Unified Quiz: Materials & Structures

December 2, 2005

Calculators allowed

No books or notes allowed

A list of equations will be provided

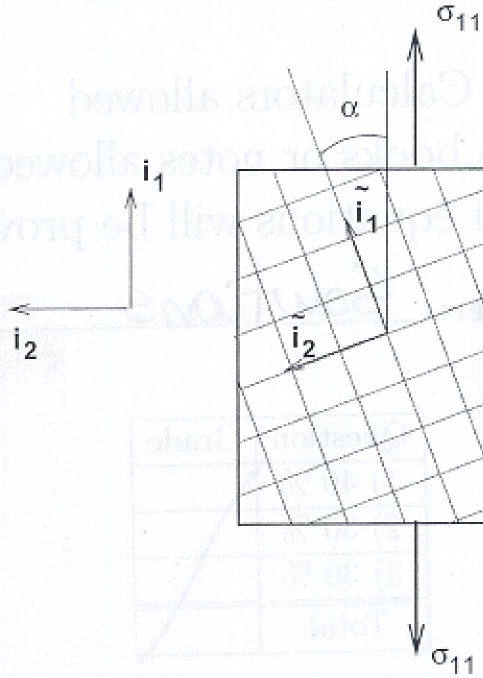
ID Number: Solutions

Question	Grade
1) 40 %	
2) 30 %	
3) 30 %	
Total:	

- Put your ID number on each page of the exam.
- Read all questions carefully.
- Do all work for each problem on the pages provided.
- Show intermediate results.
- Explain your work — don't just write equations.
- Partial credit will be given (unless otherwise noted), but only when the intermediate results and explanations are clear.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Show appropriate units with your final answers. Box your final answers.

(Questions start on the next page)

1. (40%) An orthotropic material is subjected to a uniaxial stress state as shown in the figure. The material principal axes are rotated by an angle α with respect to the system of axes on which the state of stress is given. The engineering constants of the material E_1, E_2, ν_{12} have been characterized experimentally. The goal of this problem is to explore ways of evaluating the associated state of strain.



(a) (5%) Describe in words the type of state of strain you expect to find in this problem when expressed in the same system of axes in which the state of stress is given. In particular, will the principal directions of stress and strain coincide?

A state of strain with ε_{12} and $\varepsilon_{22} \neq 0$, since the loading axes **do** not coincide with the material principal axes. the principal directions of stress are $\underline{i}_1, \underline{i}_2$ since $\sigma_{12}=0$. the principal directions of strain are not $\underline{i}_1, \underline{i}_2$ since ε_{12}. Therefore they do not coincide.

(c) (10%) Obtain the components of strain in the principal axes of the material using the results in (b). How do you justify this step?

Since the stress components in (b) correspond to the material principal axes, the constitutive equations for orthotropic materials in compliance form can be applied.

$$\tilde{\varepsilon}_{11} = \frac{1}{E_1}\left(\tilde{\sigma}_{11} - \nu_{12}\tilde{\sigma}_{22}\right) = \frac{1}{E_1}\left(\sigma_{11}\cos^2\alpha - \nu_{12}\sigma_{11}\sin^2\alpha\right)$$

$$= \frac{\sigma_{11}}{E_1}\left(\cos^2\alpha - \nu_{12}\sin^2\alpha\right)$$

$$\tilde{\varepsilon}_{22} = \frac{1}{E_2}\left(\tilde{\sigma}_{22} - \nu_{21}\tilde{\sigma}_{11}\right) = \frac{1}{E_2}\left(\sigma_{11}\sin^2\alpha - \nu_{21}\sigma_{11}\cos^2\alpha\right)$$

$$= \sigma_{11}\left(\frac{\sin^2\alpha}{E_2} - \frac{\nu_{12}}{E_1}\cos^2\alpha\right)$$, where reciprocity was used to use only known properties.

$$\tilde{\varepsilon}_{12} = \frac{\tilde{\sigma}_{12}}{2G_{12}} = \frac{-\sigma_{11}\sin 2\alpha}{4G_{12}}$$

(b) (10%) Provide the simplest possible expressions for the components of stress in the principal axes of the material

From the equations of transformation of stress components:

$$\tilde{\sigma}_{11} = \sigma_{11} \cos^2\alpha$$

$$\tilde{\sigma}_{22} = \sigma_{11} \sin^2\alpha$$

$$\tilde{\sigma}_{12} = -\frac{\sigma_{11}}{2} \sin 2\alpha$$

(d) (10%) Obtain the shear strain component ϵ_{12} in the original loading axes using the results in (c).

rotate "back", i.e., rotate by angle "$-\alpha$"
strain components $\tilde{\varepsilon}_{\alpha\beta}$ from (c)

