Unified Quiz 6F

December 15, 2003

- Put your name on each page of the exam.
- Read all questions carefully.
- Do all work for each problem on the two pages provided.
- Show intermediate results.
- Explain your work --- don't just write equations.
- Partial credit will be given, but only when the intermediate results and explanations are clear.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Show appropriate units with your final answers.
- Calculators and a 2-sided sheet of paper are allowed
- Box your final answers.

Exam Scoring

#1 (30 %)	
#2 (20%)	
#3 (50 %)	
Total	

1. (30 %)
A circular cylinder of radius R has an irrotational incompressible circular-streamline flow about it. The fluid's tangential velocity on the surface $V_\theta(R)$ is known.

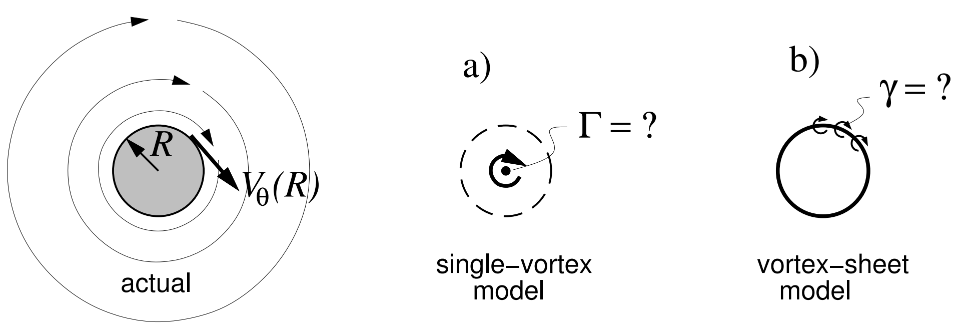
a) In terms of vorticity and conservation of mass, explain why this flow can be represented by a single vortex of strength Γ at the center of the cylinder. Determine the strength Γ required to correctly represent this flow.

b) An alternative representation is sought using a vortex sheet placed on the cylinder surface as shown. Determine the sheet strength γ required to correctly represent the flow.

c) Using the tangential-velocity jump relation for a vortex sheet

$$\Delta V_s = \gamma$$

and circulation arguments, determine the (fictitious) flow inside the cylinder for the surface-sheet model. Be sure to show your reasoning.



a) An incompressible flow has $\nabla \cdot \vec{V} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ in 2D

An irrotational flow has $\nabla \times \vec{V} = 0$ or $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ in 2D

The single point vortex has velocities... $u = \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$, $V = \frac{\Gamma}{2\pi}\frac{-x}{x^2+y^2}$

This satisfies both $\nabla \cdot \vec{V} = 0$ and $\nabla \times \vec{V} = 0$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)} \overset{\checkmark}{=} 0}$$

(factor of $\frac{\Gamma}{2\pi}$ is irrelevant)

$$\Rightarrow \boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{-2}{x^2+y^2} + \frac{2}{x^2+y^2} \overset{\checkmark}{=} 0}$$

__Alternative__: If using $\phi(x,y) = \frac{-\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right)$ or $\psi(x,y) = \frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2}$ to define the flow, then the incompressibily + irrotationality conditions become $\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, or $\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

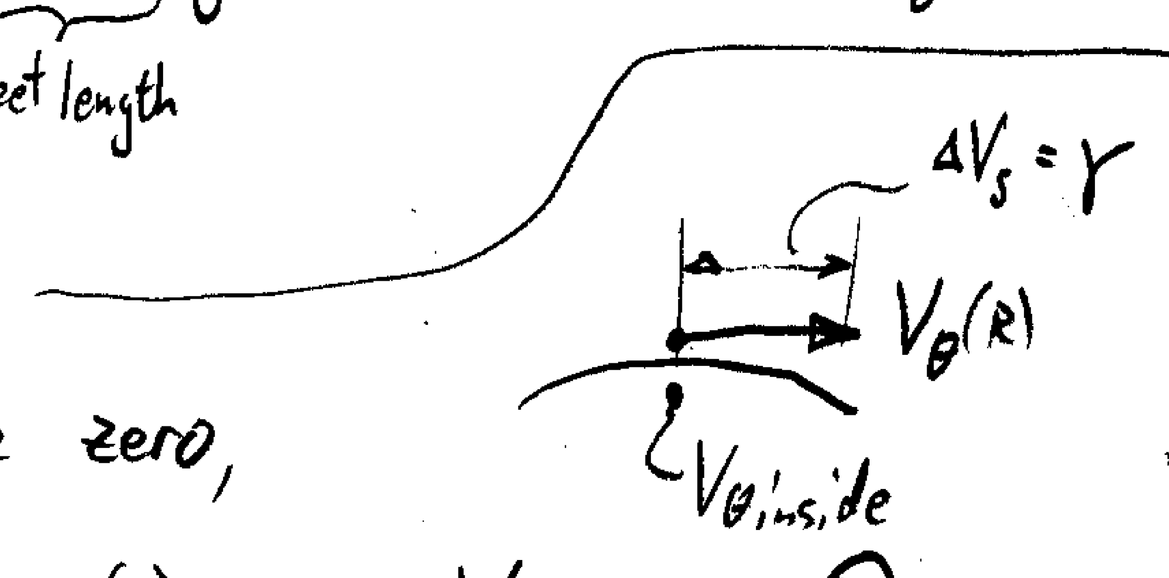
Both $\nabla^2\phi = 0$ or $\nabla^2\psi = 0$ can be confirmed for this case.

