2:45 - 3:45
Walker

MIT ID# (last four digits) SOLUTIONS

Unified Quiz 7M

December 16, 2004

- Put your MIT ID# (last four digits) on each page of the exam.
- Read all questions carefully.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit cannot be given without intermediate results.
- Show the logical path of your work. Explain clearly your reasoning and what you are doing. *In some cases, the reasoning is worth as much (or more) than the actual answers.*
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units. Intermediate answers and final answers are not correct without the units.
- Report significant digits only.
- Box your final answers.
- ***Calculators, handwritten "crib sheets", and Unified Handout #M-4 allowed.***

EXAM SCORING

#1M (25%)	
#2M (25%)	
#3M (25%)	
#4M (25%)	
FINAL SCORE	

Happy Holidays!
- Paul L.

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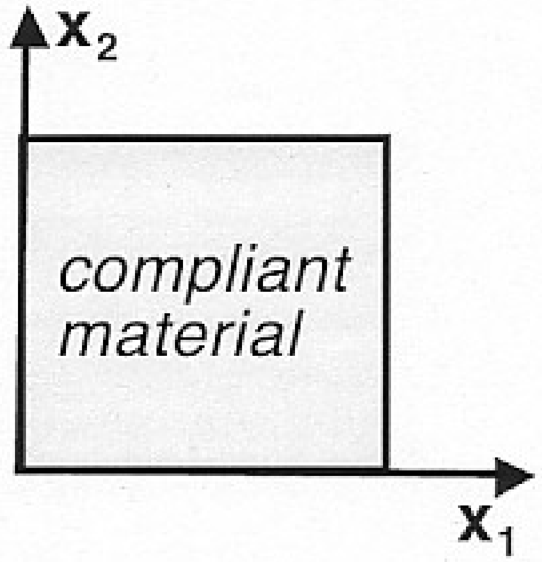
PROBLEM #1M (25%)

A square slab of a very compliant material undergoes deformation such that the in-plane strain state has a constant strain in the x_1-direction of 5000 μstrain, no strain in the x_2-direction, and an *engineering* shear strain of 16,000 μstrain.



(a) Determine a two-dimensional displacement field/vector which corresponds to this strain state. Be as general as possible.

We have: $\epsilon_{11} = 5000 \mu s$

$\epsilon_{22} = 0$

$\epsilon_{12} = \frac{\gamma_{12}}{2} = \frac{16,000 \mu s}{2} = 8,000 \mu s$ ← engineering shear strain

Use the strain-displacement relation:

$\epsilon_{11} = \partial u_1 / \partial x_1$

$\epsilon_{22} = \partial u_2 / \partial x_2$

$\epsilon_{12} = \frac{1}{2}\left(\partial u_1 / \partial x_2 + \partial u_2 / \partial x_1 \right)$

Now integrate remembering that since we are considering the 2-D case, we have u_1 and u_2 as functions of x_1 and x_2. Thus, we will have <u>functions</u> of integration, not just constants.

So: $\epsilon_{11} = 0.005 = \partial u_1 / \partial x_1$

$\Rightarrow u_1(x_1, x_2) = \int 0.005 \, dx_1$

$\Rightarrow u_1(x_1, x_2) = 0.005 x_1 + f_1(x_2)$

PROBLEM #1M (continued)

next: $\epsilon_{22} = 0 = \partial u_2 / \partial x_2$

$$\Rightarrow u_2(x_1, x_2) = g_2(x_1)$$

Finally use: $\epsilon_{12} = 0.008 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \partial u_2 / \partial x_2\right)$

and recall for small displacements that:

$$\partial u_1 / \partial x_2 = \partial u_2 / \partial x_1$$

$$\Rightarrow 0.008 = \partial u_1 / \partial x_2 = \partial u_2 / \partial x_1$$

using the results for u_1 and u_2 in the above:

$$0.008 = \partial u_1 / \partial x_2 = \frac{\partial f_1(x_2)}{\partial x_2}$$

$$\Rightarrow f_1(x_2) = 0.008 x_2 + C_1$$

and

$$0.008 = \partial u_2 / \partial x_1 = \partial u_2 / \partial x_1 = \frac{\partial g_2(x_1)}{\partial x_1}$$

$$\Rightarrow g_2(x_1) = 0.008 x_1 + C_2$$

Putting this all together:

$$u_1 = 0.005 x_1 + 0.008 x_2 + C_1$$

$$u_2 = 0.008 x_1 + C_2$$

with: $\boxed{\underline{u} = (0.005 x_1 + 0.008 x_2 + C_1)\, \underline{i}_1 + (0.008 x_1 + C_2)\, \underline{i}_2}$

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PROBLEM #1M (continued)

(b) The deformations are now increased by a factor of ten, how will the values of the in-plane strains change? Describe clearly. Use equations as necessary.

