Massachusetts Institute of Technology
Department of Aeronautics and Astronautics
Cambridge, MA 02139

Unified Engineering
Fall 2004

Problem Set #13
Solutions

Use polar coordinates to check if $\nabla^2\phi = 0$, $\nabla\phi\cdot\hat{n} = 0$

(Much simpler here than using cartesian coordinates).

$$\phi(r,\theta) = V_\infty \cos\theta\left(r + \frac{R^2}{r}\right)$$

$$\frac{\partial\phi}{\partial r} = V_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right), \quad r\frac{\partial\phi}{\partial r} = V_\infty\cos\theta\left(r - \frac{R^2}{r}\right)$$

$$\frac{\partial\phi}{\partial\theta} = -V_\infty\sin\theta\left(r + \frac{R^2}{r}\right)$$

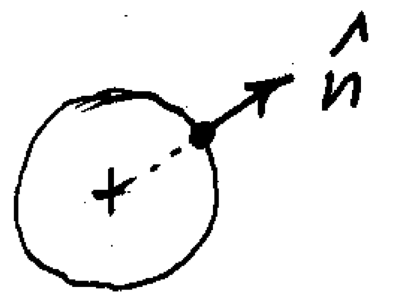
$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = V_\infty\cos\theta\left(1 + \frac{R^2}{r^2}\right)$$

$$\frac{\partial^2\phi}{\partial\theta^2} = -V_\infty\cos\theta\left(r + \frac{R^2}{r}\right)$$

1) $\Rightarrow \nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = V_\infty\cos\theta\left(\frac{1}{r} + \frac{R^2}{r^3}\right) - V_\infty\cos\theta\left(\frac{1}{r} + \frac{R^2}{r^3}\right) = 0$ ✓

(everywhere)

2) On body, $\hat{n} = \hat{r}$, $r = R$



$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$\nabla\phi\cdot\hat{n}\Big|_{r=R} = \nabla\phi\cdot\hat{r} = \frac{\partial\phi}{\partial r} = V_\infty\cos\theta\left(1 - \frac{R^2}{r^2}\right)\Big|_{r=R} = 0 \text{ ✓ (on body)}$$

3) Use cartesian form here.

$$\phi(x,y) = V_\infty\left(x + \frac{xR^2}{x^2+y^2}\right)$$

$$\frac{\partial\phi}{\partial x} = V_\infty\left(1 + R^2\frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2}\right) = V_\infty\left(1 + R^2\frac{(y^2-x^2)}{(x^2+y^2)^2}\right)$$

but $\frac{y^2-x^2}{(x^2+y^2)^2} = \frac{r^2\sin^2\theta - r^2\cos^2\theta}{r^4} = \frac{1}{r^2}(\sin^2\theta - \cos^2\theta) \rightarrow 0$ as $r\rightarrow\infty$

so $\frac{\partial\phi}{\partial x} = V_\infty\left(1 + \frac{R^2}{r^2}(\sin^2\theta - \cos^2\theta)\right) \rightarrow V_\infty$ as $r \rightarrow \infty$ ✓

OE Fluids | Problem F16 Solution | Fall '04

1) $L' = \frac{1}{2}\rho V_\infty^2 c\, C_\ell$ (definition of C_ℓ)

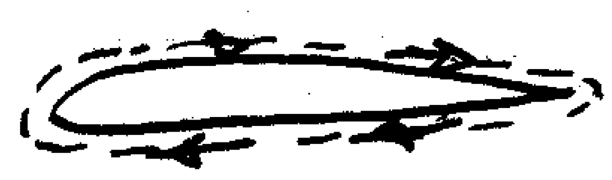
also $L' = \rho V_\infty \Gamma$ (Kutta-Joukowsky Theorem)

$$\rightarrow \rho V_\infty \Gamma = \frac{1}{2}\rho V_\infty^2 c\, C_\ell$$

$$\boxed{\Gamma = \frac{1}{2} c V_\infty C_\ell = 0.4\, c V_\infty}$$

2) Γ is the same about <u>any</u> circuit enclosing airfoil.

