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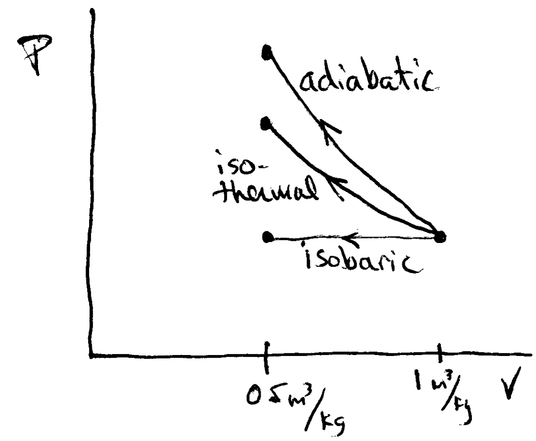
Unified Engineering
Fall 2004

Problem Set #7
Solutions

SOLUTIONS TO T4 BY WAITZ

a)



b) INITIAL CONDITIONS

$V_1 = 1\,m^3/kg$, $P_1 = 100\,kPa$ ∴ $T_1 = 348.4\,K$ by ideal gas law

1) <u>isobaric</u> $p = const.$ $p_1 = p_2 = 100\,kPa$

$V_2 = 0.5\,m^3/kg$

∴ $\boxed{T_2 = 174.2\,K}$ by ideal gas law

2) <u>isothermal</u> $T = const$ $T_1 = T_2 = 348.4\,K$

$V_2 = 0.5\,m^3/kg$

∴ $\boxed{P_2 = 200\,kPa}$ by ideal gas law

3) <u>q-s adiabatic</u> $pV^{\gamma} = const$ $\gamma = 1.4$

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = (2)^{0.4} = 1.32 \Rightarrow \boxed{T_2 = 459.7\,K}$$

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = 2.64 \quad \therefore \boxed{P_2 = 264\,kPa}$$

e) FOR EACH PROCESS CALCULATE WORK DONE BY SYSTEM &
HEAT ADDED TO SYSTEM

1) ISOBARIC

$$\omega = P \Delta V$$

$$\omega = 100\,kPa\left(0.5\,m^3/kg - 1.0\,m^3/kg\right) = ?$$

$$\boxed{\omega = -50\ kJ/kg} \quad \text{(compression)}$$

1st Law: $\Delta u = q - \omega$

$$\therefore \quad q = \Delta u + W = C_v(T_2 - T_1) + \omega$$

$$= 716.5(174.2 - 348.4) + (-50{,}000)$$

$$\boxed{= -175\ kJ/kg} \quad \text{(heat removal)}$$

2) ISOTHERMAL

$$\omega = RT \ln\left(\frac{V_2}{V_1}\right) = 287 \cdot 348.4 \ln(0.5)$$

$$\boxed{\omega = -69\ kJ/kg} \quad \text{(compression)}$$

1st LAW $\quad \Delta u = q - \omega \quad \Delta u = C_v \Delta T = 0$

so $q = W$

$$\boxed{q = -69\ kJ/kg} \quad \text{(heat removal)}$$

3) ADIABATIC $\quad \Delta u = \cancel{q}^{\,0} - \omega \quad \boxed{q = 0}$

$$\therefore \quad \omega = -\Delta u = -C_v(T_2 - T_1) = -716.5(459.7 - 348)$$

$$\boxed{\omega = -80\ kJ/kg} \quad \text{(compression)}$$

d) ENTHALPY CHANGE FOR EACH PROCESS

$$\Delta h = \underbrace{C_p \Delta T}_{} = \Delta\left[u + pv\right] = \Delta\underbrace{\left[C_v T + pV\right]}_{}$$

CAN CALC. EITHER WAY
BUT $C_p \Delta T$ is EASIER

1) ISOBARIC

$$\Delta h = 1003.5\,(174.2 - 348.4) = \boxed{-180\ kJ/kg}$$

2) ISOTHERMAL

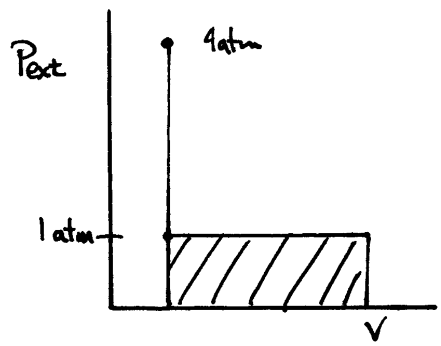
$$\boxed{\Delta h = 0}$$

3) ADIABATIC

$$\Delta h = 1003.5\,(459.7 - 348.4) = \boxed{112\ kJ/kg}$$

SOLUTIONS TO T5 BY WAITZ

a) All weights removed instantaneously



$P_{initial} = 4\,atm$, $T_i = 300K$

$$V_i = \frac{287 \cdot 300}{405.3\,kPa} = 0.21\,m^3/kg$$

WE KNOW $p_{final} = 1\,atm$ BUT WE DO NOT KNOW T_f OR V_f.

