Massachusetts Institute of Technology
Department of Aeronautics and Astronautics
Cambridge, MA 02139

Unified Engineering
Fall 2004

Problem Set #1 Solutions

Ian A. Waitz

Solution to U1 by Waitz. (Range Equation)

a) Assuming steady-level flight and no fuel reserves, estimate the range of a B-777 using the information given in the lecture notes (and/or on Boeing's web page). How well does this compare to the estimates Boeing publishes on their web page?

BASIS FOR COMPARISON: BOEING 777-200/200IGW
Max. take-off mass ≈ 275,000 kg
Typ. operating empty mass ≈ 144,000 kg
Max. fuel capacity ≈ 171,000 liters (137,000 kg kerosine)
Cargo volume ≈ 160,000 liters
Takeoff thrust ≈ 760 kN
Cruise thrust ≈ 160 kN
Design range ≈ 9600 to 13800 km
Passengers ≈ 330
Length ≈ 63.7m, wingspan = 60.9m
L/D ≈ 18, $\eta_o \approx 0.36$
Cost ≈ $140 million

Using:

$$\text{Range} = \frac{h}{g}\frac{L}{D}\eta_{overall}\ln\frac{W_{initial}}{W_{final}}$$

You get about 18000 km using the ratio of the operating empty mass and the max takeoff mass (1.9). The estimate of 18000km is more than 30% too high, but I did neglect the weight of the passengers and their cargo, food (such as it is), and reserve fuel. When these items are taken into account the estimate is within 10% of the published values.

b) Now assuming that L/D, propulsion system efficiency and final weight are unchanged, estimate the range of a B-777 if the same volume of liquid hydrogen were to be used instead of Jet-A.

To do this I wrote $W_{final}=W_{initial}-W_{fuel} = W_{initial}-\rho_{fuel} V_{fuel}$. The ratio of the two densities is 0.0875. So the initial weight is only 156,000 kg (144,000kg + 0.0875x137,000kg), and the weight ratio drops to 1.08. Of course the heating value is increased by a factor of 2.8, but it hardly makes up for the reduction in the amount of energy that is carried due to hydrogen's low density. My estimate for the range is 6100km, a reduction by a factor of three from the case with Jet-A.

c) Derive an equation for the range of a battery-powered aircraft in steady-level flight. Express the range in terms of L/D, propulsion system efficiency, battery mass and heating value, and aircraft weight. Estimate the range of a B-777 if the fuel was taken out and replaced with its equivalent weight in batteries.

The key with a battery-powered aircraft is that its mass does not change as it burns the energy. This makes the range equation more straightforward.

$m_b \; h = \text{energy available in the battery } (J)$

$\frac{T \; u_o}{\eta_{overall}} = \text{rate of energy usage to overcome drag } (J/s)$

$$time\ of\ flight = m_b\ h \Big/ \frac{T\ u_o}{\eta_{overall}} \quad (s)$$

$$Range\ of\ flight = u_o \left(m_b\ h \Big/ \frac{T\ u_o}{\eta_{overall}} \right) \quad (m)$$

or

$$Range\ of\ flight = \frac{m_b\ h\ \eta_{overall}}{T} = \frac{m_b\ h\ \eta_{overall}}{W} \left(\frac{L}{D} \right) \quad (m)$$

With m_b = 137,000kg, h=2.5MJ/kg, W=(275,000kg)(9.8m/s^2)=2695kN, I calculate the range to be: 820km. As you can see, the low energy density of the battery is a disaster for range—it is reduced by a factor of more than 20 relative to the Jet-A powered model.

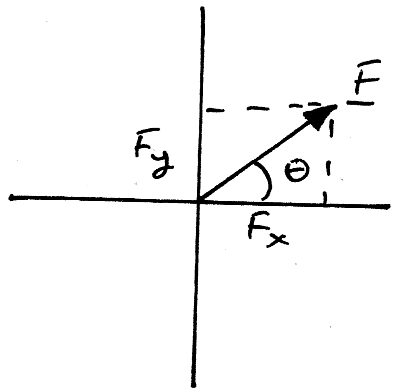
"FUEL"	Heating Value (MJ/kg)	Density (kg/m^3)
Jet-A	42.8	800
Liquid Hydrogen	120	70
Batteries	2.5	8000

Unified Engineering

Problem Set #1 - SOLUTIONS

1 (u).1 (a) Resolve each force into x and y components with the use of $\hat{i}$ and $\hat{j}$ vectors* in order to simplify plotting and for describing each as a vector

Note:



components are determined via:

$F_x = |\underline{F}| \cos\theta$

$F_y = |\underline{F}| \sin\theta$

and the magnitudes multiply the unit vectors.