With the deformation increased by a factor of 10, the displacement becomes:

$$u_1 = 0.05x_1 + 0.08x_2 + 10C_1$$
$$u_2 = 0.08x_1 + 10C_2$$

The extensional strain ϵ_{11} at first appears to simply increase by a factor of 10 (to 0.05) and there continues to be no ϵ_{22}. However, the key is to consider $\partial u_1/\partial x_2$ and $\partial u_2/\partial x_1$ which represent angular changes. Their sum is the total angular change and now sums to 0.16. So the assumption of <u>small displacements</u> and <u>small angles</u> begins to break down with larger displacements. For 0.16 radians, the $\cos\theta$ is 0.987 ($\theta = 9.2°$). So we are 1.3% off $\cos\theta = 1$. Thus, there is coupling between extension and shear that needs to be taken into account. Therefore, saying ~~simply~~ simply that the strains increase by a factor of 10 would be using an approximation that is "breaking down".

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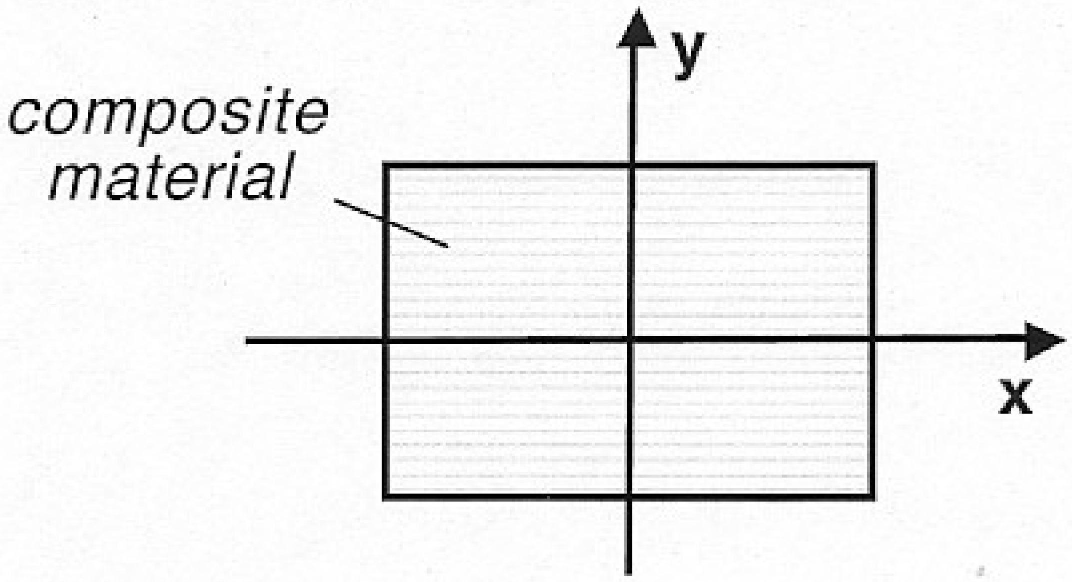
PROBLEM #2M (25%)

A rectangular slab of composite material (longitudinal modulus of 210 GPa) is loaded in a state of plane stress with a pure shear of magnitude τ.



(a) Is it possible to find a transformed axis system in the x- y plane such that the slab is loaded only by normal stresses, σ_x and σ_y? If not, describe why not. If yes, describe why, determine the angle to the transformed axis system, and determine the relative values of the normal stresses.

→ $\boxed{\text{YES}}$; it is always possible to find such stresses. They are known as principal stresses.

For the case of 2-D (plane stress) we know that:

$$\tilde{\sigma}_{12} = -\sigma_{11} \sin\theta \cos\theta + \sigma_{22} \sin\theta \cos\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)$$

Find the angle where $\tilde{\sigma}_{12}$ goes to zero. We have

$$\sigma_{11} = 0 ; \sigma_{22} = 0 ; \sigma_{12} = \tau$$

So:

$$\tilde{\sigma}_{12} = \tau(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \cos^2\theta = \sin^2\theta \Rightarrow \boxed{\theta = 45^\circ}$$

Then use the transformation equations for $\tilde{\sigma}_{11}$ and $\tilde{\sigma}_{22}$:

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PROBLEM #2M (continued)

$$\tilde{\sigma}_{11} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\cdot\cos\theta\sin\theta\,\sigma_{12}$$

$$\tilde{\sigma}_{22} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\cos\theta\sin\theta\,\sigma_{12}$$

$$\Rightarrow \tilde{\sigma}_{11} = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\tau = \tau$$

$$\tilde{\sigma}_{22} = -2\left(\frac{\sqrt{2}}{2}\right)\left(\sqrt{2}/2\right)\tau = -\tau$$

$$\Rightarrow \boxed{\begin{array}{l}\tilde{\sigma}_{11} = \tau \\ \tilde{\sigma}_{22} = -\tau\end{array}}$$

NOTE: This can also be done using Mohr's circle.

(b) Determine the principal stresses (**all three!**) for this stress state and their associated directions.

These two stresses found in part (a) are principal stresses. Since this is plane stress, the third principal stress is out-of-plane but equal to zero. Thus:

$\sigma_I = \tau$	45° from x
$\sigma_{II} = 0$	z (out-of-plane)
$\sigma_{III} = -\tau$	45° from y (or −45° from x)

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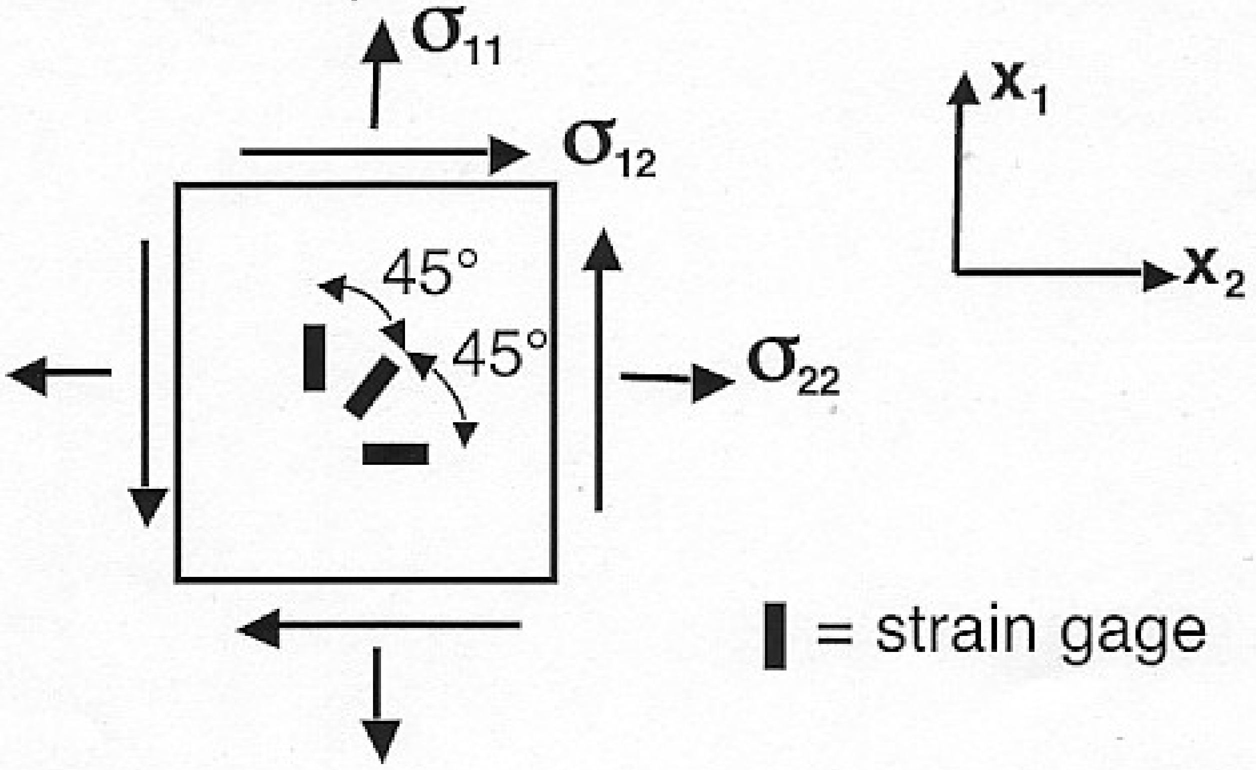
PROBLEM #3M (25%)

An *orthotropic* material has a general loading applied resulting in stresses of σ_{11}, σ_{22}, and σ_{12}, as indicated in the accompanying figure. The stresses are aligned with the main axes of the material. A three-gage strain gage rosette is placed on the material and aligned in such a way that the strain in the x_1-direction, ε_{11}, and in the x_2-direction, ε_{22}, are measured along with the strain at a 45° angle to the x_1-direction, $\varepsilon(45)$.