Set Γ to match known $V_\theta(R)$. At $(x,y) = (0,R)$ ...

$$u(0,R) = \frac{\Gamma}{2\pi}\frac{R}{0+R^2} = V_\theta(R) \Rightarrow \boxed{\Gamma = 2\pi R\, V_\theta(R)}$$

b) Must have the same total circulation: $\underbrace{2\pi R}_{\text{sheet length}}\,\gamma = \Gamma = 2\pi R\, V_\theta(R)$

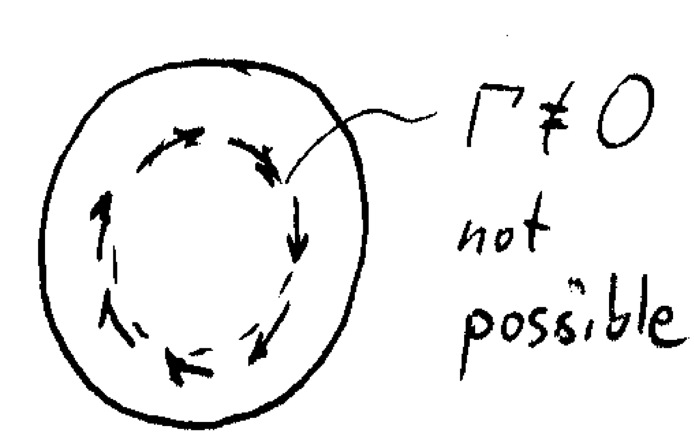
$$\boxed{\gamma = V_\theta(R)}$$



c) Flow velocity just inside surface must be zero, since $\Delta V_s = V_\theta(R) - V_{\theta\,inside} = \gamma = V_\theta(R) \Rightarrow V_{\theta\,inside} = 0$

Cannot have any velocity __anywhere__ inside, since this would imply a nonzero circulation at that radius.

This is not possible since there is no vortex inside.

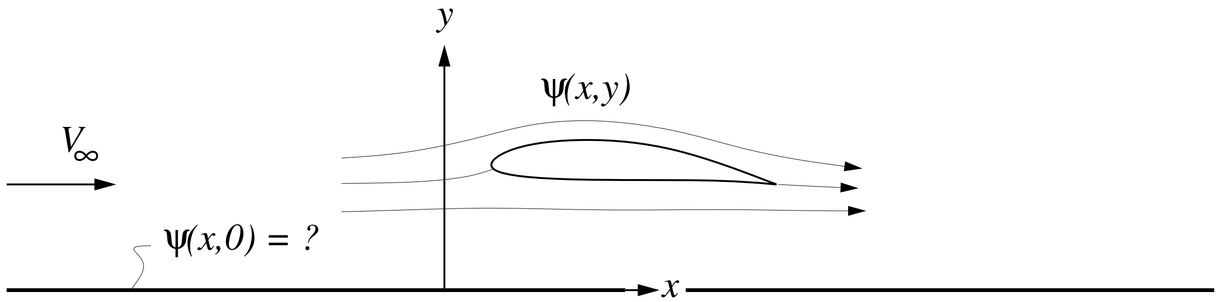


$$\rightarrow \boxed{\vec{V} = 0 \text{ inside}}$$

2. (20 %)

