Signals and Systems
Lecture 14

Properties of Laplace Transforms
Analysis of LTI Systems

April 30, 2008

Today's Topics

1. Laplace transform properties
2. LTI system Transfer functions
3. Stability

Take Away

Most, but not all, properties of Laplace transforms are the same as for Fourier transforms.

Required Reading

O&W-9.5, 9.6, 9.7.2, 9.7.3, 9.7.4, 9.9

There are many properties of Laplace transforms that are identical to those for Fourier transforms and some that differ. Today we will explore these properties and indicate where the two kinds of transforms differ.

As we did for Fourier series and transforms, we will use a specific notation to indicate a Laplace transform pair as-

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \qquad \text{ROC} = R$$

Linearity

Linearity for Laplace transforms is similar to that for Fourier transforms, except that the ROCs must be accounted for. If

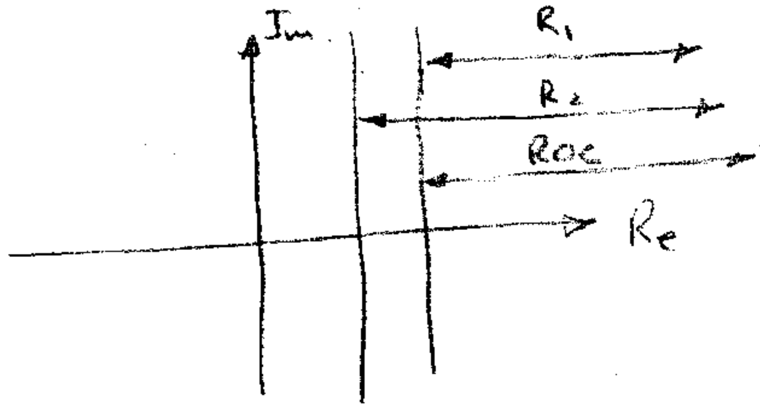
$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \qquad \text{ROC} = R_1$$

and

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \qquad \text{ROC} = R_2$$

then if there is a linear relationship in the time domain, the same relationship holds in the Laplace domain, but with additional conditions on the region of convergence

$$a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) + b X_2(s) \qquad \text{ROC contain } R_1 \cap R_2$$



The ROC for a linear combination can be different from the intersection of the two individual ROCs. For example, suppose we have the difference between $x_1(t)$ and $x_2(t)$ so

$$x(t) = x_1(t) - x_2(t)$$

and the Laplace transforms for $x_1(t)$ and $x_2(t)$ are

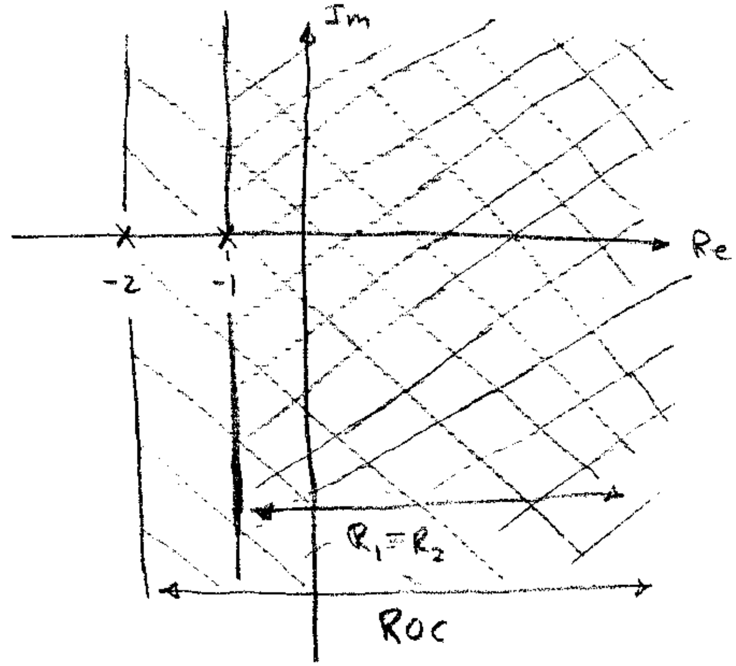
$$X_1(s) = \frac{1}{s+1} \qquad R_1 \equiv \sigma > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)} \qquad R_2 \equiv \sigma > -1$$

then the Laplace transform for difference is

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{1}{s+2} \qquad ROC \equiv \sigma > -2$$

and the ROC for *x(t)* contains the intersection of ROCs for $x_1(t)$ and $x_2(t)$.