Pick circuit just off the surface:



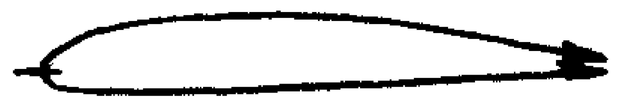
$$\Gamma = \oint \vec{V}\cdot d\vec{s} = \int_0^{s_{u_{max}}} V_u\, ds - \int_0^{s_{\ell_{max}}} V_\ell\, ds$$

$$\Gamma = s_{u_{max}} \overline{V_u} - s_{\ell_{max}} \overline{V_\ell} = 0.4\, c V_\infty$$

since $s_{u_{max}} \simeq s_{\ell_{max}} \simeq c$ for thin airfoils,

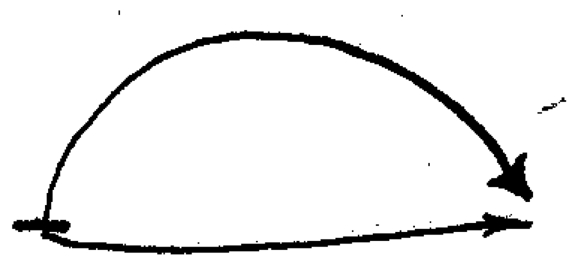
$$\boxed{\frac{\overline{V_u} - \overline{V_\ell}}{V_\infty} \simeq 0.4}$$

3) According to the "theory", a ~40% difference in $s_{u_{max}}$ and $s_{\ell_{max}}$ would be required to generate the estimated 40% $\overline{V_u}$ and $\overline{V_\ell}$ difference. On most airfoils, $s_{u_{max}}$ and $s_{\ell_{max}}$ differ by only a few %.



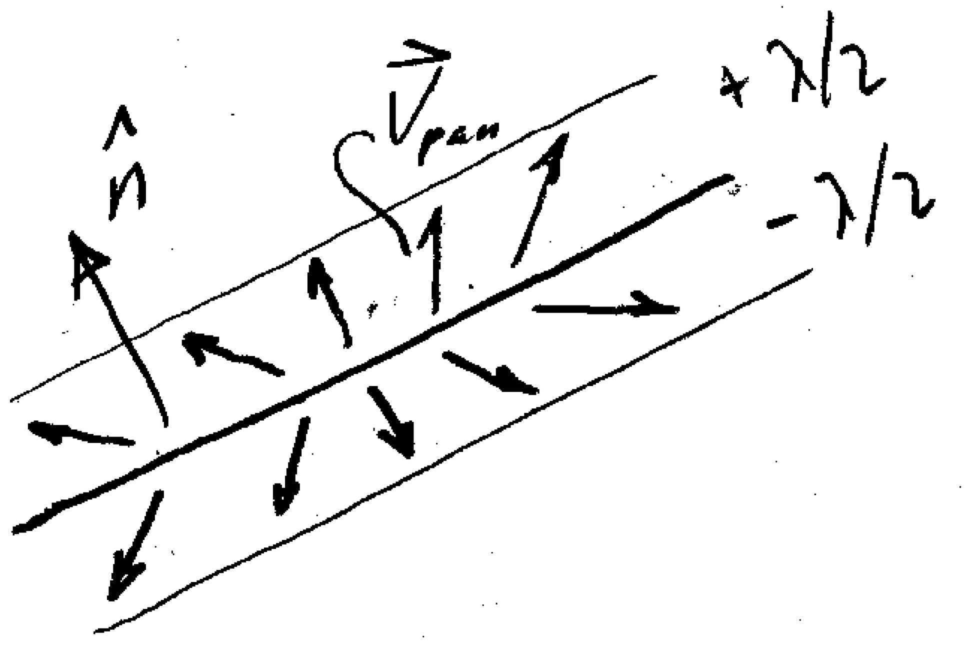
~3%

✓



~40%

(nuts)

