Uniform Engineering Problem Set
Week 7    Fall 2007

SOLUTIONS

M 7.1

for all cases recall rule for
tensorial/indicical notation:

- Latin subscript take on values 1, 2, 3
- Greek subscript take on values 1, 2

- when subscript is repeated in one term it is a "dummy index" and is summed on

- when subscript appears only once on left side of equation in one term it is a "free index" and represents separate equations

So......

(a) \( C_{mn} = R_{mjk} \delta^j_z \delta^k_z \) (for \( m=1, n=3 \))

\[ \Rightarrow C_{13} = C_{13 jk} \delta^j_z \delta^k_z \]

- \( j \) and \( k \) are dummy indices and are summed on from 1 to 3 (they are latin subscripts)
\[ C_{ij} \text{ same as: } \sum_{j=1}^{3} \sum_{k=1}^{3} C_{ijk} \delta_{j} \delta_{k} \]

Thus:
\[
C_{i3} = C_{1311} \delta_{1} \delta_{1} + C_{1312} \delta_{1} \delta_{2} + C_{1313} \delta_{1} \delta_{3} + C_{1321} \delta_{2} \delta_{1} + C_{1322} \delta_{2} \delta_{2} + C_{1323} \delta_{2} \delta_{3} + C_{1331} \delta_{3} \delta_{1} + C_{1332} \delta_{3} \delta_{2} + C_{1333} \delta_{3} \delta_{3}
\]

\[(b) \quad E = \frac{1}{2} \sum_{\alpha} \sum_{\beta} \sigma_{\alpha \beta} \varepsilon_{\alpha \beta} \]

- \(\alpha\) and \(\beta\) are dummy indices and are summed on from 1 to 3 (they are greek subscripts).

\[ \Rightarrow \text{ same as: } E = \frac{1}{2} \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \sigma_{\alpha \beta} \varepsilon_{\alpha \beta} \]

Thus:
\[
E = \frac{1}{2} \left\{ \sigma_{11} \varepsilon_{11} + \sigma_{12} \varepsilon_{12} + \sigma_{21} \varepsilon_{21} + \sigma_{33} \varepsilon_{33} \right\}
\]

\[(c) \quad H_{ij} = \sigma_{ij} \varepsilon_{ij} \]

- \(\alpha\) and \(\beta\) are repeated in one term and thus are dummy indices and are summed on from 1 to 3 (they are greek subscripts).
• i is a free index and indicates separate equation (as it is a latin subscript)

\[ H_i = \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} b_{\alpha \beta} p_{\alpha \beta} n_i \]

Thus:

\begin{align*}
(i = 1) & \quad H_1 = (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_1 \\
(i = 2) & \quad H_2 = (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_2 \\
(i = 3) & \quad H_3 = (b_{11} P_{11} + b_{12} P_{12} + b_{21} P_{21} + b_{22} P_{22}) n_3
\end{align*}

(c) \[ \sigma_{31} = l_{3m'} l_{1n'} \sigma_{m'n'} \]

• m' and n' are repeated in one term and are dummy indices are summed on from 1 to 3 (they are latin subscripts)

\[ \sigma_{31} = \sum_{m'=1}^{3} \sum_{n'=1}^{3} l_{3m'} l_{1n'} \sigma_{m'n'} \]

Thus:

\[ \sigma_{31} = l_{31'} (l_{11'} \sigma_{11'} + l_{12'} \sigma_{12'} + l_{13'} \sigma_{13'}) \]
\[ + l_{32'} (l_{11'} \sigma_{21'} + l_{12'} \sigma_{22'} + l_{13'} \sigma_{23'}) \]
\[ + l_{33'} (l_{11'} \sigma_{31'} + l_{12'} \sigma_{32'} + l_{13'} \sigma_{33'}) \]
(e) \( f_{pq} \left( \frac{\partial g_{p}}{\partial t} \right) + x_{p} = 0 \)

- \( g \) is repeated in the first term and
- it is a dummy index and is
- summed on from 1 to 3 (it is a Latin
- subscript)
- \( p \) is a free index (not repeated in terms)
  - and indicates there are 3 equations
  - (Latin subscript \( \Rightarrow 1, 2, 3 \))

\[ \Rightarrow \text{same as: } \sum_{q=1}^{3} f_{pq} \left( \frac{\partial g_{q}}{\partial t} \right) + x_{p} = 0 \]

Thus:

\[
\begin{align*}
\text{\( p=1 \):} & \quad f_{11} \frac{\partial g_{1}}{\partial t} + f_{12} \frac{\partial g_{2}}{\partial t} + f_{13} \frac{\partial g_{3}}{\partial t} + x_{1} = 0 \\
\text{\( p=2 \):} & \quad f_{21} \frac{\partial g_{1}}{\partial t} + f_{22} \frac{\partial g_{2}}{\partial t} + f_{23} \frac{\partial g_{3}}{\partial t} + x_{2} = 0 \\
\text{\( p=3 \):} & \quad f_{31} \frac{\partial g_{1}}{\partial t} + f_{32} \frac{\partial g_{2}}{\partial t} + f_{33} \frac{\partial g_{3}}{\partial t} + x_{3} = 0
\end{align*}
\]
\[ \begin{cases} M_1 \\ M_2 \\ M_3 \end{cases} = \begin{bmatrix} \alpha_{111} & \alpha_{221} & 2\alpha_{121} \\
\alpha_{112} & \alpha_{222} & 2\alpha_{122} \\
\alpha_{113} & \alpha_{223} & 2\alpha_{123} \end{bmatrix} \begin{bmatrix} T_{11} \\
T_{22} \\
T_{12} \end{bmatrix} \]

