Signals and Systems
Lecture (S1)

Response of LTI Systems to Complex Exponentials

March 13, 2008

Today's Topics

1. Response of LTI systems to complex exponentials
2. Representing sinusoids as complex exponentials
3. Examples

Take Away

A sinusoidal input to a stable LTI system produces a sinusoid response at the input frequency. Both the amplitude and phase of the input sinusoid are modified by the LTI system to produce the output. The input frequency completely determines how the amplitude and phase are modified.

Required Reading

O&W-3.0, 3.1, 3.2

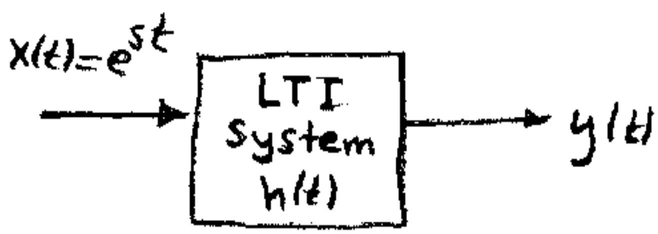
LTI System Responses

Complex exponentials are an extremely useful class of functions for representing signals in LTI systems. This utility stems, in part, from-

1. a very wide class of real world signals can be represented, to virtually any desired level of accuracy, by complex exponentials.

2. the responses of LTI systems to this broad class of signals can be represented and analyzed quite effectively using complex exponentials.

Of particular significance is the fact that the response of an LTI system, to a complex exponential, is that same complex exponential multiplied by another complex exponential.



$x(t) = e^{st}$ = complex exponential input

s = complex variable = $\sigma + j\omega$

$y(t)$ = output

$h(t)$ = LTI system impulse response

where the output is obtained by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

Now, the exponential of the difference can be written as a product, so

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

and we define

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

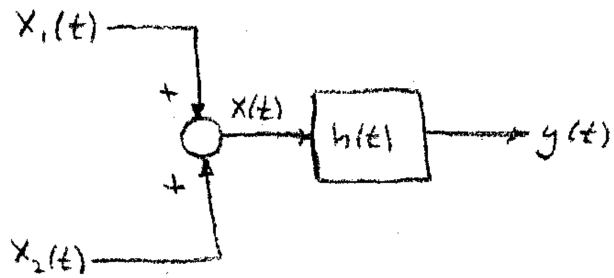
This integral is the transfer function of the LTI system and it will play a very important role in what we will be doing in future lectures. Note, in particular, that the transfer function is a function of the complex variable *s* and is not a function of time. Also, for the time being, we tacitly assume that the infinite integral converges. This question will be important later in our work.

Substituting back into the equation for the output we obtain a simple expression for the output in terms of the input

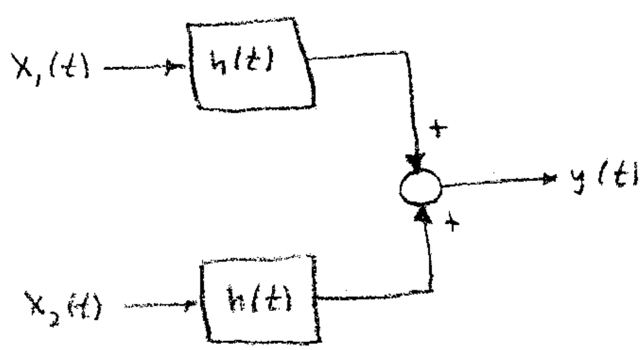
$$y(t) = H(s)\, e^{st}$$

Such signals, for which the system output is simply a multiplication of the input by another complex variable, are called eigenfunctions and the multiplicative factor is the called the eigenvalue. Thus complex exponentials are eigenfunctions of LTI systems.

We will also draw on your LTI systems work last term using the principle of superposition. Recall that if the input to a system is made up of the sum of two signals, then the total output is the sum of the outputs that result from the system operating on each of the two inputs separately.
In particular, the system