Can any of the engineering constants of this material be determined? If so, explain why and do so. If not, explain why not. Equations can be utilized in the explanations in either case.

First consider the stress-strain equations for the case of:

① plane stress (only $\sigma_{11}, \sigma_{22}, \sigma_{12}$ -- as here)

② orthotropic material (stresses aligned with main material axes -- as here)

These are:

$$\epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{12}}{E_1}\sigma_{22}$$

$$\epsilon_{22} = -\frac{\nu_{21}}{E_2}\sigma_{11} + \frac{1}{E_2}\sigma_{22}$$

$$2\epsilon_{12} = 1/G_{12}\ \sigma_{12}$$

(Again) note that shear stresses do not cause normal strains and normal stresses do not cause shear strains since the material main axes are aligned with the loading axes.

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PROBLEM #3M (continued)

Also note that straingages measure extensional strains. So, ϵ_{11} and ϵ_{22} are measured.

$\Rightarrow$ This gives two equations in four unknowns
(E_1, E_2, ν_{12}, ν_{21})

Reciprocity gives a third equation of:

$$\nu_{12} E_2 = \nu_{21} E_1$$

<u>Question</u>: Does $\epsilon(45°)$ give a fourth equation?

$\rightarrow$ Transformation of strains helps here. Use:

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

with $\tilde{\epsilon}_{11}$ being $\epsilon(45°)$ and the base system being x_1, x_2

$$\Rightarrow \epsilon(45°) = \cos^2 45°\, \underbrace{\epsilon_{11} + \sin^2 45°\, \epsilon_{22}}_{\text{measured}} + 2\cos 45° \sin 45°\, \epsilon_{12}$$

So we get a fourth equation, but a <u>fifth</u> unknown (G_{12}), since:

$$2\epsilon_{12} = \frac{1}{G_{12}} \sigma_{12}$$

So the situation gives 4 equations in 5 unknowns.

<u>BUT</u>, one equation is uncoupled. We know σ_{12} and can find ϵ_{12} from the equation for $\epsilon(45°)$

$$\epsilon_{12} = \epsilon(45°) - \tfrac{1}{2}(\epsilon_{11} + \epsilon_{22})$$

So: $\boxed{\text{YES, can find } G_{12}}$

$$G_{12} = \frac{\sigma_{12}}{2\epsilon(45°) - \epsilon_{11} - \epsilon_{22}}$$

<u>and</u> Can determine the relative values of the other four <u>in-plane</u> engineering constants to one another, but $\boxed{\text{NO}}$ cannot determine any of the values of E_1, E_2, ν_{12}, ν_{21}

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PROBLEM #4M (25%)

In order of their importance (as best as possible), **list** the factors that contribute at lower levels of length scale to the material stiffness characterized at the macromechanical level. Clearly indicate the relative importance of each, the length scale at which they operate, and any associated limitations including the types of materials that are affected.

There are 3 basic factors and these are listed in order of importance:

1. Atomic Bonding Most basic and important is the stiffness of the atomic bond(s) within the material. Materials like metals and ceramics have relatively stiff bonds (primary: ionic, covalent, metallic), whereas other materials are governed by less stiff secondary bonds (hydrogen, Van der Waals). For example, although polymer chains have covalent bonds, the chains are linked together by van der Waal bonds making the overall behavior less stiff.
   LENGTH SCALE: On order of basic atomic distances (10^{-10} - 10^{-9} m)

2. Atomic/Molecular Structure/Arrangement Atoms and molecules set themselves in various arrangements depending on the material. Thus bond stiffness(es) act(s) in different arrangements and directions thereby affecting macromechanical stiffness. Crystalline materials (metals, ceramics) have various crystal arrangements (e.g. cubic, hexagonal) whereas polymers can have the polymer chains oriented in different directions.
   LENGTH SCALE: One to three orders above atomic distances (10^{-9} - 10^{-7} m)

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PROBLEM #4M (continued)

3. Material (Composite) Structure Overall material can be made up of different materials. For composites, fibers or particulates can be embedded in a matrix. In metals (and ceramics) grains oriented in different directions make up the overall material. The direction and nature of these materials and how they interact affect the stiffness of the overall material.

LENGTHSCALE: Four to five orders above atomic distances (up to and above microns) (10^{-6} - 10^{-5} m)