* NOTE THAT WE EXPECT $T_f < T_i$ SINCE IT IS A THERMALLY-INSULATED (ADIABATIC) CYLINDER AND WE ARE GETTING WORK OUT SO ΔU SHOULD BE NEGATIVE.

CONSIDER 1st LAW FOR ADIABATIC PROCESS:

$\Delta u = \cancel{q}^{0} - w$ so $C_v \Delta T = -P_{ext} \Delta V$

$$\left.\begin{array}{l} C_v (T_f - T_i) = -p_{ext}(V_f - V_i) \\ \text{ideal gas: } p_f V_f = RT_f \end{array}\right\} \begin{array}{l} 2 \text{ eqns} \\ 2 \text{ unknowns} \\ (V_f \,\&\, T_f) \end{array}$$

LET $V_f = \dfrac{RT_f}{p_f}$ $\quad C_v(T_f - T_i) = -p_{ext}\left(\dfrac{RT_f}{p_f} - V_i\right)$

AT THE FINAL STATE $p_f = p_{ext}$ (it comes to thermodynamic equilibrium)

SO $C_v T_f - C_v T_i = -RT_f + p_{ext} V_i$

$$(C_v + R) T_f = p_{ext} V_i + C_v T_i$$

$$\boxed{T_f = \frac{p_{ext} V_i + C_v T_i}{C_v + R}} \; *$$

PLUG IN SOME NUMBERS

$$T_f = \frac{(101325)(0.21) + (716.5)(300)}{716.5 + 287} = 235.4\text{ K}$$

$$P_f = P_{ext} = 101325\text{ N/m}^2$$

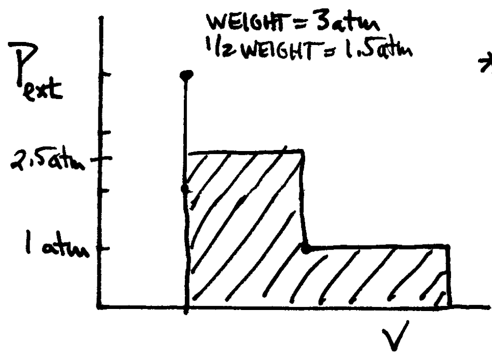
$$PV = RT \Longrightarrow V_f = 0.67\text{ m}^3/\text{kg}$$

$$\Delta u = -w$$

$$w = -C_v(235.4 - 300) = +46.3\text{ kJ/kg}$$

work by system

b) Two step process (expect more work out of system!)



* Define subscript "m" as middle condition (½ weight removed)

USE SAME EQUATIONS

$$T_m = \frac{P_{ext①} v_i + C_v T_i}{C_v + R} = \frac{253312 \cdot 0.21 + 716.5 \cdot 300}{716.5 + 287}$$

$$= 267.2\text{K}$$

$$P_m = 253312\text{ N/m}^2$$

$$V_m = 0.30\text{ m}^3/\text{kg} \qquad W_① = 23.5\text{ kJ/kg}$$

$$T_f = \frac{P_{ext②} V_m + C_v T_m}{C_v + R} = \frac{101325 \cdot 0.30 + 716.5 \cdot 267.2}{716.5 + 287}$$

$T_f = 221\ K$, $p_f = 101325\ Pa$, $V_f = 0.63\ m^3/kg$

$w_{②} = -716.5\,(221 - 267.2) = 33.1\ kJ/kg$

$w_{total} = w_{①} + w_{②} = 56.6\ kJ/kg$

c) Quasi-static process (expect the <u>most</u> work)
adiabatic

$\Rightarrow pv^{\gamma} = const.$ $\quad \dfrac{T_f}{T_i} = \left(\dfrac{p_f}{p_i}\right)^{\gamma - 1/\gamma} = \left(\dfrac{1}{4}\right)^{0.4/1.4}$

$\therefore\ T_f = 202\,K$ $\qquad = 0.673$

$p_f = 101325\ Pa$

$V_f = 0.572\ m^3/kg$

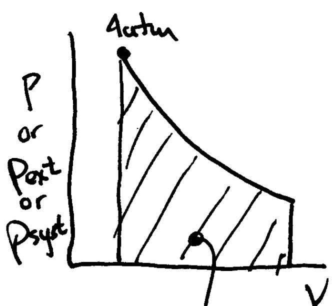
$\Delta u = \cancel{q}^{\,0} - w \qquad C_v \Delta T = -w$