So: indicates (x, y) location with units in [meters]

$\underline{F}_1 (1,1) = (2N)\{\hat{i}\cos(0°) + \hat{j}\sin(0°)\} = (2N)\hat{i}$

$\underline{F}_2 (1,-4) = (5N)\{\hat{i}\cos(63.4°) + \hat{j}\sin(63.4°)\} = (2.24N)\hat{i} + (4.47N)\hat{j}$

$\underline{F}_3 (2,-3) = (5N)\{\hat{i}\cos(-116.6°) + \hat{j}\sin(-116.6°)\} = (-2.24N)\hat{i} - (4.47N)\hat{j}$

$\underline{F}_4 (-5,5) = (3N)\{\hat{i}\cos(45°) + \hat{j}\sin(45°)\} = (2.12N)\hat{i} + (2.12N)\hat{j}$

$\underline{F}_5 (2,4) = (3N)\{\hat{i}\cos(251.5°) + \hat{j}\sin(251.5°)\} = (-0.952N)\hat{i} - (2.84N)\hat{j}$

$\underline{F}_6 (-5,5) = (4N)\{\hat{i}\cos(315°) + \hat{j}\sin(315°)\} = (2.83N)\hat{i} - (2.83N)\hat{j}$

---

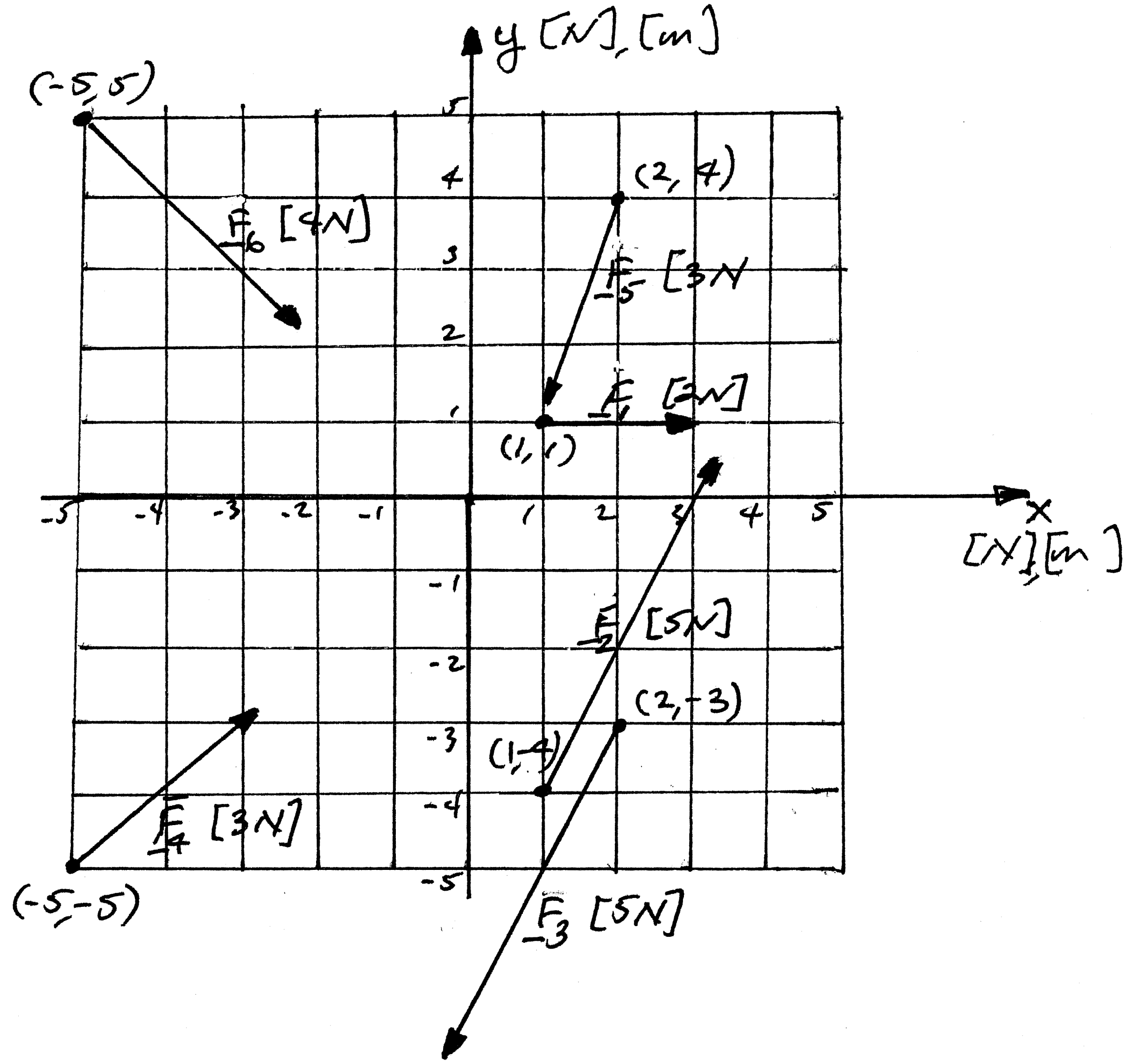
* NOTE: Sometimes unit vectors are noted via "hats" - ^ rather than an underline (or overbar) for general vector.