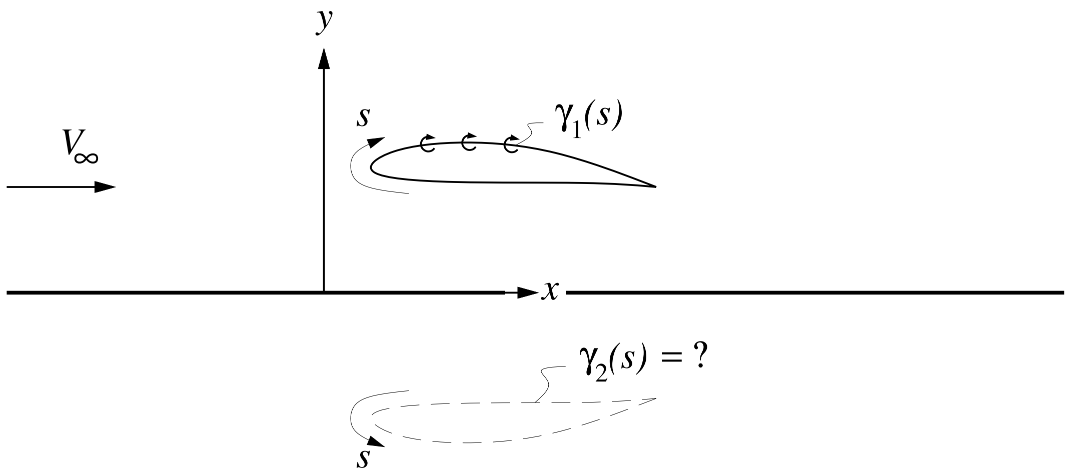
A lifting airfoil is positioned over an infinite flat ground plane. The air velocity far away is V_∞. The flowfield is described by a stream function $\psi(x, y)$.

a) How does the stream function vary along the ground plane (i.e. what does $\psi(x, 0)$ look like)?



The stream function $\psi(x, y)$ is to be represented by placing on the airfoil surface a vortex sheet of strength $\gamma_1(s)$. At the same time, a sheet of strength $\gamma_2(s)$ is placed on the fictitious mirror image airfoil of the same shape, as shown.

b) How must $\gamma_2(s)$ be related to $\gamma_1(s)$ so that the flow is correctly represented? Explain.



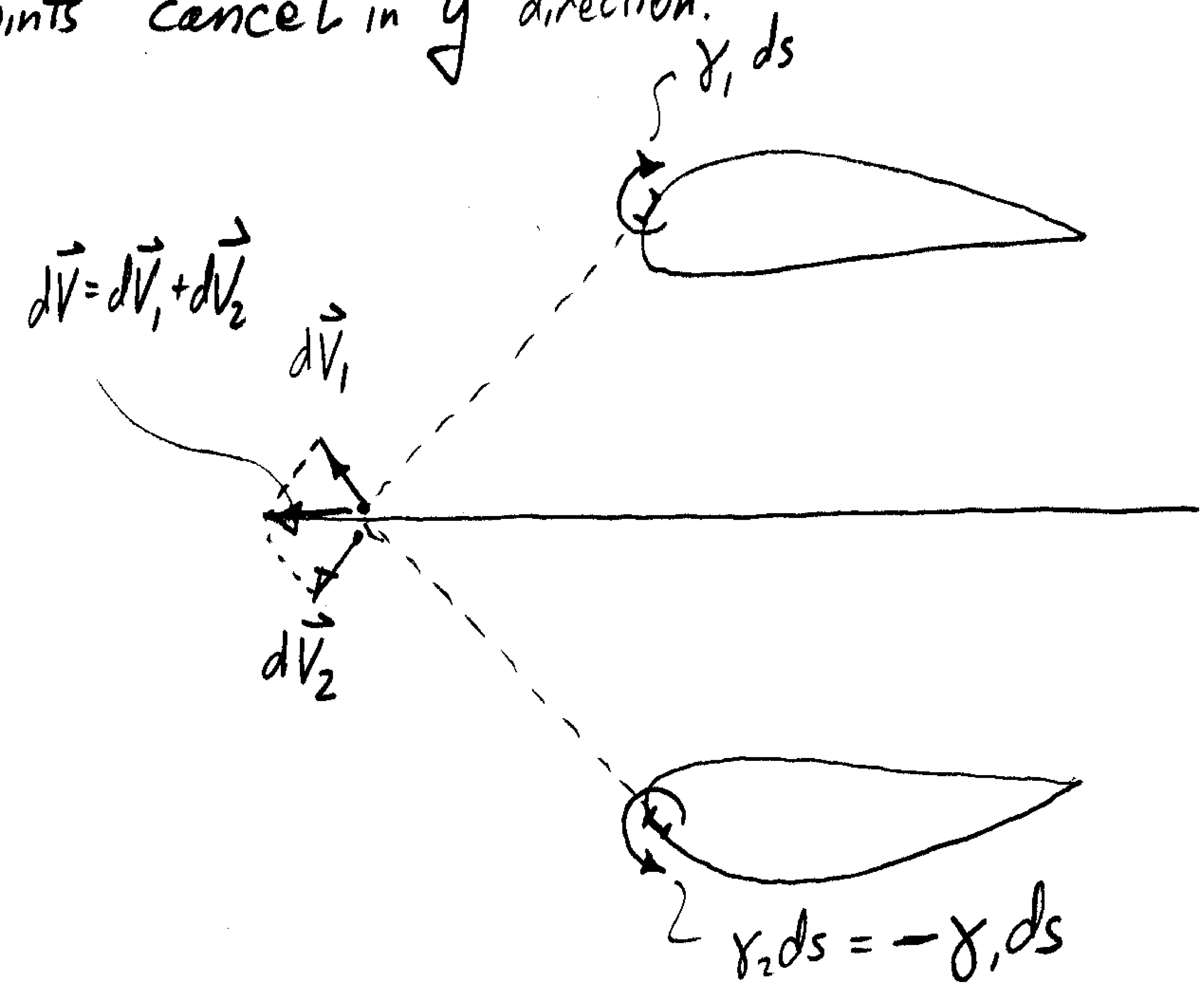
a) Ground plane is a streamline.

