Problem S7.1: Look-back to Lectures 9, 10, 11 (15 points)

The satellite and scientific probe for the Gravity Probe-B experiment (launched in 2004) can be modeled as shown in the schematic below. The mass $m_1$ represents the mass of the spacecraft plus helium tank. The mass $m_2$ is the probe. The coupling between the probe and spacecraft is modeled using a spring with spring constant $k$ and a dashpot with damping coefficient $c$. A force $f(t)$ is applied to the spacecraft body as shown. The displacement of the spacecraft and probe with respect to the inertial reference are respectively $d_1(t)$ and $d_2(t)$.

![Figure 1: Schematic for the Gravity Probe-B model, Question S7.1.](image)

(a) Derive the differential equations governing this system.

(b) If the input is the applied force $f(t)$ and the output of interest is the probe position $d_2(t)$, derive the state-space matrices $A$, $B$, $C$, and $D$, using state variables $d_1$, $d_1'$, $d_2$, and $d_2'$.

(c) The motion of the probe is expected to be very small, so your team lead prefers you to use the following state variables: $d_1'' = 10^6 d_1$, $d_1''', d_2'' = 10^6 d_2$, $d_2'''$. What are the state-space matrices $A'$, $B'$, $C'$, and $D'$ for this choice of state?

(d) What is the transformation matrix $T$ that relates the state in part (c) to the state in part (b), i.e.

$$\tilde{x}' = T \tilde{x}?$$

Show that the matrices you obtained in part (c) could also be obtained through the relationships derived in Lecture S11, i.e.

$$A' = TAT^{-1}, B' = TB, C' = CT^{-1}, D' = D.$$
(e) For the choice of state

\[ \tilde{x}'' = \begin{bmatrix} d_1 & \dot{d}_1 & (d_2 - d_1) & (\dot{d}_2 - \dot{d}_1) \end{bmatrix}^T, \]

give the transformation matrix \( T \) and the state-space matrices \( A'' \), \( B'' \), \( C'' \), and \( D'' \).

(f) For the values \( m_1 = 2000\text{kg}, m_2 = 1000\text{kg}, k = 3.2 \times 10^6\text{Nm}, \) and \( c = 4.6 \times 10^3\text{Nm/s}, \) compute the eigenvalues of your state dynamics matrices \( A \), \( A' \), and \( A'' \).

(g) For each eigenvalue (or each complex pair of eigenvalues) write one sentence explaining the physical meaning of the eigenvalue with reference to the satellite-probe system.
Problem S7.2: Look-ahead to Lectures S14 and S15 (15 points)

You saw in SL9 that the Quanser roll subsystem acts largely as a lightly damped second-order system. To simplify the analysis here, we will ignore the damping, and model the Quanser roll dynamics as a second-order undamped system. The state equations in state-space form therefore have the form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-w_n^2 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 
\begin{bmatrix}
0 \\
k
\end{bmatrix} u,
\]

where the states have been chosen to be the roll angle and the roll rate, i.e. $x_1 = \phi$ and $x_2 = \dot{\phi}$. The input $u(t)$ is the voltage.

We will use a typical value for the Quanser natural frequency of $\omega_n = 1 \text{ rad/s}$. The constant $k$ takes a value $k = 1$.

(a) Compute the eigenvectors, $v_1$ and $v_2$, and the eigenvalues, $\lambda_1$ and $\lambda_2$, of the state dynamics matrix $A$.

(b) One possible choice for a state transformation is to use the eigenvectors of $A$. Use the following state transformation:

\[
\begin{bmatrix}
\bar{x}_1 \\
\bar{x}_2
\end{bmatrix} = V \begin{bmatrix}
\bar{x}_1' \\
\bar{x}_2'
\end{bmatrix},
\]

where $V = [v_1 \ v_2]$ is the matrix whose columns contain the eigenvectors of $A$. (So we have chosen $V = T^{-1}$, using the notation in Lecture S11.) Compute the transformed state matrices $A'$ and $B'$. (Hint: if you first normalize your eigenvectors, then since the eigenvectors form an orthonormal set, $V^{-1}$ can be easily computed as the complex conjugate transpose of $V$.)

(c) What do you notice about the form of your transformed matrix $A'$?

(d) We wish to compute the state response of the Quanser to a step input in voltage, i.e.

\[
v(t) = \begin{cases}
0, & t < 0 \\
\bar{v}, & t > 0,
\end{cases}
\]

where $\bar{v}$ is some constant voltage value. The Quanser is initially at rest, i.e. $x_1(0) = 0$, $x_2(0) = 0$. First, compute the response of the transformed state $\bar{x}'(t)$. (Hint: you will first need to transform the initial conditions to the new state.) Then transform your resulting solution for $\bar{x}'(t)$ to obtain $\bar{x}(t)$.

(e) Verify that your solution for $\bar{x}(t)$ in part (d) satisfies the original state-space system and given initial conditions.
**Extra Practice Problems**

1. Sketch the block diagrams for your three state-space systems in Question S7.1.

2. Sketch the block diagrams for your two state-space systems in Question S7.2.

More questions will be posted over the weekend...