$w = -716.5\,(202 - 300) = 70.2\ kJ/kg$

MESSAGES: (1) THE AMOUNT OF WORK EXTRACTED DEPENDS ON PATH

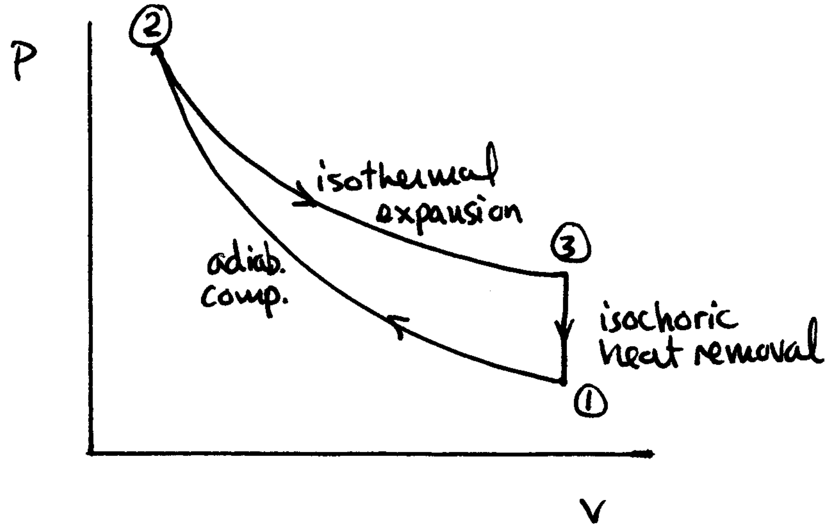
(2) IN CALCULATING WORK FOR <u>NON</u>-QUASI-STATIC PROCESSES, NEED TO USE p_{ext}.

(3) THE CLOSER WE APPROACH A QUASI-STATIC PROCESS, THE MORE WORK WE GET OUT OF THE SYSTEM. THEREFORE IN DESIGNING AN ENGINE WE WANT TO LIMIT NON-QUASI-STATIC PROCESSES TO EVERY EXTENT POSSIBLE.



we get the most area under the curve

SOLUTIONS TO T5 BY WAITZ



a) LEG 1-2: ADIABATIC COMPRESSION $\quad Q=0 \quad W \Rightarrow (-)$

LEG 2-3: ISOTHERMAL EXPANSION

$$\cancel{\Delta U}^{\,0} = Q - W \quad \text{so } Q = W \quad W \Rightarrow (+)$$