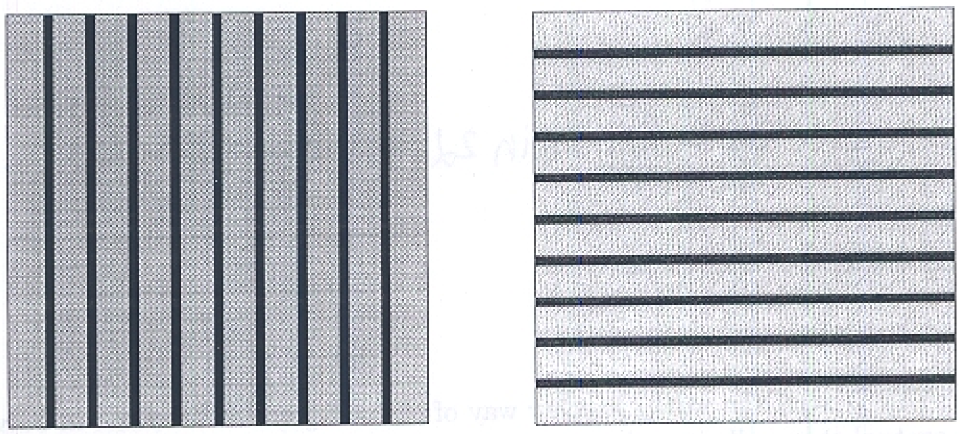
$$\varepsilon_{12} = \frac{\tilde{\varepsilon}_{22} - \tilde{\varepsilon}_{11}}{2} \sin(-2\alpha) + \tilde{\varepsilon}_{12} \cos(-2\alpha)$$

$$= \frac{\tilde{\varepsilon}_{11} - \tilde{\varepsilon}_{22}}{2} \sin 2\alpha + \tilde{\varepsilon}_{12} \cos 2\alpha$$

(e) (5%) Describe in words another way of obtaining the same result in only one conceptual step—albeit perhaps requiring lengthy algebraic manipulations–.

Rotate compliance matrix directly from material axes $\tilde{\underline{i}}_1, \tilde{\underline{i}}_2$ to loading axes $\underline{i}_1, \underline{i}_2$, which is a rather lengthy operation. Find that rotated compliance will have coefficients coupling shear stresses with extensional strains and vice versa, i.e., full matrix. This explains why a shear strain ε_{12} appears under the uniaxial stress σ_{11}.

2. (30%) A laminated composite material is made of an equal number of two different types of plies as shown in the figure. Each ply consists of unidirectional fibers embedded in a matrix. In the first ply, the elastic modulus of the fibers is $E_{f1} = 400GPa$, the elastic modulus of the matrix is $E_{m1} = 100GPa$ and the volume fraction of fibers is $\xi_{f1} = 0.25$. In the second ply, the elastic modulus of the fibers is $E_{f2} = 100GPa$, the elastic modulus of the matrix is $E_{m2} = 10GPa$ and the volume fraction of fibers is $\xi_{f1} = 0.4$. The composite is assembled by alternating layers of the two different types of plies and such that the fibers of the two types of plies are orthogonal.



Ply 1   Ply 2

(a) (15%) Describe a method to estimate the effective elastic modulus of the resulting composite in the direction of the fibers of the plies of the first type.

- consider two phases with same volume fraction = 1/2
- first phase has a modulus equal to longitudinal modulus of ply 1 (E_{L1})
- second phase has a modulus equal to transverse modulus of ply 2 (E_{T2})
- use estimate of composite modulus

$$E_c = E_{L1} f + E_{T2}(1-f) \quad \text{where } f = \frac{1}{2}$$

(long. direction ≡ ply1)

(b) (15%) Apply your method and compute this estimate.

$$E_{L1} = (400 \times 0.25 + 100 \times 0.75)\,GPa = 175\,GPa$$

$$E_{T2} = \frac{100\,GPa \times 10\,GPa}{10\,GPa \times 0.4 + 100\,GPa \times 0.6} = \frac{1000}{64}\,GPa = 15.6\,GPa$$

$$E_C = \frac{1}{2}\left(175 + \frac{1000}{64}\right) GPa$$

$$= 95.3\,GPa$$

3. (30%) In one of the homework assignments you estimated the elastic modulus of an ionically-bonded material with cubic crystal structure and whose atomic potential energy was given by:

$$U(r) = -\frac{C}{r} + \frac{D}{r^9} + U_i$$

where $C = 1.1 \times 10^{-27} Jm$ and $D = 4.5 \times 10^{-106} Jm^9$, $r_0 = 2.1 \times 10^{-10}\, m$

(a) (20%) Use this same model to estimate an upper bound of the fracture stress σ_f and strain ϵ_f for this material.

The estimate will be based on finding the maximum of the force:

$$F(r) = \frac{\partial U}{\partial r} = \frac{C}{r^2} - \frac{9D}{r^{10}}$$ as follows

$$\frac{\partial F}{\partial r} = 0 = -\frac{2C}{r^3} + \frac{90D}{r^{11}}$$, solve for r_f

$$\Rightarrow r_f^8 = \frac{90D}{2C} \qquad r_f = \sqrt[8]{\frac{45D}{C}}$$. then

$$\epsilon_f = \frac{r_f - r_0}{r_0} \qquad \sigma_f = \frac{F(r_f)}{r_0^2}$$

$$r_f = \sqrt[8]{\frac{45 \times 4.5 \times 10^{-106}\, J\, m^{\cancel{9}8}}{1.1 \times 10^{-27}\, J\, \cancel{m}}} = 2.56 \times 10^{-10}\, m$$

$$\epsilon_f = 0.22 \qquad \sigma_f = \frac{1.64 \times 10^{-8} N}{(2.1 \times 10^{-10})^2} = 373\, GPa$$

(b) (10%) Do you expect this estimate to be good? Why or why not?

The estimate is poor for several reasons:

- idealizations of bonding energy: only nearest neighbor interactions considered, phenomenological expression of energy function.
- ideal lattice arrangement assummed but real material full of different types of defects at various scales: dislocations, vacancies, stacking faults; microvoids, microcracks, inclusions, impurities, etc.
- stress concentrations produced by these defects are cause of failure.