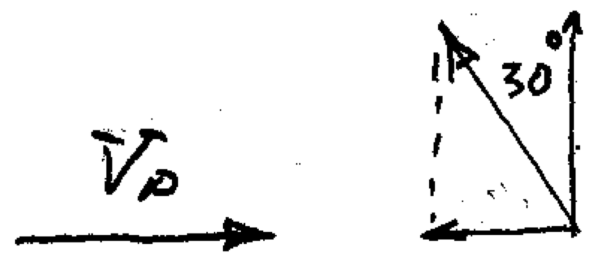


Isolated panel: $\vec{V}_{pan} \cdot \hat{n} = \pm \lambda/2$

Including freestream: $\quad \xrightarrow{V_\infty}$

$$\vec{V} = \vec{V}_\infty + \vec{V}_{pan}$$

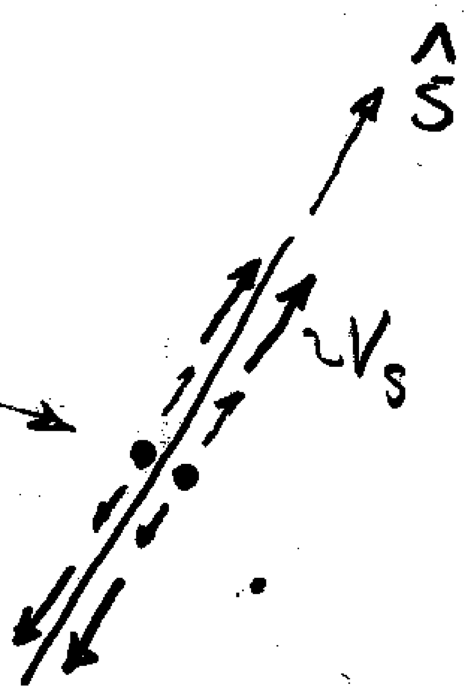
1) required: $\vec{V} \cdot \hat{n} = 0$ on upstream face



or $\vec{V}_\infty \cdot \hat{n} + \vec{V}_{pan} \cdot \hat{n} = 0$

$$\lambda/2 = - V_\infty \cdot \hat{n} = V_\infty \sin 30° = \tfrac{1}{2} V_\infty$$

$$\boxed{\lambda = V_\infty}$$



2) On panel: $\vec{V}_{pan} = \pm(\lambda/2)\,\hat{n} + V_s\,\hat{s}$

<u>on center point</u>: $V_s = 0$ by symmetry

so $\vec{V}_{pan} = \pm(\lambda/2)\,\hat{n}$ (center)

$\vec{V}_{pan} = -(\lambda/2)\,\hat{n}$ (center, rear)

with $\hat{n} = -\hat{\imath}\sin 30° + \hat{\jmath}\cos 30°$

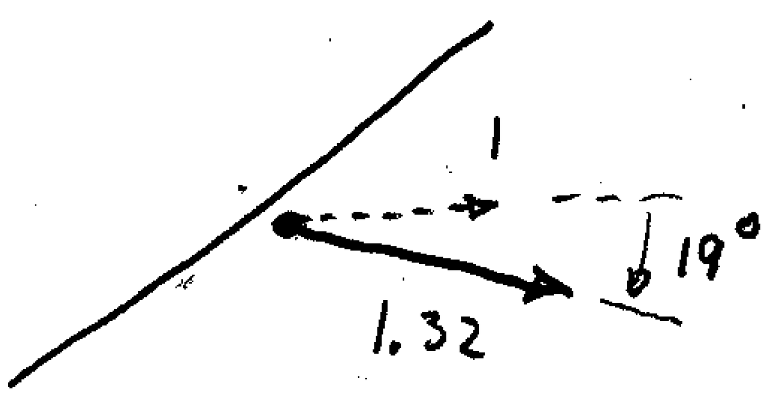
$\hat{n} = -\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$

$$\vec{V} = \vec{V}_\infty + \vec{V}_{pan} = V_\infty \hat{\imath} + \left(-\lambda/2\right)\left[-\tfrac{1}{2}\hat{\imath} + \tfrac{\sqrt{3}}{2}\hat{\jmath}\right] \quad ; \lambda = V_\infty$$

$$\boxed{\vec{V} = V_\infty\left[\tfrac{5}{4}\hat{\imath} - \tfrac{\sqrt{3}}{4}\hat{\jmath}\right]}$$

or $\vec{V} = V_\infty\left[1.25\,\hat{\imath} - 0.433\,\hat{\jmath}\right]$

$= 1.32\,V_\infty\left[\hat{\imath}\cos 19° - \hat{\jmath}\sin 19°\right]$



BAK
11/19/04
①

UNIFIED ENGINEERING

Problem Set #13 -- SOLUTIONS

M16 Start with:

$$\epsilon_{mn} = S_{mnpq} \sigma_{pq}$$

This has the same form as the elasticity equation:

$$\sigma_{mn} = E_{mnpq} \epsilon_{pq}$$

and since the same symmetries exist for the compliance tensor as for the elasticity tensor, the full anisotropic equations have the same form. Thus:

$$\boxed{\begin{aligned}
\epsilon_{11} &= S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33} + 2S_{1123}\sigma_{23} + 2S_{1113}\sigma_{13} + 2S_{1112}\sigma_{12} \\
\epsilon_{22} &= S_{1122}\sigma_{11} + S_{2222}\sigma_{22} + S_{2233}\sigma_{33} + 2S_{2223}\sigma_{23} + 2S_{2213}\sigma_{13} + 2S_{2212}\sigma_{12} \\
\epsilon_{33} &= S_{1133}\sigma_{11} + S_{2233}\sigma_{22} + S_{3333}\sigma_{33} + 2S_{3323}\sigma_{23} + 2S_{3313}\sigma_{13} + 2S_{3312}\sigma_{12} \\
\epsilon_{23} &= S_{1123}\sigma_{11} + S_{2223}\sigma_{22} + S_{3323}\sigma_{33} + 2S_{2323}\sigma_{23} + 2S_{1323}\sigma_{13} + 2S_{1223}\sigma_{12} \\
\epsilon_{13} &= S_{1113}\sigma_{11} + S_{2213}\sigma_{22} + S_{3313}\sigma_{33} + 2S_{1323}\sigma_{23} + 2S_{1313}\sigma_{13} + 2S_{1213}\sigma_{12} \\
\epsilon_{12} &= S_{1112}\sigma_{11} + S_{2212}\sigma_{22} + S_{3312}\sigma_{33} + 2S_{1223}\sigma_{23} + 2S_{1213}\sigma_{13} + 2S_{1212}\sigma_{12}
\end{aligned}}$$

Likewise, these can be grouped into the 3 groups similar to that for E_{mnpq}:

extensional stresses to extensional strains $\left\{ \begin{matrix} S_{1111} & S_{1122} \\ S_{2222} & S_{1133} \\ S_{3333} & S_{2233} \end{matrix} \right.$

shear stresses to shear strains $\left\{ \begin{array}{ll} S_{1212} & S_{1213} \\ S_{1313} & S_{1323} \\ S_{2323} & S_{1223} \end{array} \right.$

COUPLING TERMS

extensional stresses to shear strains
or
shear stresses to extensional strains $\left\{ \begin{array}{lll} S_{1112} & S_{2212} & S_{3312} \\ S_{1113} & S_{2213} & S_{3313} \\ S_{1123} & S_{2223} & S_{3323} \end{array} \right.$

M17 start with the tensorial form:

$$\sigma_{11} = E_{1111}\epsilon_{11} + E_{1122}\epsilon_{22} + E_{1133}\epsilon_{33} + 2E_{1123}\epsilon_{23} + 2E_{1113}\epsilon_{13} + 2E_{1112}\epsilon_{12}$$
$$\sigma_{22} = E_{1122}\epsilon_{11} + E_{2222}\epsilon_{22} + E_{2233}\epsilon_{33} + 2E_{2223}\epsilon_{23} + 2E_{2213}\epsilon_{13} + 2E_{2212}\epsilon_{12}$$
$$\sigma_{33} = E_{1133}\epsilon_{11} + E_{2233}\epsilon_{22} + E_{3333}\epsilon_{33} + 2E_{3323}\epsilon_{23} + 2E_{3313}\epsilon_{13} + 2E_{3312}\epsilon_{12}$$
$$\sigma_{23} = E_{1123}\epsilon_{11} + E_{2223}\epsilon_{22} + E_{3323}\epsilon_{33} + 2E_{2323}\epsilon_{23} + 2E_{1323}\epsilon_{13} + 2E_{1223}\epsilon_{12}$$
$$\sigma_{13} = E_{1113}\epsilon_{11} + E_{2213}\epsilon_{22} + E_{3313}\epsilon_{33} + 2E_{1323}\epsilon_{23} + 2E_{1313}\epsilon_{13} + 2E_{1213}\epsilon_{12}$$
$$\sigma_{12} = E_{1112}\epsilon_{11} + E_{2212}\epsilon_{22} + E_{3312}\epsilon_{33} + 2E_{1223}\epsilon_{23} + 2E_{1213}\epsilon_{13} + 2E_{1212}\epsilon_{12}$$