$$\therefore \; Q \Rightarrow (+)$$

LEG 3-1: CONSTANT VOLUME HEAT EXTRACTION

$$W = 0, \quad Q \Rightarrow (-)$$

b) LEG 1-2

$$\Delta u = \cancel{q}^{\,0} - w \qquad c_v \Delta T = \Delta u = -w$$

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = (8)^{0.4} = 2.3$$

$$T_1 = 300\text{K} \quad \text{so } T_2 = 690\text{ K}$$

$$\Delta u = 716.5(690-300) = 280 \text{ kJ/kg}$$

$$w = -280 \text{ kJ/kg}$$

$$q = 0$$

$$\Delta h = c_p \Delta T = 391 \text{ kJ/kg}$$

LEG ② - ③ ISOTHERMAL $\cancel{\Delta u}^{0} = q - w$

$q = w$

$w = RT \ln\left(\frac{V_2}{V_1}\right) = 287\,(690) \ln(8)$

$w = 412\ \text{kJ/kg}$

$\Delta u = 0$

$\Delta h = 0$

$q = 412\ \text{kJ/kg}$

LEG ③ - ① CONST. VOLUME

$w = 0 \qquad \Delta u = q \qquad C_v \Delta T = q$

$q = 716.5\,(300 - 690) = - \quad \text{kJ/kg}$

$\Delta h = C_p \Delta T = 1003.5\,(300 - 690) = -391\ \text{kJ/kg}$

c) NET WORK

$W_{cycle} = W_{①-②} + W_{②-③} + W_{③-①}$

$= -280 + 412 + 0 = 132\ \text{kJ/kg}$

✓ NOTE FOR CYCLE, $W_{cycle} = q_{cycle} = q_{①-②} + q_{②-③} + q_{③-①}$

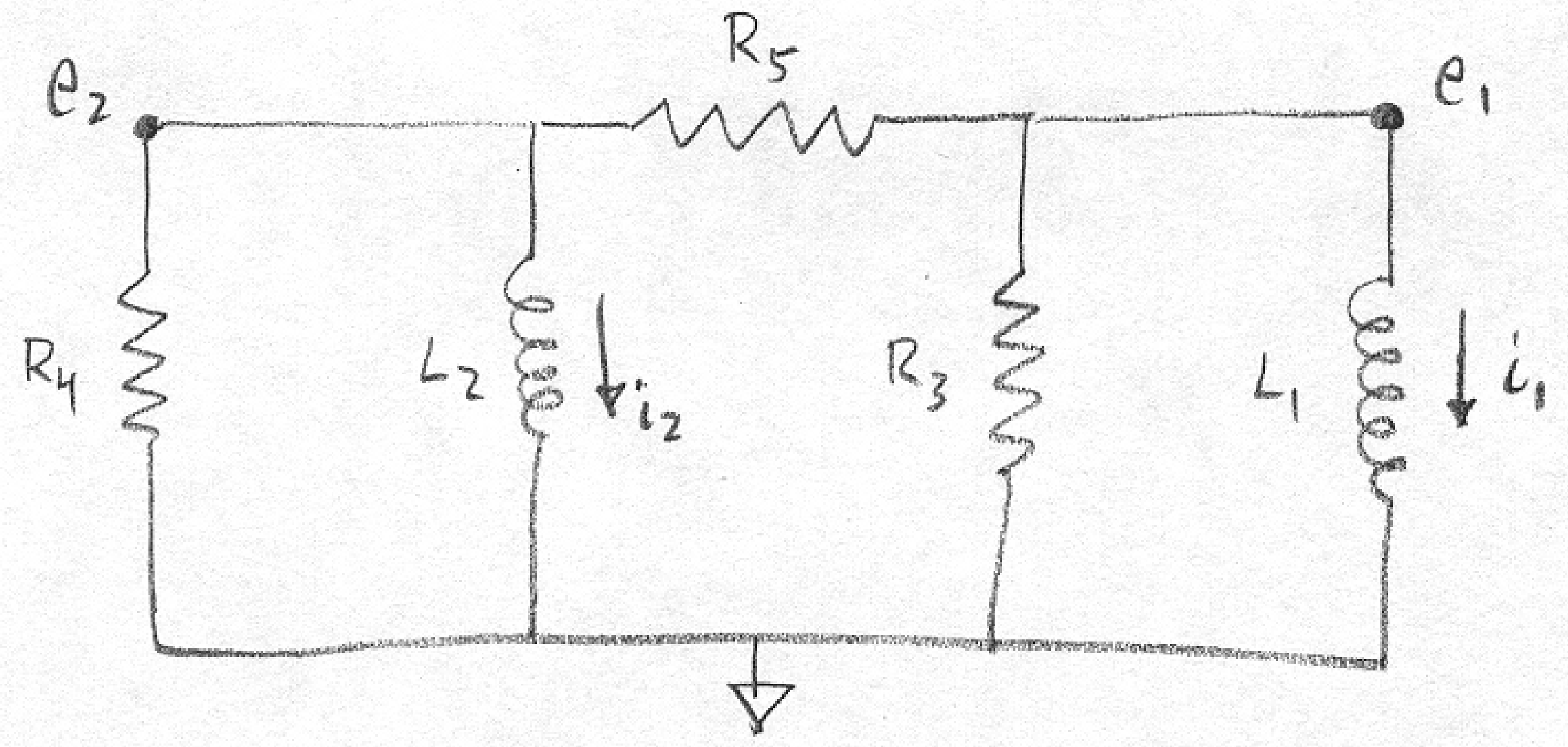
$= 0 + 412 - 280$

$= 132\ \text{kJ/kg}$

d) THERMAL EFFICIENCY = $\frac{\text{WHAT YOU GET}}{\text{WHAT YOU PAY FOR}} = \frac{\text{NET WORK}}{\text{HEAT IN}}$

