Unified Quiz 6F
December 15, 2003

- Put your name on each page of the exam.
- Read all questions carefully.
- Do all work for each problem on the two pages provided.
- Show intermediate results.
- Explain your work --- don’t just write equations.
- Partial credit will be given, but only when the intermediate results and explanations are clear.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Show appropriate units with your final answers.
- Calculators and a 2-sided sheet of paper are allowed
- Box your final answers.

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1. (30 %)
A circular cylinder of radius $R$ has an irrotational incompressible circular-streamline flow about it. The fluid’s tangential velocity on the surface $V_\theta(R)$ is known.

a) In terms of vorticity and conservation of mass, explain why this flow can be represented by a single vortex of strength $\Gamma$ at the center of the cylinder. Determine the strength $\Gamma$ required to correctly represent this flow.

b) An alternative representation is sought using a vortex sheet placed on the cylinder surface as shown. Determine the sheet strength $\gamma$ required to correctly represent the flow.

c) Using the tangential-velocity jump relation for a vortex sheet
\[
\Delta V_s = \gamma
\]
and circulation arguments, determine the (fictitious) flow inside the cylinder for the surface-sheet model. Be sure to show your reasoning.
a) An incompressible flow has $\nabla \cdot \vec{V} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ in 2D.

An irrotational flow has $\nabla \times \vec{V} = 0$ or $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ in 2D.

The single point vortex has velocities: $u = \frac{1}{2\pi} \frac{y}{x^2 + y^2}, \quad v = \frac{1}{2\pi} \frac{-x}{x^2 + y^2}$.

This satisfies both $\nabla \cdot \vec{V} = 0$ and $\nabla \times \vec{V} = 0$.

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\frac{2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} = 0 \quad \text{(factor of } \frac{1}{2\pi} \text{ is irrelevant)} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= -\frac{1}{x^2 + y^2} + \frac{2x}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2} + \frac{2y}{(x^2 + y^2)^2} = -\frac{2}{x^2 + y^2} + \frac{2}{x^2 + y^2} = 0 \end{cases}$$

Alternative: If using $\phi(x,y) = \frac{-1}{2\pi} \arctan \left( \frac{y}{x} \right)$ or $\psi(x,y) = \frac{x}{2\pi} \ln \sqrt{x^2 + y^2}$

to define the flow, then the incompressibility + irrotationality conditions become $\nabla^2 \phi = \frac{2x^2}{x^2 + y^2} + \frac{2y^2}{x^2 + y^2} = 0$, or $\nabla^2 \psi = \frac{2}{x^2 + y^2} + \frac{2}{x^2 + y^2} = 0$

Both $\nabla^2 \phi = 0 \land \nabla^2 \psi = 0$ can be confirmed for this case.

Set $\Gamma$ to match known $V_0(\Gamma)$. At $(x,y) = (0,R)$, $U(0,R) = \frac{\Gamma}{2\pi} \frac{R}{0+R^2} = V_0(\Gamma) \Rightarrow \Gamma = 2\pi R V_0(\Gamma)$.

b) Must have the same total circulation: $2\pi R \gamma = \Gamma = 2\pi R V_0(\Gamma)$.

\[ \gamma = V_0(\Gamma) \]

\[ V = \partial_0 V_0(\Gamma) \]

\[ V_{\text{side}} = \frac{V_0}{\Gamma} \]

\[ V_{\text{side}} = \frac{V_0}{\Gamma} \neq 0 \]

\[ \nabla \times \vec{V} = 0 \]

\[ \nabla \cdot \vec{V} = 0 \]

c) Flow velocity just inside surface must be zero, since $\nabla V = V_0(\Gamma) - V_{\text{side}} = \gamma = V_0(\Gamma) \Rightarrow V_{\text{side}} = 0$.

Cannot have any velocity anywhere inside, since this would imply a nonzero circulation at that radius.

This is not possible since there is no vortex inside.

\[ \vec{V} = 0 \text{ inside} \]
2. (20 %)
A lifting airfoil is positioned over an infinite flat ground plane. The air velocity far away is \( V_\infty \). The flowfield is described by a stream function \( \psi(x, y) \).

a) How does the stream function vary along the ground plane (i.e. what does \( \psi(x, 0) \) look like)?