Now to go from tensorial to engineering notation recall:

→ for strains (shear strains):

$$2\epsilon \to \gamma$$

→ subscripts:

$$
\begin{array}{ll}
11 \rightarrow x & 12 \rightarrow xy \\
22 \rightarrow y & 13 \rightarrow xz \\
33 \rightarrow z & 23 \rightarrow yz
\end{array}
$$

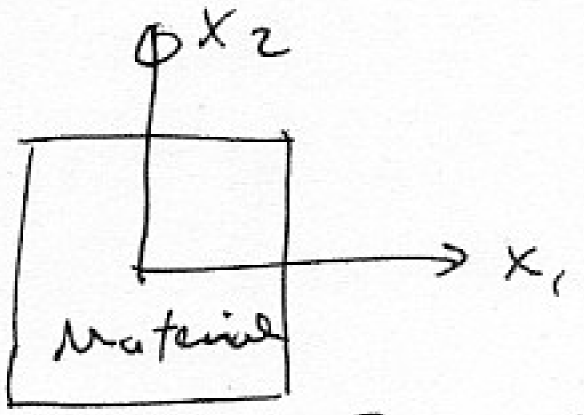
So:

$$
\boxed{
\begin{aligned}
\sigma_x &= E_{xx}\epsilon_x + E_{xy}\epsilon_y + E_{xz}\epsilon_z + E_{x,yz}\gamma_{yz} + E_{x,xz}\gamma_{xz} + E_{x,xy}\gamma_{xy} \\
\sigma_y &= E_{xy}\epsilon_x + E_{yy}\epsilon_y + E_{yz}\epsilon_z + E_{y,yz}\gamma_{yz} + E_{y,xz}\gamma_{xz} + E_{y,xy}\gamma_{xy} \\
\sigma_z &= E_{xz}\epsilon_x + E_{yz}\epsilon_y + E_{zz}\epsilon_z + E_{z,yz}\gamma_{yz} + E_{z,xz}\gamma_{xz} + E_{z,xy}\gamma_{xy} \\
\sigma_{yz} = \tau_{yz} &= E_{x,yz}\epsilon_x + E_{y,yz}\epsilon_y + E_{z,yz}\epsilon_z + E_{yz,yz}\gamma_{yz} + E_{xz,yz}\gamma_{xz} + E_{xy,yz}\gamma_{xy} \\
\sigma_{xz} = \tau_{xz} &= E_{x,xz}\epsilon_x + E_{y,xz}\epsilon_y + E_{z,xz}\epsilon_z + E_{xz,yz}\gamma_{yz} + E_{xz,xz}\gamma_{xz} + E_{xy,xz}\gamma_{xy} \\
\sigma_{xy} = \tau_{xy} &= E_{x,xy}\epsilon_x + E_{y,xy}\epsilon_y + E_{z,xy}\epsilon_z + E_{xy,yz}\gamma_{yz} + E_{xy,xz}\gamma_{xz} + E_{xy,xy}\gamma_{xy}
\end{aligned}
}
$$

Note that all symmetries (e.g. $E_{xy} = E_{yx}$) are retained since the implied symmetries still exist in the stress and strain notation.

<u>Note</u>: This form is very seldom used. <u>Engineering constants</u> are more often seen and these are <u>different</u> from what are written here.

M18



Experiment A: $\sigma_{11} = 125$ MPa
$\epsilon_{11} = 6220\,\mu$strain
$\epsilon_{22} = -1990\,\mu$strain

Experiment B: $\sigma_{12} = 75$ MPa
$\epsilon_{12} = 4900\,\mu$strain

Experiment C: $\sigma_{22} = 100$ MPa
$\epsilon_{11} = -1600\,\mu$strain
$\epsilon_{22} = 5000\,\mu$strain

stresses and strains not specified are zero.

(a) ~~Experiments~~ Experiments A and C show that extensional stresses cause only extensional stresses and Experiment B shows that shear stress causes only shear strain. Thus, this material behaves at most as an orthotropic material.

