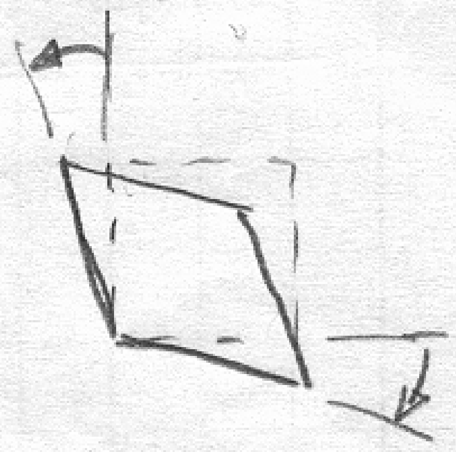
Both sides rotate clockwise at the same rate

b) $$\frac{d\Delta\theta_1}{dt} = -\left.\frac{\partial u}{\partial y}\right|_{0,1} = -\left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right|_{0,1} = 1$$

$$\frac{d\Delta\theta_2}{dt} = \left.\frac{\partial v}{\partial x}\right|_{0,1} = \left(\frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2}\right|_{0,1} = -1$$

For small Δt: $\Delta\theta_1 \simeq \dfrac{d\Delta\theta_1}{dt}\Delta t = \Delta t$

$$\Delta\theta_2 \simeq \frac{d\Delta\theta_2}{dt}\Delta t = -\Delta t$$



Sides rotate at equal and opposite rates.

c) Flow A is rotational, since $\dfrac{d\Delta\theta_1}{dt} + \dfrac{d\Delta\theta_2}{dt} = -2 \neq 0$

Flow B is irrotational, since $\dfrac{d\Delta\theta_1}{dt} + \dfrac{d\Delta\theta_2}{dt} = 0$

Conclusive only at our sampled point $(0,1)$.

Can also evaluate the curl of the velocity:

A: $\vec{\zeta} = \nabla \times \vec{V} = \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)\hat{k} = -2\hat{k} \neq 0$ rotational everywhere

B: $$\vec{\zeta} = \nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = \left(\frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right)\hat{k}$$

$$= \left(\frac{-2}{x^2+y^2} + \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right)\hat{k}$$

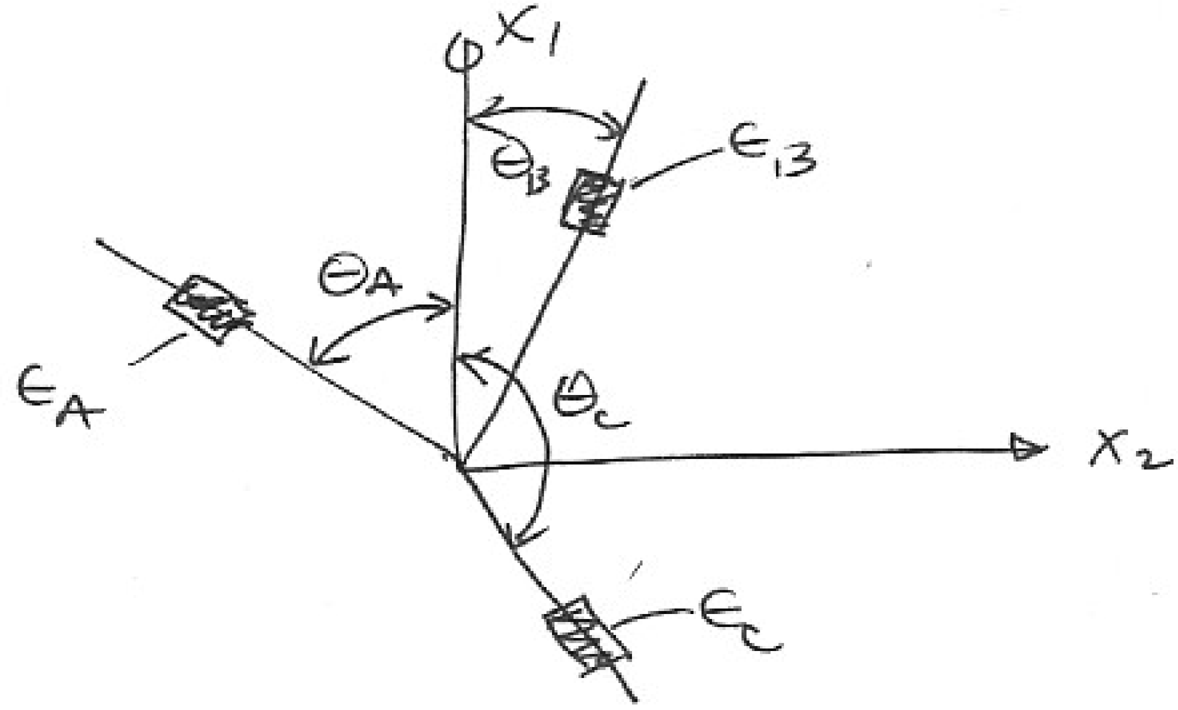
$$= 0$$ irrotational everywhere, except at $x,y = 0,0$ (ζ blows up)

UNIFIED ENGINEERING

Problem Set #11 -- SOLUTIONS

[11(M).1] M12.