So the vector description is given by the magnitudes of the forces in each direction, times the unit vectors, with indication of the (x, y) location from which the force vector acts:

$$\underline{F}_1\,(1,1) = (2N)\,\hat{i}$$
$$\underline{F}_2\,(1,-4) = (2.24N)\,\hat{i} + (4.47N)\,\hat{j}$$
$$\underline{F}_3\,(2,-3) = (-2.24N)\,\hat{i} - (4.47N)\,\hat{j}$$
$$\underline{F}_4\,(-5,-5) = (2.12N)\,\hat{i} + (2.12N)\,\hat{j}$$
$$\underline{F}_5\,(2,4) = (-0.952N)\,\hat{i} - (2.84N)\,\hat{j}$$
$$\underline{F}_6\,(-5,5) = (2.83N)\,\hat{i} - (2.83N)\,\hat{j}$$

In order to plot this one must choose a scale for the force magnitude as well as the x,y location. For convenience I choose a coincident grid where 1N equals 1 ~~meter~~ of location.

Draw a grid with the vectors on it:



(b) $\underline{F}_{total} = \sum \underline{F}_i$

easiest to sum $\hat{i}$ and $\hat{j}$ components:

$$\underline{F}_{total} = \hat{i}(2N + 2.24N - 2.24N + 2.12N - 0.952N + 2.83N) + \hat{j}(4.47N - 4.47N + 2.12N - 2.84N - 2.83N)$$

$$\Rightarrow \boxed{\underline{F}_{total\,(net)} = (6N)\hat{i} - (3.55N)\hat{j}}$$

(c) The definition of a couple is ~~it~~ results from two parallel (coplanar) forces of equal magnitude and opposite directions such that there is a moment, but no net force

Notice that $\underline{F}_2$ and $\underline{F}_3$ are equal in magnitude and opposite in direction and thus satisfy this definition.

So, **YES -- $\underline{F}_2$ and $\underline{F}_3$**

Now the expression for the couple is:

$$\underline{C} = \underline{r} \times \underline{F}$$

where:

$\underline{F}$ is one of the vectors

$\underline{r}$ is the position vector from one to the other

So to get $\underline{r}$, subtract one (x, y) position from the other and use the unit vectors for each component: [from $\underline{F}_3$ to $\underline{F}_2$]

$$\Rightarrow \underline{r} = (2-1)\,\hat{i} + (-3-(-4))\,\hat{j}$$
$$= \hat{i} + \hat{j} \qquad \text{units in [m]}$$

Thus:

$$\underline{C} = (\hat{i} + \hat{j}) \times (-2.24\text{N}\,\hat{i} - 4.47\text{N}\,\hat{j})$$

Recall: $\hat{i} \times \hat{j} = \hat{k}$ (z-direction)

$$\hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{i} \times \hat{i} = 0$$
$$\hat{j} \times \hat{j} = 0$$

$$\Rightarrow \quad \underline{C} = (-4.47\ N{\cdot}m)\hat{k} + (-2.24\ N{\cdot}m)(-\hat{k})$$

$$\Rightarrow \boxed{\underline{C} = -2.23\ N{\cdot}m\ \hat{k}}$$

(d) Determine the moment about the origin of each force ($\underline{F}_i$) $= \underline{M}_{O_i}$ and sum these up to determine the net moment.

$$\underline{M} = \underline{r} \times \underline{F}$$
$$\underline{M}_{O_i} = (x_i \hat{i} + y_i \hat{j}) \times (F_{x_i} \hat{i} + F_{y_i} \hat{j})$$

location vector for force $\underline{F}_i$ — force vector $\underline{F}_i$

$$\underline{M}_{O1} = (\hat{i} + \hat{j})m \times (2N)\hat{i} = (-2\ N{\cdot}m)\hat{k}$$

$$\underline{M}_{O2} = (\hat{i} - 4\hat{j})m \times \{(2.24N)\hat{i} + (4.47N)\hat{j}\}$$
$$= (4.47\ N{\cdot}m)\hat{k} + (8.96\ N{\cdot}m)\hat{k} = 13.43\ N{\cdot}m\ \hat{k}$$