<u>Time Shifting</u>

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

then

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s) \quad ROC = R$$

<u>Shifting in s-Domain</u>

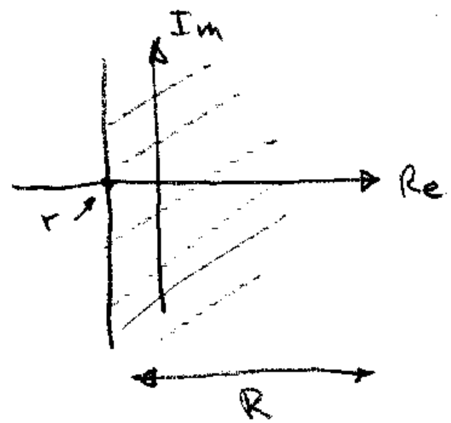
If

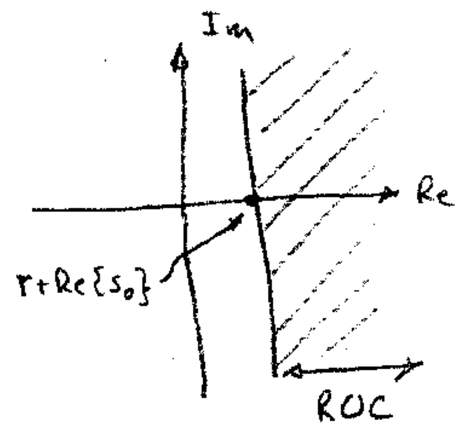
$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$$

then

$$x(t)e^{s_0 t} = X(s-s_0)$$

and the ROC associated with $X(s-s_0)$ is that of $X(s)$, but shifted by the real part of s_0. The following diagrams illustrate this shift of the ROC.





Time Scaling

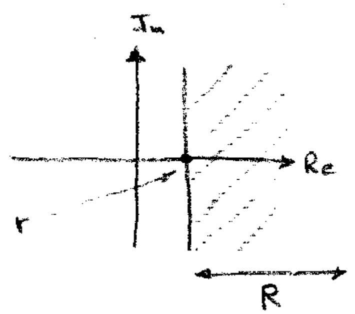
If

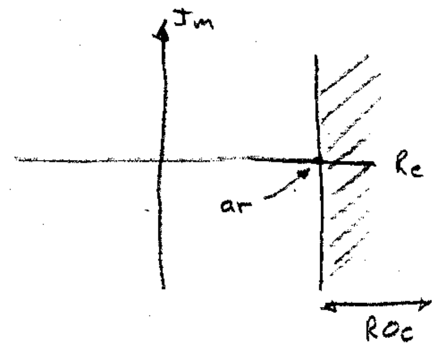
$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$$

then

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad ROC = aR$$

The effect of time scaling on the ROC is shown in the following figure





For positive values of a the ROC simply expands or contracts in proportion to a. For negative values of a the ROC is reversed and its size expands or contracts according to the magnitude of a.

<u>Conjugation</u>
If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

then

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$$

Furthermore, if $x(t)$ is real it must equal its complex conjugate,

$$x(t) = x^*(t)$$

so

$$X(s) = X^*(s^*)$$

It follows that if $x(t)$ is real then all its poles and zeros must have complex conjugates.

Convolution

If

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \qquad ROC = R_1$$

and

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \qquad ROC = R_2$$

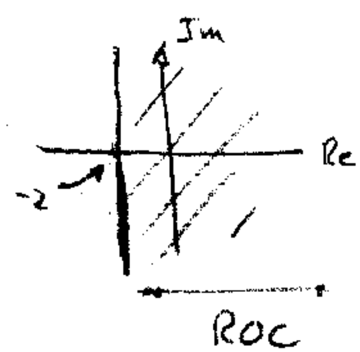
then

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \cdot X_2(s)$$

ROC will contain $R_1 \cap R_2$

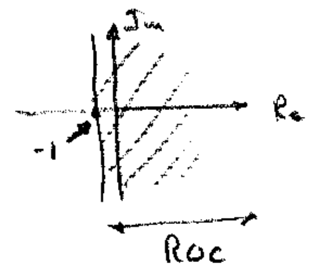
If there is pole-zero cancellation then the ROC may be larger than the intersection. For example if

$$X_1(s) = \frac{s+1}{s+2}$$



and

$$X_2(s) = \frac{s+2}{s+1}$$



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Then the zeros cancel with poles and the ROC is the entire complex plane.