$\rightarrow \psi(x,0) = \text{Constant}$

This also implies $v(x,0) = -\left.\frac{\partial \psi}{\partial x}\right|_{x,0} = 0$

Flow is tangent to wall.

b) Must have $v(x,0) = 0$. This is only possible if $\boxed{\gamma_2 = -\gamma_1}$ so that contributions at the wall from matching points cancel in y direction.

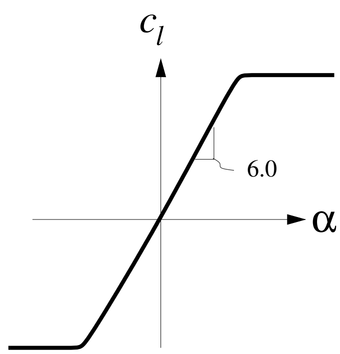


3. (50 %)

The symmetrical NACA 0015 airfoil has a lift coefficient function approximated by

$$c_\ell(\alpha) = \begin{cases} 1.2 & , \quad \alpha > 0.2 & \text{(positive stall)} \\ 6.0\alpha & , \quad 0.2 > \alpha > -0.2 & \text{(unstalled)} \\ -1.2 & , \quad -0.2 > \alpha & \text{(negative stall)} \end{cases}$$

where α is in radians.

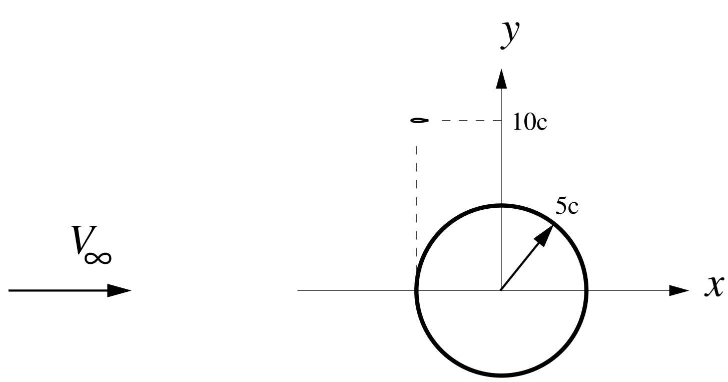


A NACA 0015 airfoil of chord c is held parallel to the x-axis, in a flow with freestream velocity V_∞ along the x-axis. The air density ρ is effectively constant. A large circular cylinder of radius $R = 5c$ is now stuck in the flow, such that the airfoil is at $x, y = (-5c, 10c)$ relative to the cylinder center, as shown.

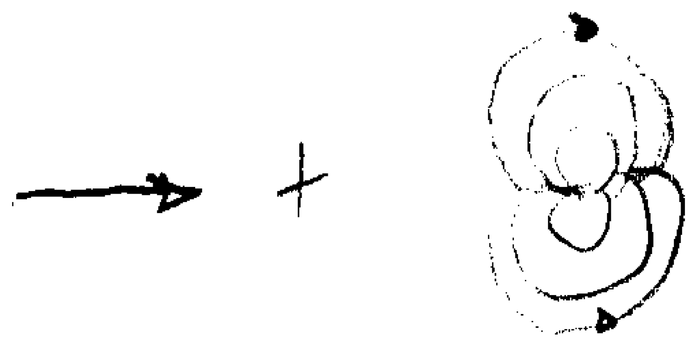
a) Construct a velocity field $u(x, y)$ and $v(x, y)$ for this configuration, for the purpose of estimating the apparent velocity seen by the little airfoil. Assume the cylinder has inviscid flow about it, and that the airfoil itself has a negligible effect on the overall flow.