The stream function \( \psi(x, y) \) is to be represented by placing on the airfoil surface a vortex sheet of strength \( \gamma_1(s) \). At the same time, a sheet of strength \( \gamma_2(s) \) is placed on the fictitious mirror image airfoil of the same shape, as shown.

b) How must \( \gamma_2(s) \) be related to \( \gamma_1(s) \) so that the flow is correctly represented? Explain.
a) Ground plane is a streamline.
   \[ \forall (x,0) = \text{constant} \]
   This also implies \[ V(x,0) = -\frac{\partial \psi}{\partial x}|_{x,0} = 0 \]
   Flow is tangent to wall.

b) Must have \( V(x,0) = 0 \). This is only possible if \[ \gamma_2 = -\delta \]
so that contributions at the wall from matching points cancel in y direction.
   \[ \gamma_1, ds \]
3. (50 %)
The symmetrical NACA 0015 airfoil has a lift coefficient function approximated by

\[ c_l(\alpha) = \begin{cases} 
1.2 & , \alpha > 0.2 \\
6.0\alpha & , 0.2 > \alpha > -0.2 \\
-1.2 & , -0.2 > \alpha 
\end{cases} \] (positive stall) (unstalled) (negative stall)

where \( \alpha \) is in radians.

\[ c_l \]

\[ 6.0 \]

\[ \alpha \]

A NACA 0015 airfoil of chord \( c \) is held parallel to the \( x \)-axis, in a flow with freestream velocity \( V_\infty \) along the \( x \)-axis. The air density \( \rho \) is effectively constant. A large circular cylinder of radius \( R = 5c \) is now stuck in the flow, such that the airfoil is at \( x, y = (-5c, 10c) \) relative to the cylinder center, as shown.

a) Construct a velocity field \( u(x, y) \) and \( v(x, y) \) for this configuration, for the purpose of estimating the apparent velocity seen by the little airfoil. Assume the cylinder has inviscid flow about it, and that the airfoil itself has a negligible effect on the overall flow.

b) Evaluate the velocity components \( u \) and \( v \) at the airfoil location. Is the airfoil stalled?

c) Determine the lift \( L' \) on the airfoil.
a) Freestream + Doublet: 
\[ \phi(x,y) = V_\infty \times + \frac{K}{2\pi} \frac{x}{xy^2} \]
\[ u(x,y) = \frac{\delta \phi}{\delta y} = V_\infty + \frac{K}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2} \]
\[ v(x,y) = \frac{\delta \phi}{\delta x} = -\frac{K}{2\pi} \frac{2xy}{(x^2 + y^2)^2} \]

Set doublet strength \( K \) so point \((-5c, 0)\) is a stagnation point:

\[ U(-5c, 0) = V_\infty - \frac{K}{2\pi} \frac{25c^2}{(25c^2)^2} = 0 \]
\[ \text{or} \quad \frac{K}{2\pi} = 25c^2 V_\infty \]

So
\[ u(x,y) = V_\infty \left( 1 + 25c^2 \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) \]
\[ v(x,y) = V_\infty \left( -25c^2 \frac{2xy}{(x^2 + y^2)^2} \right) \]

b) At \( x, y = (-5c, 10c) \):

\[ U(-5c, 10c) = V_\infty \left( 1 + 25c^2 \frac{100 - 25}{(25 + 100)^2} \right) = 1.12 V_\infty \]
\[ V(-5c, 10c) = V_\infty \left( -25c^2 \frac{2 \cdot (-5) \cdot 10}{(25 + 100)^2} \right) = 0.16 V_\infty \]

Airfoil sees \( \alpha = \arctan \frac{V}{U} = \arctan \left( \frac{0.16}{1.12} \right) = 8.13^\circ = 0.1419 \text{ rad} \) \( < 0.2 \) not stalled

c) \( C_\alpha(k) = 6.0 \cdot 0.1419 = 0.8514 \)

Dynamic pressure at airfoil:

\[ \frac{1}{2} \rho V^2 = \frac{1}{2} \rho (u^2 + v^2) = \frac{1}{2} \rho V_\infty^2 \left( 1.12^2 + 0.16^2 \right) = \frac{1}{2} \rho V_\infty^2 \cdot 1.28 \]

\[ L = \frac{1}{2} \rho V^2 c C_\alpha = \frac{1}{2} \rho V_\infty^2 \cdot 1.28 \cdot 0.8514 = \frac{1}{2} \rho V_\infty^2 \cdot 1.0898 = 0.5449 \rho V_\infty^2 c \]