$\eta_{TH} = \frac{132\ \text{kJ/kg}}{412\ \text{kJ/kg}} = 32\%$

UNIFIED ENGINEERING PROBLEM S10 SOLUTIONS FALL 2004



$L_1 = 1\,H$

$L_2 = 1\,H$

$R_3 = 1\,\Omega$

$R_4 = 1\,\Omega$

$R_5 = \frac{1}{2}\,\Omega$

APPLY NODE METHOD AT e_1 & e_2:

$$e_1: \quad i_1 + (G_3 + G_5)e_1 - G_5 e_2 = 0$$

$$e_2: \quad i_2 - G_5 e_1 + (G_4 + G_5)e_2 = 0$$

ASSUME EXPONENTIAL SOLUTIONS & WRITE i_1 & i_2 IN TERMS OF IMPEDANCES:

$$\frac{1}{L_1 s}E_1 + (G_3 + G_5)E_1 - G_5 E_2 = 0$$

$$\frac{1}{L_2 s}E_2 - G_5 E_1 + (G_4 + G_5)E_2 = 0$$

PLUG IN VALUES AND WRITE IN FORM $M(s)\,\underline{E} = \underline{0}$

$$\begin{bmatrix} 3 + \frac{1}{s} & -2 \\ -2 & 3 + \frac{1}{s} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \underline{0}$$

SOLVE FOR CHARACTERISTIC VALUES USING $\det(M(s))$

$$\left(3 + \frac{1}{s}\right)\left(3 + \frac{1}{s}\right) - 4 = 0$$

$$9 + \frac{6}{s} + \frac{1}{s^2} - 4 = 0$$

$$5s^2 + 6s + 1 = 0$$

$$\therefore s = -1 \text{ and } -\frac{1}{5}$$

FIND CHARACTERISTIC VECTOR $\underline{E}$ FOR EACH CHARACTERISTIC VALUE

$\underline{S_1 = -1}$ $\quad M(-1) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$M(s)\underline{E} = \underline{0} \Rightarrow \underline{E} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{S_2 = \frac{-1}{5}}$ $\quad M(\frac{-1}{5}) = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$

$$M(s)\underline{E} = \underline{0} \Rightarrow \underline{E} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

SO $\quad e(t) = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-1t} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-\frac{1}{5}t}$

NOW USE CONSTITUTIVE LAW FOR INDUCTOR TO FIND $i_1(t)$ & $i_2(t)$

$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int v(t)\,dt$$

SINCE L_1 & $L_2 = 1\,H$,

$$i_1(t) = -ae^{-t} - 5be^{-\frac{1}{5}t}$$
$$i_2(t) = -ae^{-t} + 5be^{-\frac{1}{5}t}$$