b) Evaluate the velocity components u and v at the airfoil location. Is the airfoil stalled?

c) Determine the lift L' on the airfoil.



a) Freestream + Doublet: $\phi(x,y) = V_\infty x + \frac{K}{2\pi}\frac{x}{x^2+y^2}$

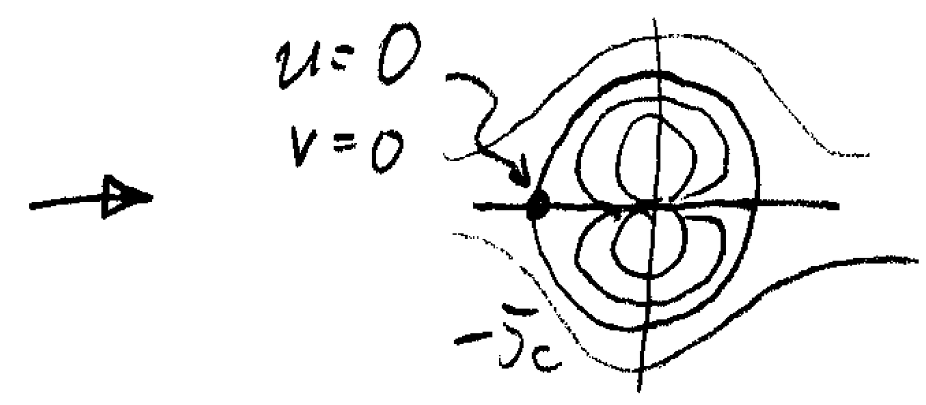


$$u(x,y) = \frac{\partial \phi}{\partial x} = V_\infty + \frac{K}{2\pi}\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v(x,y) = \frac{\partial \phi}{\partial y} = \frac{K}{2\pi}\frac{-2xy}{(x^2+y^2)^2}$$

Set doublet strength K so point $(-5c, 0)$ is a stagnation point.

$$u(-5c,0) = V_\infty - \frac{K}{2\pi}\frac{25c^2}{(25c^2)^2} = 0$$



or $\boxed{\frac{K}{2\pi} = 25c^2 V_\infty}$

So
$$\boxed{\begin{aligned} u(x,y) &= V_\infty\left(1 + 25c^2\frac{y^2-x^2}{(x^2+y^2)^2}\right)\\ v(x,y) &= V_\infty\left(-25c^2\frac{2xy}{(x^2+y^2)^2}\right)\end{aligned}}$$

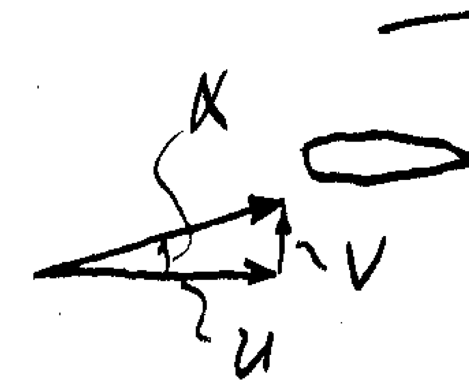
b) At $x,y = (-5c, 10c)$ ...

$$u(-5c,10c) = V_\infty\left(1 + 25c^2\frac{100-25}{(25+100)^2}\frac{1}{c^2}\right) = 1.12\,V_\infty$$

$$v(-5c,10c) = V_\infty\left(-25c^2\frac{2\cdot(-5)\cdot 10}{(25+100)^2}\frac{1}{c^2}\right) = 0.16\,V_\infty$$

Airfoil sees $\boxed{\alpha = \arctan\frac{v}{u} = \arctan\left(\frac{0.16}{1.12}\right) = 8.13^\circ = 0.1419 \text{ rad}} < 0.2$ not stalled

c) $C_\ell(\alpha) = 6.0\cdot 0.1419 = 0.8514$



Dynamic pressure at airfoil:

$$\frac{1}{2}\rho V^2 = \frac{1}{2}\rho(u^2+v^2) = \frac{1}{2}\rho V_\infty^2(1.12^2+0.16^2) = \frac{1}{2}\rho V_\infty^2\cdot 1.28$$

$$\boxed{L' = \frac{1}{2}\rho V^2 c\, c_\ell = \frac{1}{2}\rho V_\infty^2\cdot 1.28\cdot c\cdot 0.8514 = \frac{1}{2}\rho V_\infty^2 c\cdot 1.0898}$$

$$\boxed{= 0.5449\,\rho V_\infty^2 c}$$