is equivalent to the system



Hence if the input signals are complex exponentials, so

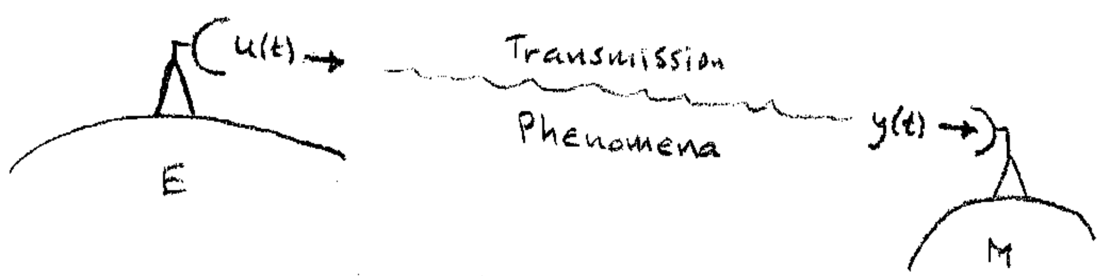
$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

where C_1 and C_2 are constants. Then

$$y(t) = H(s_1) C_1 e^{s_1 t} + H(s_2) C_2 e^{s_2 t}$$

We now have enough tools to do an aerospace problem.

Example 1
We wish to send a signal from our Earth based communications facility to a lunar lander on the surface of the Moon.



The transmitted signal is a tone (sinusoid) that we designate as *u(t),* so

$$u(t) = A \sin \omega_0 t$$

The signal *u(t)* is transmitted from Earth and a signal *y(t)* is received at the Moon.

In order to analyze the effects of transmission we will use a complex exponential representation of *u(t)*. We can write Euler's Equation as

$$e^{j\theta} = \cos\theta + j\sin\theta$$

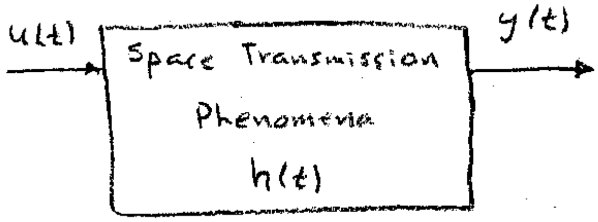
and if we define

$$s_1 = +j\omega_0 \qquad s_2 = -j\omega_0$$

then *u(t)* can be written as the difference of two complex exponentials

$$u(t) = A\left(\frac{e^{s_1 t} - e^{s_2 t}}{2j}\right)$$

Also, the transmission phenomena can be modeled as a LTI system



The transmission of the signal through space, from Earth to Moon, has two effects. The signal is attenuated to a very significant amount and it is also delayed in time. Hence the impulse response of the transmission is

$$h(t) = k\,\delta(t - T)$$

where *k* is an attenuation factor that models the loss of signal strength and *T* is the time delay incurred, as the signal travels at the speed of light from Earth to Moon

$$T > 0 \qquad 0 < k \ll 1$$

To determine the output *y(t)* we need the system transfer function. From above the system transfer function is

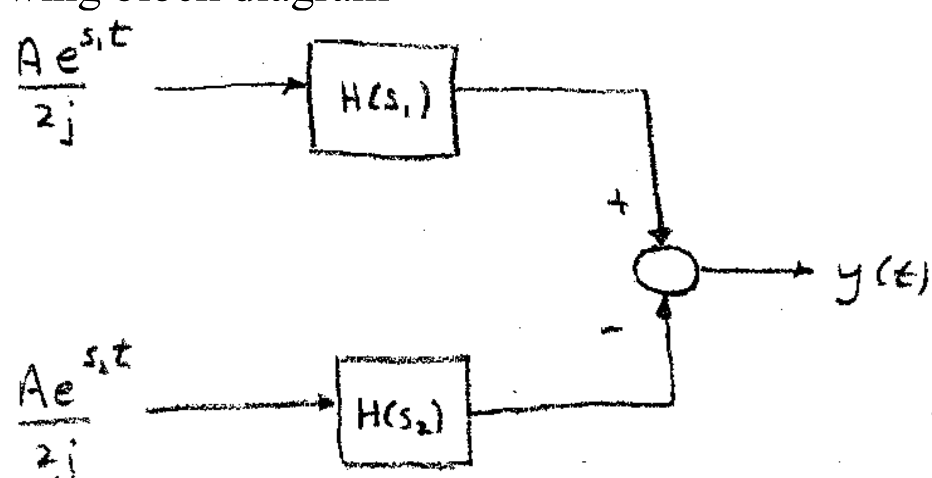
$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau = \int_{-\infty}^{\infty} k\, \delta(\tau - T)\, e^{-s\tau}\, d\tau$$

$$= k e^{-sT}$$

Also, we know that the sinusoidal input is the difference of two complex exponentials. From above