USE INITIAL CONDITIONS TO SOLVE FOR a & b

$i_1(0) = 10\,A \qquad i_2(t) = 0\,A$

$10 = -a - 5b \qquad 0 = -a + 5b$

$a = 5b$

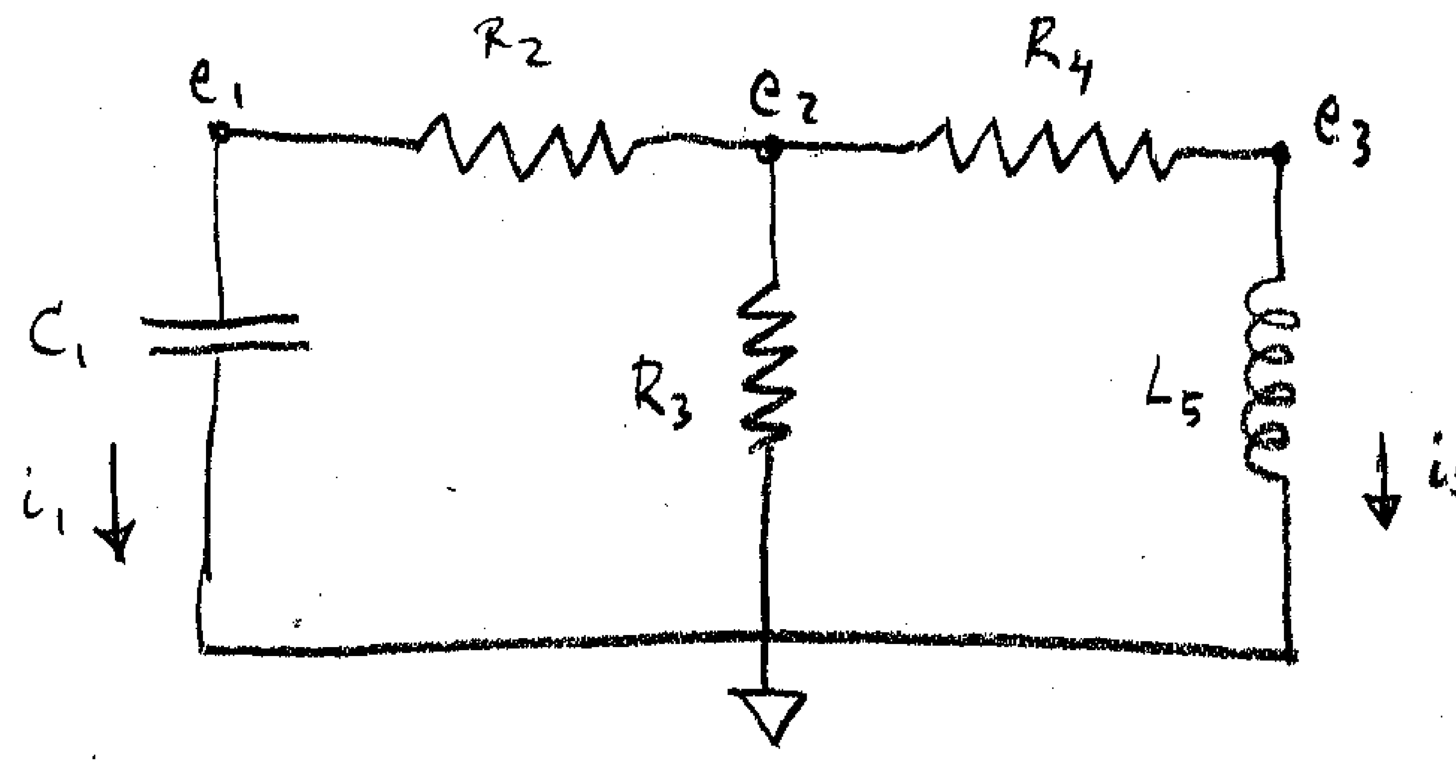
$10 = -5b - 5b$

$10 = -10b$

$\boxed{b = -1} \longrightarrow \boxed{a = -5}$

$$\therefore \boxed{\begin{aligned} i_1(t) &= 5e^{-t} + 5e^{-\frac{1}{5}t} \\ i_2(t) &= 5e^{-t} - 5e^{-\frac{1}{5}t} \end{aligned}}$$

UNIFIED ENGINEERING PROBLEM S11 SOLUTIONS FALL 2004



$C_1 = \frac{1}{4}F$

$R_2 = 4\Omega$

$R_3 = 4\Omega$

$R_4 = 1\Omega$

$L_5 = 1H$

APPLY NODE METHOD AT $e_1, e_2, \& \ e_3$. ASSUME EXPONENTIAL SOLUTIONS AND WRITE $i_1 \ \& \ i_5$ IN TERMS OF IMPEDANCES TO OBTAIN THE FOLLOWING:

$$\begin{bmatrix} C_1 s + G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3+G_4 & -G_4 \\ 0 & -G_4 & G_4 + \frac{1}{L_5 s} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \underline{0}$$

PLUG IN COMPONENT VALUES.

$$\begin{bmatrix} \frac{1}{4}s + \frac{1}{4} & \frac{-1}{4} & 0 \\ \frac{-1}{4} & \frac{3}{2} & -1 \\ 0 & -1 & \frac{1}{s}+1 \end{bmatrix} \underline{E} = \underline{0}$$

SOLVE FOR CHARACTERISTIC VALUES USING $\det(M(s)) = 0$

$$\left(\frac{1}{4}s + \frac{1}{4}\right)\left[\frac{3}{2}\left(\frac{1}{s}+1\right) - 1\right] + \frac{1}{4}\left[\frac{-1}{4}\left(\frac{1}{s}+1\right)\right] = 0$$

$$2s^2 + 7s + 5 = 0$$

$$\therefore \ s = -1 \quad \text{AND} \quad -2.5$$

FIND CHARACTERISTIC VECTOR $\underline{E}$ FOR EACH CHARACTERISTIC VALUE THAT SATISFIES $M(s)\underline{E} = \underline{0}$

$\underline{s_1 = -1}$ $\quad M(-1) = \begin{bmatrix} 0 & \frac{-1}{4} & 0 \\ \frac{-1}{4} & \frac{3}{2} & -1 \\ 0 & -1 & 0 \end{bmatrix}$ $\quad \therefore \underline{E_1} = \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{4} \end{bmatrix}$

$\underline{s_2 = -2.5}$ $\quad M(-2.5) = \begin{bmatrix} \frac{-3}{8} & \frac{-1}{4} & 0 \\ \frac{-1}{4} & \frac{3}{2} & -1 \\ 0 & -1 & \frac{3}{5} \end{bmatrix}$ $\quad \therefore \underline{E_2} = \begin{bmatrix} 1 \\ \frac{-3}{2} \\ \frac{-5}{2} \end{bmatrix}$