First write out in full (as it may help):

\[
M_1 = \alpha_{111} \overline{T}_{11} + \alpha_{221} \overline{T}_{22} + 2\alpha_{121} \overline{T}_{12} \\
M_2 = \alpha_{112} \overline{T}_{11} + \alpha_{222} \overline{T}_{22} + 2\alpha_{122} \overline{T}_{12} \\
M_3 = \alpha_{113} \overline{T}_{11} + \alpha_{223} \overline{T}_{22} + 2\alpha_{123} \overline{T}_{12} 
\]

Look at/consider this piece by piece:

1. The subscript on \( M \) must be a free index because it changes with the equation and represents separate equations. It must be latex since it takes on the values 1, 2, 3, ..., \( ( = M_n ) \)
(2) The subscripts on $T$ take on the values 1 and 2 and therefore must be Greek. They change independently and thus must be different.

\[ (T_\alpha \beta) \]

(3) The third subscript on $\alpha$ matches the subscript on $M$.

\[ (M_{\alpha} = \alpha?) \]

(4) The second and first subscripts on $\alpha$ match those on $T$. By making them the same, they are also summed on (as occurs in the equation)

\[ \Rightarrow \quad M_{\alpha} = \alpha_\beta \gamma \Gamma_{\alpha \beta} \]

But one must also make the assumption that $T_\alpha \beta$ is symmetric ($T_\alpha \beta = T_\beta \alpha$) and $\alpha_\beta \gamma \mu$ is symmetric in the first two indices ($\alpha_\beta \gamma \mu = \alpha_\gamma \beta \mu$) to get the factor of $2$ in the final equation on the $T_{12}$ term with $\alpha_1 \alpha_2 \mu$ as multipliers.
\[ r = 4 \hat{e}_1 - 3 \hat{e}_2 - 5 \hat{e}_3 \]

(note: unit vector)

(0) Rotation is in \( x_1-x_2 \) plane about \( x_3 \) axis by angle of \(-25^\circ\).

Draw this:

\[ + \text{CCW} \quad \phi = -25^\circ \]

with \( x_3 \) out of the paper.

\( \hat{x}_3 \) represents the rotated system:

\[ \hat{x}_3 = x_3 \]

Know that the rotation can be represented via the direction cosines:

\[ \tilde{e}_i = b_{ij} \hat{f}_j \]

Determine the direction cosines:
\[ l_{11} = \cos (\phi) = \cos (-25^\circ) = 0.906 \]
\[ l_{12} = \cos (-90 + \phi) = \cos (90 - \phi) = \sin \phi = \sin (-25^\circ) = -0.423 \]
\[ l_{13} = \cos (-90^\circ) = 0 \]
\[ l_{21} = \cos (90 + \phi) = -\sin \phi = 0.423 \]
\[ l_{22} = \cos (\phi) = \cos (-25^\circ) = 0.906 \]
\[ l_{23} = \cos (90^\circ) = 0 \]
\[ l_{31} = \cos (+90^\circ) = 0 \]
\[ l_{32} = \cos (+90^\circ) = 0 \]
\[ l_{33} = \cos (0^\circ) = 1 \]

**Writing out the rotational equation:**

\[ \tilde{r}_i = l_{ij} r_j \]

\[ \Rightarrow \]
\[ \tilde{r}_1 = l_{11} r_1 + l_{12} r_2 + l_{13} r_3 \]
\[ = (0.906)(4) + (-0.423)(-3) = 4.893 \]
\[ \tilde{r}_2 = l_{21} r_1 + l_{22} r_2 + l_{23} r_3 \]
\[ = (0.423)(4) + (0.906)(-3) = -1.026 \]
\[ \tilde{r}_3 = l_{31} r_1 + l_{32} r_2 + l_{33} r_3 \]
\[ = r_3 = 5 \]
To prove that these expressions are equivalent, one can look at the overall magnitude of the vector (along with directions).

**Step 1**  Note that the magnitude in the 3-direction is the same since there is no rotation there (that axis is not rotated about).

**Step 2**  The 2-D vector in the $x_1 - x_2$ plane creates a right triangle with the $x_2$-magnitude as one side and the $x_1$-magnitude as the other.

With this, determine the overall magnitude of the vector via:

$$|r_{1-2}| = \sqrt{(r_1)^2 + (r_2)^2} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

And the angle relative to $x_1 - x_2$:

$$\tan^{-1} \left( \frac{r_2}{r_1} \right) = \tan^{-1} \left( -\frac{3}{4} \right) = -37^\circ$$
Step 3: Do the same for the rotated $\tilde{x}_1 - \tilde{x}_2$ system

$$|\tilde{r}_{1-2}| = \sqrt{(\tilde{r}_1)^2 + (\tilde{r}_2)^2}$$

$$= \sqrt{(4.8\sqrt{3})^2 + (-1.026)^2}$$

$$= \sqrt{24.99} \quad \text{for } x = 5$$

the same

And then the angle relative to $\tilde{x}_1 - \tilde{x}_2$:

$$\tan^{-1}\left(\frac{\tilde{r}_2}{\tilde{r}_1}\right) = \tan^{-1}\left(\frac{-1.026}{4.8\sqrt{3}}\right)$$

$$= \tan^{-1}(0.210)$$

$$= -11.8^\circ$$

Now include the rotation of $-25^\circ$ from $\tilde{x}_1$:

$$\tan^{-1}\left(\frac{r_2}{r_1}\right) = -11.8^\circ + (-25^\circ) = -36.8^\circ$$

Same as $\phi = 37^\circ$

Proven

Q.E.D.