3 gages located at three positions (for parts)



(a) Three strains completely characterize the in-plane strain state. These are the two elongational strains (ϵ_{11} and ϵ_{22}) and the one in-plane shear strain (ϵ_{12}). Measuring three strains and their angular locations ~~are~~ sufficient to characterize this state (3 unknowns) via the transformation equations:

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{22} = \sin^2\theta\, \epsilon_{11} + \cos^2\theta\, \epsilon_{22} - 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{12} = -\sin\theta\cos\theta\, \epsilon_{11} + \sin\theta\cos\theta\, \epsilon_{22} + (\cos^2\theta - \sin^2\theta)\, \epsilon_{12}$$

One takes the first equation and applies it three times as gages measure elongational strains. The angle at which the gage is located is used for θ and the measured strain is used for $\tilde{\epsilon}_{11}$. This gives 3 equations in the 3 unknowns (ϵ_{11}, ϵ_{22}, ϵ_{12}) that are subsequently solved

(5) we measure: $\tilde{\epsilon}_{11} = \epsilon_A$ for $\theta = \theta_A$

$\tilde{\epsilon}_{11} = \epsilon_B$ for $\theta = \theta_B$

$\tilde{\epsilon}_{11} = \epsilon_C$ for $\theta = \theta_C$

using the equation

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

three times gives:

$$\epsilon_A = \cos^2\theta_A\, \epsilon_{11} + \sin^2\theta_A\, \epsilon_{22} + 2\cos\theta_A \sin\theta_A\, \epsilon_{12}$$

$$\epsilon_B = \cos^2\theta_B\, \epsilon_{11} + \sin^2\theta_B\, \epsilon_{22} + 2\cos\theta_B \sin\theta_B\, \epsilon_{12}$$

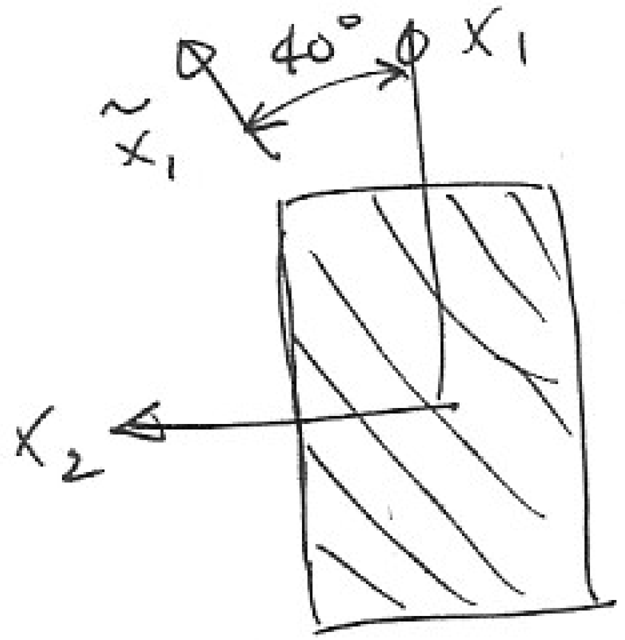
$$\epsilon_C = \cos^2\theta_C\, \epsilon_{11} + \sin^2\theta_C\, \epsilon_{22} + 2\cos\theta_C \sin\theta_C\, \epsilon_{12}$$

solve these 3 equations for the 3 unknowns (ϵ_{11}, ϵ_{22}, ϵ_{12})

<u>NOTE</u>: Be sure to use the appropriate signs (θ is + CCW) so θ_A is actually positive and θ_B and θ_C are negative

[11(M).2]

M13.



$\sigma_{11} = 30$ MPa

$\sigma_{22} = -20$ MPa

$\sigma_{12} = 15$ MPa

(a) The stress state in the "composite fiber axes" are found by transforming the stress state in the loading axes $(x_1 - x_2)$ to the fiber axes $(\tilde{x}_1 - \tilde{x}_2)$.