SO $\underline{e}(t) = a\begin{bmatrix} 1 \\ 0 \\ \frac{-1}{4} \end{bmatrix} e^{-t} + b\begin{bmatrix} 1 \\ \frac{-3}{2} \\ \frac{-5}{2} \end{bmatrix} e^{-2.5t}$

LOOKING AT CIRCUIT, $v_1(t) = e_1(t) - 0 = e_1(t)$

AND $v_5(t) = L_5 \frac{di_5(t)}{dt} \Rightarrow i_5(t) = \frac{1}{L_5}\int v_5(t)\,dt = \frac{1}{L_5}\int e_3(t)\,dt$

$\therefore v_1(t) = ae^{-t} + be^{-2.5t}$

$i_5(t) = \frac{a}{4}e^{-t} + be^{-2.5t}$

USE INITIAL CONDITIONS TO SOLVE FOR a & b:

$v_1(0) = 2V = a + b$ $\qquad i_5(0) = 1A = \frac{a}{4} + b$

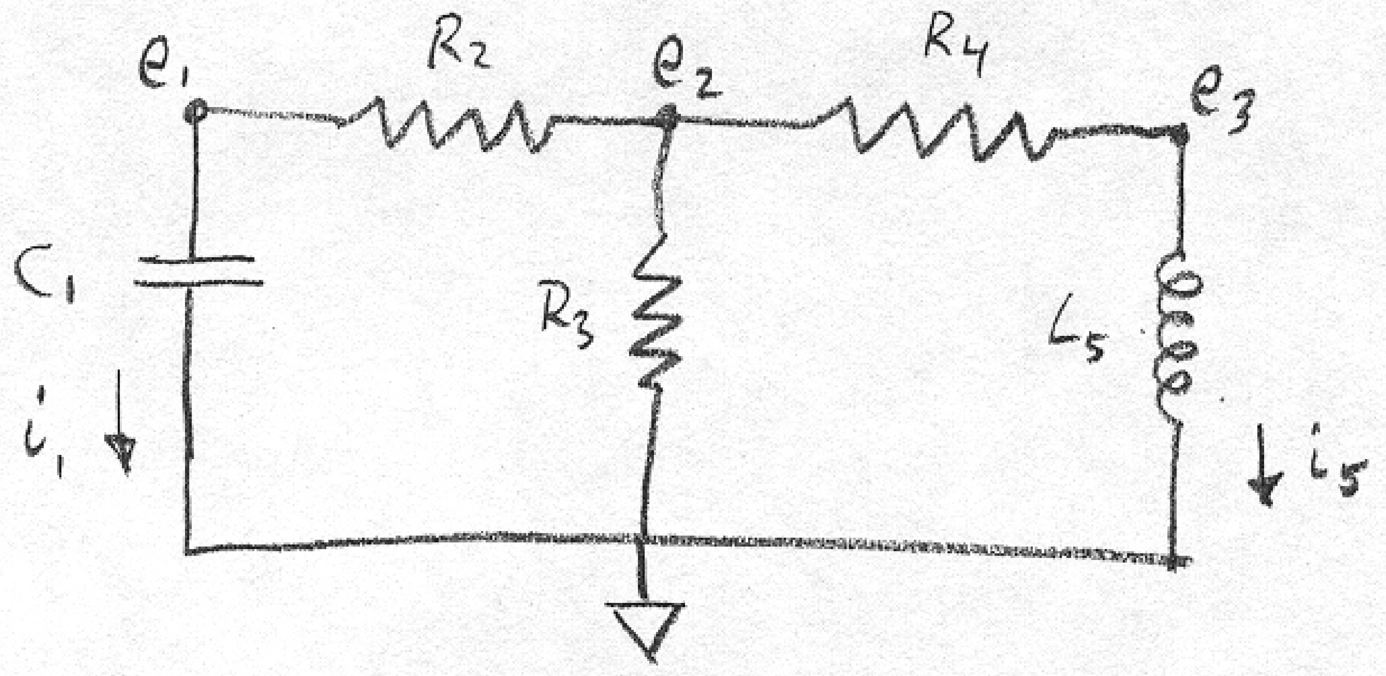
$2 = a + (1 - \frac{a}{4}) \longleftarrow b = 1 - \frac{a}{4}$

$1 = \frac{3}{4}a$

$\boxed{a = \frac{4}{3}} \longrightarrow b = 1 - \frac{4}{3 \cdot 4} = \boxed{\frac{2}{3}}$

$$\boxed{\begin{aligned} \therefore v_1(t) &= \frac{4}{3}e^{-t} + \frac{2}{3}e^{-2.5t} \\ i_5(t) &= \frac{1}{3}e^{-t} + \frac{2}{3}e^{-2.5t} \end{aligned}}$$