$$X_1(s)X_2(s) = \frac{s+1}{s+2}\,\frac{s+2}{s+1} = 1$$

ROC = entire s-plane

<u>Differentiation in the Time Domain</u>

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

and we take the Laplace transform of the derivative of *x(t),* and integrate by parts

$$\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty x(t)e^{-st} dt$$

Thus we have

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$$

so

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s) - x(0)$$

Thus differentiation in the time domain becomes simply multiplication by *s* in the Laplace domain, along with a subtraction of the initial condition on *x(t)*. The ROC may be larger than R if there is pole/zero cancellation (e.g., *X(s)=1/s*). Recall that there was no *x(0)* term in the equivalent formula for Fourier transforms so this is an example where the properties of the two kinds of transforms differ.

Repeating this procedure yields the transform of the second derivative.

$$\frac{d^2x(t)}{dt} \xleftrightarrow{\mathcal{L}} s^2X(s) - sX(0) - \frac{dx}{dt}(0)$$

and similarly for higher derivatives.

<u>Differentiation in the s-Domain</u>

If we write the equation for the Laplace transform of *x(t)*

$$X(s) = \int_0^\infty x(t)\, e^{-st} dt$$

and differentiate with respect to *s*

$$\frac{dX(s)}{ds} = \int_0^\infty (-t)\, x(t) e^{-st} dt$$

So, if

$$X(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$$

then

$$-t\,x(t) \xleftrightarrow{\mathcal{L}} \frac{d\mathbf{X}(s)}{ds} \quad \text{ROC} = R$$

This property is useful in determining inverse transforms for cases where there are repeated poles. For example if *x(t)* is

$$x(t) = t\,e^{-at}\,u(t)$$

then since

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \underbrace{\mathrm{Re}\{s\} > -a}_{\text{ROC}}$$

so, from above

$$t\,e^{-at}\,u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$$

Repeated application of this method yields the more general formula

$$\frac{t^{n-1}}{(n-1)!}\,e^{-at}\,u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n}$$

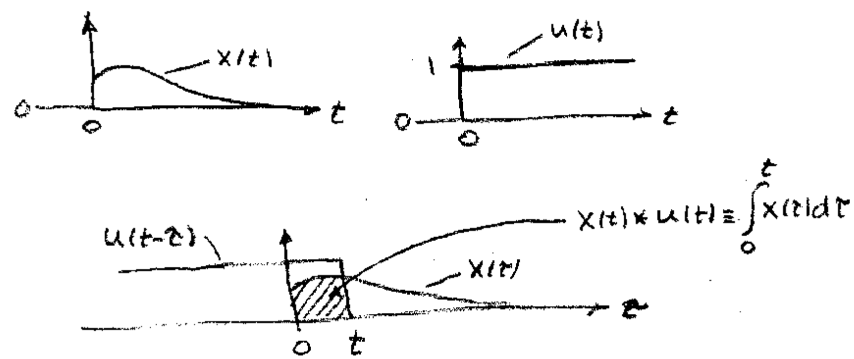
<u>Integration in the Time Domain</u>
If

$$x(t) \xleftrightarrow{\mathcal{L}} \mathbf{X}(s)$$

then, since the time integral of *x(t)* can be written as

$$\int_0^t x(\tau)\,d\tau \equiv \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau = x(t) * u(t)$$

which can be depicted as



and using the Laplace transform for *u(t)* and the property that convolution in the time domain implies multiplication in the Laplace domain

$$\int_0^t x(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} X(s)\,\mathcal{L}[u(t)] = \frac{1}{s}X(s)$$

<u>Initial and Final Value Theorems</u>

Laplace transforms can also be used to determine the initial value of a time function. If we write out the formula for the Laplace transform of the derivative of *x(t)*

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$$

Then in the region of convergence we can take the limit as *s* goes to infinity.

$$\lim_{s\to\infty} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to\infty} s\,X(s) - x(0) \qquad s \in ROC$$

and

$$\lim_{s\to\infty} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to\infty} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \int_0^\infty \frac{dx(t)}{dt} \lim_{s\to\infty} e^{-st} dt = 0$$

which yields the initial value theorem

$$x(0) = \lim_{s\to\infty} s\,X(s)$$

This equation is often useful for determining the initial value of a signal or system response.