Since the out-of-plane stresses (σ_{3j}) are also zero, we only need to use the in-plane form. We end up with:

$$\epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} \qquad (1)$$

$$\epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} \qquad (2)$$

$$2\epsilon_{12} = \frac{1}{G_{12}}\sigma_{12} \qquad (3)$$

→ Using the results of Experiment A in (1):

$$6222\times10^{-6} = \frac{1}{E_1}(125\times10^{6}\,Pa)$$

$$\Rightarrow \boxed{E_1 = 20.1\times10^{9}\,Pa = 20.1\,GPa}$$

and in (2) along with the value for E_1:

$$-1990\times10^{-6} = \frac{-\nu_{12}}{20.1\times10^{9}\,Pa}(125\times10^{6}\,Pa)$$

$$\Rightarrow \boxed{\nu_{12} = 0.320}$$

→ Using the results of Experiment B in (3):

$$2(4900\times10^{-6}) = \frac{1}{G_{12}}(75\times10^{6})\,Pa$$

$$\Rightarrow \boxed{G_{12} = 7.65\times10^{9}\,Pa = 7.65\,GPa}$$

→ Using the results of Experiment C in (2):

$$5000\times10^{-6} = \frac{1}{E_2}(100\times10^{6}\,Pa)$$

$$\Rightarrow \boxed{E_2 = 20.0\times10^{9}\,Pa = 20.0\,GPa}$$

and in (1) along with the value for E_2:

$$-1600 \times 10^{-6} = \frac{-\nu_{21}}{(20.0 \times 10^9\ Pa)} (100 \times 10^6\ Pa)$$

$$\Rightarrow \boxed{\nu_{21} = 0.320}$$

we can see that: $\nu_{12} = \nu_{21}$

and $E_1 = E_2$ within experimental error

SUMMARY:
Therefore, this indicates 2 independent constants and $\boxed{\text{isotropic}}$ behavior with $\boxed{\begin{array}{l} E = 20.0\ GPa \\ \nu = 0.32 \end{array}}$

check through the relationship with shear modulus:

$$G = \frac{E}{2(1+\nu)} = \frac{20.0 \times 10^9\ Pa}{2(1 + 0.320)}$$

$$\Rightarrow G = 7.576 \times 10^9\ Pa = 7.576\ GPa$$

and got 7.65 GPa before

✓ checks well within experimental error

(b) Rewrite the stress-strain equations in matrix form:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Now look at the compliance tensor relations also written in matrix form and with the zero stresses ($\sigma_{13}, \sigma_{23}, \sigma_{33}$) and strains ($\epsilon_{13}, \epsilon_{23}, \epsilon_{33}$) ignored/eliminated:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1112} \\ S_{2211} & S_{2222} & S_{2212} \\ S_{1211} & S_{1222} & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Comparing the two and using the values from part (a):

$$S_{1111} = 1/E = \frac{1}{20.0 \times 10^9 \, Pa} = 5.0 \times 10^{-11} \frac{1}{Pa}$$

$$S_{1122} = S_{2211} = -\nu/E = -\frac{0.32}{20.0 \times 10^9 \, Pa} = -1.6 \times 10^{-11} \frac{1}{Pa}$$

$$S_{2222} = S_{1111} = \frac{1}{E} = 5.0 \times 10^{-11} \frac{1}{Pa}$$

coupling terms are zero ($S_{1211}, S_{1112}, S_{1222}, S_{2212}$)

$$2 S_{1212} = \frac{1}{2G_{12}} = \frac{1}{2(7.576 \times 10^9 \, Pa)} = 0.066 \times 10^{-11} \frac{1}{Pa}$$

$$\Rightarrow S_{1212} = 3.3 \times 10^{-11} \frac{1}{Pa}$$

Summarizing:

$$\boxed{\begin{aligned} S_{1111} &= S_{2222} = 5.0 \times 10^{-11} \frac{1}{Pa} \\ S_{1122} &= S_{2211} = -1.6 \times 10^{-11} \, 1/Pa \\ S_{1212} &= 3.3 \times 10^{-11} \, 1/Pa \\ S_{1112} &= S_{2212} = S_{1211} = S_{1222} = 0 \end{aligned}}$$

In matrix form:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 5.0 & -1.6 & 0 \\ -1.6 & 5.0 & 0 \\ 0 & 0 & 3.3 \end{bmatrix} \left[\times 10^{-11} \frac{1}{Pa}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Note that no components of the compliance tensor with subscript 3 can be determined.