Stress transformation is done via (for two-dimensional stress):

$$\tilde{\sigma}_{11} = \cos^2\theta\,\sigma_{11} + \sin^2\theta\,\sigma_{22} + 2\cos\theta\sin\theta\,\sigma_{12}$$

$$\tilde{\sigma}_{22} = \sin^2\theta\,\sigma_{11} + \cos^2\theta\,\sigma_{22} - 2\cos\theta\sin\theta\,\sigma_{12}$$

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

The convention is +ccw for the angle. Here, it is counterclockwise to go from $x_1 - x_2$ to $\tilde{x}_1 - \tilde{x}_2$ and thus $\theta = +40°$

This gives:

$\sin\theta = 0.643$

$\cos\theta = 0.766$

$\sin^2\theta = 0.413$

$\cos^2\theta = 0.587$

$\sin\theta\cos\theta = 0.493$

So:

$\tilde{\sigma}_{11} = 0.587(30\,MPa) + 0.413(-20\,MPa) + 0.986(15\,MPa)$

$\tilde{\sigma}_{22} = 0.413(30\,MPa) + 0.587(-20\,MPa) - 0.986(15\,MPa)$

$\tilde{\sigma}_{12} = -0.493(30\,MPa) + 0.493(-20\,MPa)$
$\quad + (0.587 - 0.413)(15\,MPa)$

$$\Rightarrow \boxed{\begin{aligned} \tilde{\sigma}_{11} &= 24.1\ MPa \\ \tilde{\sigma}_{22} &= -14.1\ MPa \\ \tilde{\sigma}_{12} &= -22.0\ MPa \end{aligned}}$$

Verify by checking invariants:

Does $\sigma_{11} + \sigma_{22} = \tilde{\sigma}_{11} + \tilde{\sigma}_{22}$?

$30\,MPa + (-20\,MPa) = 24.1\,MPa + (-14.1\,MPa)$?

$10\,MPa = 10\,MPa$ ✓ $\underline{\underline{\text{Yes}}}$

(b) for plane stress, the principal stresses are the roots of the equation:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

using the stress values (in loading axes):

$$\tau^2 - \tau(30\,MPa - 20\,MPa) + [(30\,MPa)(-20\,MPa) - (15\,MPa)^2] = 0$$

$$\Rightarrow \tau^2 - (10\,MPa)\tau - 825(MPa)^2 = 0$$

solve for the quadratic formula

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for } ax^2 + bx + c = 0$$

$$\Rightarrow \tau = \frac{-(-10\,MPa) \pm \sqrt{(10\,MPa)^2 - 4(1)(-825[MPa]^2)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 + 3300}}{2}\ MPa$$

$$= \frac{10 \pm \sqrt{3400}}{2}\ MPa$$

$$\Rightarrow \tau = \frac{10 \pm 58.31}{2}\ MPa$$

$$= 34.2\,MPa,\ -24.2\,MPa$$

$$\Rightarrow \boxed{\begin{array}{l}\sigma_I = 34.2\,MPa \\ \sigma_{II} = -24.2\,MPa\end{array}}$$

to find the associated directions, use the expression:

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)$$

$$\Rightarrow \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2(15\,MPa)}{30\,MPa - (-20\,MPa)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{30}{50}\right)$$

$$= \tfrac{1}{2}\tan^{-1}(0.6) \Rightarrow$$

$$\theta_p = \tfrac{1}{2}(30.96^\circ)$$

$$\Rightarrow \theta_p = 15.5^\circ \quad \text{for } \sigma_I$$

with σ_{II} rotated 90° from that

so: $\boxed{\begin{aligned} \theta_{p_I} &= 15.5^\circ \\ \theta_{p_{II}} &= 105.5^\circ \end{aligned}}$

Check the angles via the transformation equation for shear and that shear stress goes to zero (definition of principal stress):

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

$$\Rightarrow 0 \stackrel{?}{=} (-0.257)(30\,MPa) + (0.257)(-20\,MPa) + (0.858)(15\,MPa)$$

$$0 \stackrel{?}{=} -12.85\,MPa + 12.87\,MPa$$

✓ yes

(same for 105.5°)

(c) Maximum shear stress(es) occur along planes/directions that are at 45° to the principal axes:

So: direction of maximum shear stress:

$$\theta_{p_I} + 45^\circ \text{ from } x_1 = 60.5^\circ$$

$$\theta_{p_{II}} + 45^\circ \text{ from } x_1 = 150.5^\circ$$

$$\boxed{\text{Directions of maximum shear: } 60.5^\circ, 150.5^\circ}$$

The value(s) of the maximum shear stress(es) in the $x_1 - x_2$ plane is:

$$\frac{\sigma_I - \sigma_{II}}{2}$$

$$= \frac{34.2\,MPa - (-24.2\,MPa)}{2}$$

$$\Rightarrow \boxed{\text{Value of Maximum shear stress} = 29.2\,MPa}$$

(this takes on a sign of + and −)

There are two other maximum shear stresses out of the $x_1 - x_2$ plane. For the case of plane stress, the out-of-plane principal stress is zero ($\sigma_{III} = 0$). So we have:

$$|\tau_{max}| = \frac{\sigma_{I} - \sigma_{III}}{2} = \frac{34.2\,MPa - 0}{2} = 17.1\,MPa$$

$$|\tau_{max}| = \frac{\sigma_{II} - \sigma_{III}}{0} = \frac{-24.2\,MPa - 0}{2} = -12.1\,MPa$$

These stresses are in planes at 45° to the $x_1 - x_2$ planes (one rotated about x_1, one rotated about x_2)

(d) Draw the Mohr's circle as per Handout #M-3 (see figure)

① ~~Plot~~ σ_{11}, σ_{12} (30 MPa, -15 MPa) as Point A

② Plot σ_{22}, σ_{21} (-20 MPa, 15 MPa) as Point B

③ Connect A and B

④ Draw a circle of diameter of the line AB about the point where the line AB crosses the horizontal axis (denote this as point C.

point $C = \frac{\sigma_{11} + \sigma_{22}}{2} = \frac{30\,MPa - 20\,MPa}{2} = 5\,MPa$)

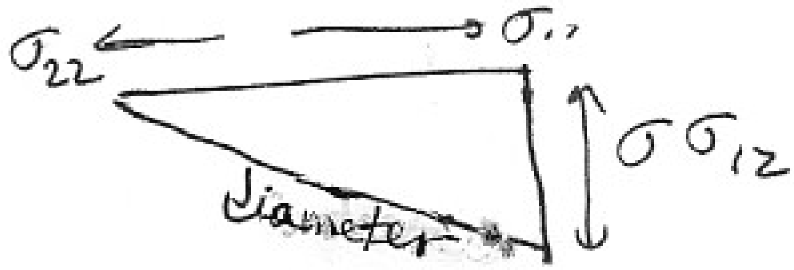
→ Principal stresses and directions

The principal stresses correspond to points F and G. By sight, these are approximately at 35 MPa and -25 MPa which correspond to the results in (b). One can be a bit more formal by noting that:

$$F = C + \frac{\text{diameter}}{2}$$

$$G = C - \frac{\text{diameter}}{2}$$

by geometry: $(\text{diameter})^2 = (30+20)^2 + (2(15))^2$ [MPa]



$\Rightarrow$ diameter = 58.3 MPa

So: $F = 5\text{ MPa} + 29.2\text{ MPa} = 34.2\text{ MPa}$

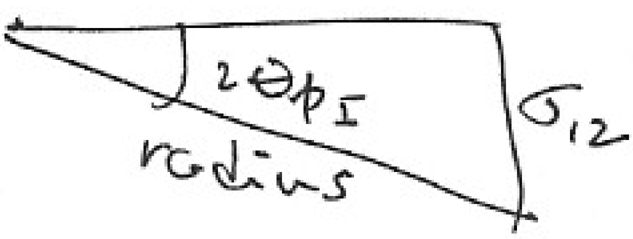
$G = 5\text{ MPa} - 29.2\text{ MPa} = -24.2\text{ MPa}$

as in (5)!

The angle can be measured from Mohr's circle recalling the convention that measured angle is twice the rotation and + ccw.