Alternatively we can take a limit as *s* goes to zero

$$\lim_{s\to 0} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to 0} s\,X(s) - x(0)$$

and the limit on the left becomes

$$\lim_{s\to 0} \mathcal{L}\left[\frac{dx(t)}{dt}\right] = \lim_{s\to 0} \int_0^\infty \frac{dx(t)}{dt} e^{-st} = \int_0^\infty \frac{dx(t)}{dt} \lim_{s\to 0} e^{-st} dt$$

$$= \int_0^\infty \frac{dx(t)}{dt} dt = x(\infty) - x(0)$$

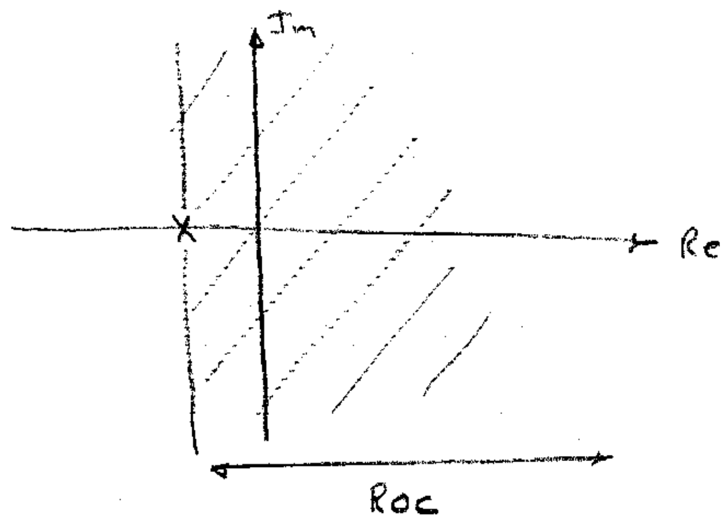
which yields the final value theorem.

Stability

The ROC of a system is useful for determining the stability of the system. As we learned last term, the stability of a system requires that its impulse response be absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Also, we know that the Laplace transform evaluated at any point $j\omega$ on the imaginary axis is the Fourier transform. Thus, if the ROC of a Laplace transform of a system impulse response includes the $j\omega$ axis, then the Fourier transform of the impulse response converges.



Furthermore, we know that for the Fourier transform of a function to converge then the function must be absolutely integrable. Hence, existence of the Fourier transform is equivalent to system stability. Whence, if the ROC of a system impulse response includes the $j\omega$ axis then the system is stable. Furthermore, we know that the ROC is always to the right of the right most pole of *H(s),* Thus, finally, a stable system must have all of its poles in the left half plane.

As an example of an unstable system consider the transfer function for the rocket that we derived last time

$$H(s) = \frac{k}{s^2 - c_1^2} \qquad k = \frac{T\ell_2}{J} \qquad c_1 = \sqrt{\frac{c_n \ell_1}{J}}$$

$$s_p = \pm c_1$$

The region of convergence, which is to the right of the right most pole at $+c_1$, does not include the $j\omega$ axis. We can readily obtain the system impulse response by doing a partial fraction expansion

$$H(s) = \frac{k}{(s+c_1)(s-c_1)} = \frac{k_1}{s+c_1} + \frac{k_2}{s-c_1}$$

$$k_1 = H(s)\cdot(s+c_1)\Big|_{s=-c_1} = \frac{k}{s-c_1}\Big|_{s=-c_1} = \frac{k}{-2c_1}$$

$$k_2 = \frac{k}{s+c_1}\Big|_{s=c_1} = \frac{k}{2c_1}$$

so

$$H(s) = \frac{\left(\frac{k}{2c_1}\right)}{s-c_1} - \frac{\left(\frac{k}{2c_1}\right)}{s+c_1}$$

Taking the inverse Laplace transform of each term obtains

$$h(t) = \frac{k}{2c_1} e^{c_1 t} - \frac{k}{2c_1} e^{-c_1 t}$$

and thus the impulse response diverges as time goes to infinity and the system is unstable.

As we stated earlier, if the system transfer function *H(s)* is rational, so it is a ratio of polynomials in *s*, then the stability criterion requires that all poles of *H(s)* must be strictly to the left of the $j\omega$ axis.