$$\underline{M}_{O3} = (2\hat{i} - 3\hat{j})m \times \{(-2.24N)\hat{i} - (4.47N)\hat{j}\}$$
$$= (-8.94\ N{\cdot}m)\hat{k} - (6.72\ N{\cdot}m)\hat{k} = -15.66\ N{\cdot}m\ \hat{k}$$

$$\underline{M}_{O4} = (-5\hat{i} - 5\hat{j})m \times \{(2.12N)\hat{i} + (2.12N)\hat{j}\}$$
$$= (-10.6\ N{\cdot}m\ \hat{k} + 10.6\ N{\cdot}m\ \hat{k}) = \underline{0}$$

$$\underline{M}_{05} = (2\hat{i} + 4\hat{j})m \times \{(-0.952N)\hat{i} - (2.84N)\hat{j}\}$$
$$= -5.68 N\cdot m\,\hat{k} + 3.808 N\cdot m\,\hat{k} = -1.87 N\cdot m\,\hat{k}$$
$$\underline{M}_{06} = (-5\hat{i} + 5\hat{j})m \times \{(2.83N)\hat{i} - (2.83N)\hat{j}\}$$
$$= 14.15 N\cdot m\,\hat{k} - 14.15 N\cdot m\,\hat{k} = \underline{0}$$

So: $\underline{M}_{o\,net} = \sum \underline{M}_{oi}$

$$\Rightarrow \boxed{\underline{M}_{o\,net} = -6.10\, N\cdot m\,\hat{k}}$$

(e) This is the same set of calculations as in part (d), except with the position vector in each case, $\underline{r}$, is from (5m, 5m) to the vector (rather than from the origin).

Thus:
$$\underline{r}_i = (x_i - 5m)\hat{i} + (y_i - 5m)\hat{j}$$
and again:
$$\underline{M}_i = \underline{r}_i \times \underline{F}_i$$

So:
$$\underline{M}_1 = (-4m\hat{i} - 4m\hat{j}) \times (2N)\hat{i} = 8\, N\cdot m\,\hat{k}$$
$$\underline{M}_2 = (-4m\hat{i} - 9m\hat{j}) \times \{(2.24N)\hat{i} + (4.47N)\hat{j}\}$$
$$= (-17.88 N\cdot m)\hat{k} + (20.16 N\cdot m)\hat{k} = 2.28 N\cdot m\,\hat{k}$$
$$\underline{M}_3 = (-3m\hat{i} - 8m\hat{j}) \times \{(-2.24N)\hat{i} - (4.47N)\hat{j}\}$$
$$= (13.41 N\cdot m)\hat{k} - (17.92 N\cdot m)\hat{k} = -4.51 N\cdot m\,\hat{k}$$
$$\underline{M}_4 = (-10m\hat{i} - 10m\hat{j}) \times \{(2.12N)\hat{i} + (2.12N)\hat{j}\}$$
$$= (-21.2 N\cdot m)\hat{k} + (21.2 N\cdot m)\hat{k} = \underline{0}$$

$$\underline{M}_5 = (-3\,m\,\hat{i} - 1\,m\,\hat{j}) \times \{(-0.952\,N)\hat{i} - (2.84\,N)\hat{j}\}$$
$$= (8.52\,N\cdot m)\hat{k} - (0.952\,N\cdot m)\hat{k} = 7.57\,N\cdot m\,\hat{k}$$

$$\underline{M}_6 = (-10\,m\,\hat{i}) \times \{(2.83\,N)\hat{i} - (2.83\,N)\hat{j}\}$$
$$= 28.3\,N\cdot m\,\hat{k}$$

So with: $\underline{M}_{net} = \sum \underline{M}_i$

$$\Rightarrow \boxed{\underline{M}_{net} = 41.64\,N\cdot m\,\hat{k}}$$

(f) By examination, one can see there are no components acting about the $\hat{y}$-axis and x-axis since the only unit vector in the expression for the moments is $\hat{k}$ (z-direction)

More generally, one can find:

Moment about axis = (unit vector about axis) $\cdot$ ($\underline{r} \times \underline{F}$)

So:

M_x = Moment component about x-axis
$= \hat{i} \cdot \underline{M}_{net}$

with $\hat{i} \cdot \hat{k} = 0$

$$\Rightarrow \boxed{M_x = 0}$$

Similarly:

M_y = Moment component about y-axis

$= \hat{j} \cdot \underline{M}_{net}$

with $\hat{j} \cdot \hat{k} = 0$

$\Rightarrow \boxed{M_y = 0}$