UNIFIED ENGINEERING PROBLEM S12 SOLUTIONS FALL 2004



$C_1 = \frac{1}{4} F$
$R_2 = 4\Omega$
$R_3 = 4\Omega$
$R_4 = 1\Omega$
$R_5 = 1H$

STATES CORRESPOND TO ENERGY STORAGE UNITS IN CIRCUIT

SO STATES ARE $x_1 = v_1$

$x_2 = i_5$

USING CONSTITUTIVE RELATIONS FOR INDUCTOR & CAPACITOR,

$$\dot{x}_1 = \frac{dv_1}{dt} = \frac{i_1}{C_1} \Rightarrow f(v_1, i_5)$$

$$\dot{x}_2 = \frac{di_5}{dt} = \frac{v_5}{L_5} \Rightarrow f(v_1, i_5)$$

→ WANT STATE DERIVATIVES IN TERMS OF STATE VARIABLES

APPLY NODE METHOD AT e_1, e_2, & e_3:

e_1: $i_1 + G_2 e_1 - G_2 e_2 = 0$

e_2: $-G_2 e_1 + (G_2 + G_3 + G_4) e_2 - G_4 e_3 = 0$

e_3: $i_5 - G_4 e_2 + G_4 e_3 = 0$

PLUG IN VALUES AND LET $e_1 = v_1$ & $e_3 = v_5$

$$\left.\begin{aligned} &i_1 + \tfrac{1}{4} v_1 - \tfrac{1}{4} e_2 = 0 \\ &-\tfrac{1}{4} v_1 + \tfrac{3}{2} e_2 - v_5 = 0 \\ &i_5 - e_2 + v_5 = 0 \end{aligned}\right\}$$

TREAT STATES v_1 & i_5 AS CONSTANT
3 EQUATIONS, 3 UNKNOWNS (e_2, i_1, v_5)

SOLVING FOR i_1 & v_5 IN ORDER TO FIND $\dot{\underline{x}}$ FROM ABOVE:

$$i_1 = -\frac{1}{8} v_1 - \frac{1}{2} i_5$$

$$v_5 = \frac{1}{2} v_1 - 3 i_5$$

SINCE $\dot{x}_1 = \frac{i_1}{C_1}$ & $\dot{x}_2 = \frac{v_5}{L_5}$

$$\boxed{\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{bmatrix} -\frac{1}{2} & -2 \\ \frac{1}{2} & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_1 \\ i_5 \end{bmatrix}}_{\underline{x}}}$$

FIND $\det(sI-A)$ TO SOLVE FOR EIGENVALUES OF MATRIX A

$$[sI-A] = \begin{bmatrix} s+\frac{1}{2} & 2 \\ -\frac{1}{2} & s+3 \end{bmatrix}$$

$$\det[sI-A] = (s+\tfrac{1}{2})(s+3)+1 = 0$$

$$s^2+\tfrac{7}{2}s+\tfrac{5}{2} = 0 \qquad \therefore \boxed{s=-1 \text{ AND } -2.5} \rightarrow \text{EIGENVALUES}$$

SO NON-TRIVIAL SOLUTION $\underline{v}$ TO THE EQUATION

$$[sI-A]\,\underline{v} = 0$$

IS THE EIGEN VECTOR

$\underline{s_1=-1}$ $\quad [sI-A]\underline{v}=0$

$$\begin{bmatrix} -\frac{1}{2} & 2 \\ -\frac{1}{2} & 2 \end{bmatrix}\underline{v} = 0 \quad \Rightarrow \quad \underline{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$\underline{s_2=-2.5}$

$$\begin{bmatrix} -2 & 2 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\underline{v} = 0 \quad \Rightarrow \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \underline{x} = a\begin{bmatrix} 4 \\ 1 \end{bmatrix}e^{-t} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2.5t}$$

USE INITIAL CONDITIONS TO SOLVE FOR a & b

$v_1(0) = x_1(0) = 2V = 4a+b \qquad i_s(0) = x_2(0) = 1A = a+b$

$2 = 4a+(1-a) \leftarrow b = 1-a$

$1 = 3a$

$a = \frac{1}{3} \rightarrow b = 1-\frac{1}{3}$

$b = \frac{2}{3}$

$$\therefore \boxed{\begin{bmatrix} v_1(t) \\ i_s(t) \end{bmatrix} = \underline{x} = \frac{1}{3}\begin{bmatrix} 4 \\ 1 \end{bmatrix}e^{-t} + \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2.5t}}$$