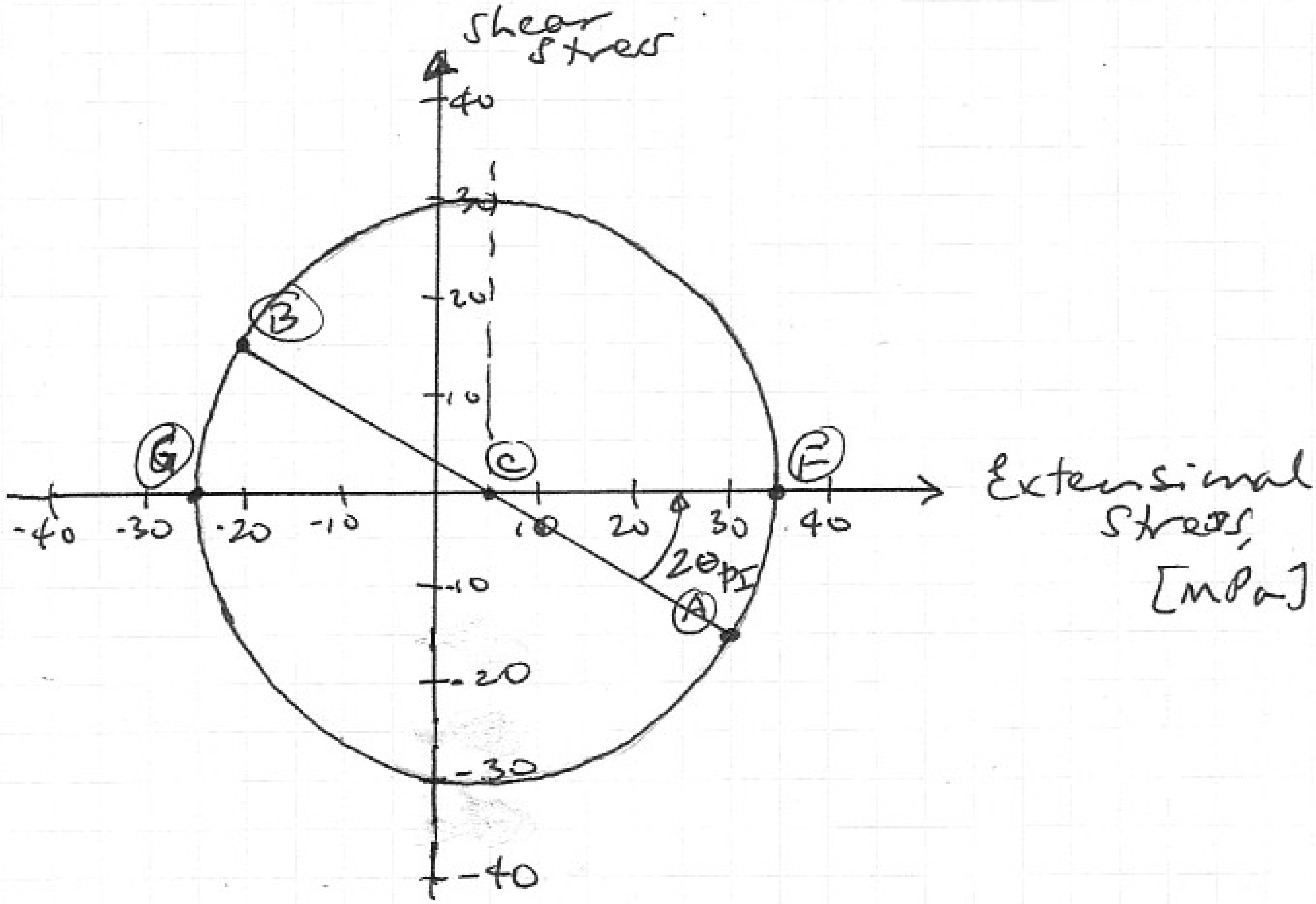
This corresponds to the counterclockwise rotation from line AB to line FG $= 2\theta_{pI}$

using geometry:



So: radius $\sin(2\theta_{pI}) = \sigma_{12}$

$\Rightarrow 29.2\text{ MPa} \sin(2\theta_{pI}) = 15\text{ MPa}$


Shear Stress
40
30
20
10
-10
-20
-30
-40
B
G
C
F
A
-40
-30
-20
-10
10
20
30
40
2θp1
Extensional Stress, [MPa]

$2\theta_{p_I} = \sin^{-1}(0.514)$

$2\theta_{p_I} = 30.9^\circ$

$\Rightarrow \theta_{p_I} = 15.4^\circ$

same as in (b)

The other angle is 180° from that on Mohr's circle giving $180^\circ/2$ in actuality

$\Rightarrow \theta_{p_{II}} = 105.4^\circ$ same as in (b)

→ Maximum shear stress

This can be read off to be just under 30 MPa. This is exactly the radius of the circle

$\Rightarrow$ maximum shear stress = 29.2 MPa

same as in (c)

The direction of the maximum shear stress is 90° on Mohr's circle from that of the principal stress. Divide by 2 and that is 45° added on to θ_{p_I}.

$\Rightarrow$ direction: $15.4^\circ + 45^\circ = 60.4^\circ$

same as in (c)

The other direction is 180° on Mohr's circle and thus 90° (150.4°) in actuality.

Same as in (c)

<u>Note</u>: only the maximum shear stress in the x_1-x_2 plane can be determined since Mohr's circle only allows rotation in the x_1-x_2 plane