$$u(t) = \frac{Ae^{s_1 t}}{2j} - \frac{Ae^{s_2 t}}{2j}$$

and using superposition once again we can draw the following block diagram



Each of the two input branches is a complex exponential input to the system transfer function so

$$y(t) = \frac{Ae^{s_1 t}}{2j} \cdot H(s_1) - \frac{Ae^{s_2 t}}{2j} H(s_2)$$

$$= \frac{Ae^{s_1 t}}{2j} \cdot ke^{-s_1 T} - \frac{Ae^{s_2 t}}{2j} ke^{-s_2 T}$$

$$= kA\left[\frac{e^{j\omega_0 t} e^{-j\omega_0 T} - e^{-j\omega_0 t} e^{j\omega_0 T}}{2j}\right]$$

$$= kA\left[\frac{e^{j\omega_0 (t-T)} - e^{-j\omega_0 (t-T)}}{2j}\right]$$

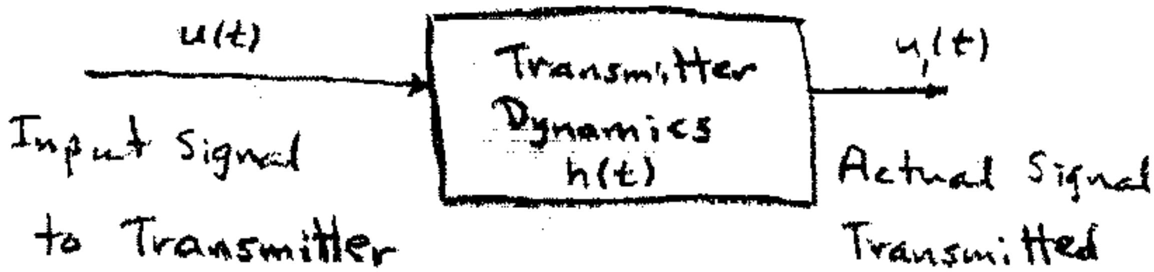
$$= \underset{\text{Attenuation}}{kA} \sin \underset{\substack{\text{Transmitted}\\ \text{Frequency}}}{\omega_0} (t - \underset{\substack{\text{Time}\\ \text{Delay}}}{T})$$

Thus our complex exponential analysis of this simple problem yields he correct result, namely that the transmitted signal is attenuated by the factor *k* and delayed by the time *T*.

This example was kind of a long process to realize an answer that was rather obvious from the beginning. Now let's do a problem for which the complex analysis is more useful.

Example 2

The transmission system for our previous problem was assumed to transmit the desired sinusoidal signal precisely. However, typically there are significant dynamic effects in the electronics of such a transmitter. Thus, we are interested in what happens if we transmit signals of various frequencies. The transmitter is modeled as a LTI system with input *u(t)* and output *y(t)*. Of course the transmitter output is the input to our previous LTI system. Hence we now have the following block diagram-



where now the LTI transmitter system is a first order causal system

$$h(t) = a e^{-at} \quad t > 0 \qquad a > 0$$
$$= 0 \quad t < 0$$

and the coefficient *a* is the inverse time constant of the transmitter. We can find the system transfer function as

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = \int_{0}^{\infty} a e^{-a\tau} e^{-s\tau} d\tau$$

$$= a \int_{0}^{\infty} e^{-(s+a)\tau} d\tau = \frac{-a}{(s+a)} e^{-(s+a)\tau} \Big|_{0}^{\infty}$$

$$= \frac{a}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

Now recall that the desired signal that we wish to transmit is the sinusoid

$$u(t) = A \sin \omega_o t$$

which has the complex exponential representation

$$u(t) = A \left(\frac{e^{s_1 t} - e^{s_2 t}}{2j} \right) \qquad \begin{array}{l} s_1 = j\omega_o \\ s_2 = -j\omega_o \end{array}$$

Also, if we apply superposition, once again the transmitted signal is-

$$u_1(t) = \frac{Ae^{s_1 t}}{2j} H(s_1) - \frac{Ae^{s_2 t}}{2j} H(s_2)$$

$$= \frac{Ae^{s_1 t}}{2j} \cdot \left(\frac{a}{s_1+a}\right) - \frac{Ae^{s_2 t}}{2j} \left(\frac{a}{s_2+a}\right)$$

$$= \frac{aA\, e^{j\omega_o t}}{2j(j\omega_o + a)} - \frac{aA\, e^{-j\omega_o t}}{2j(-j\omega_o + a)}$$

$$= \frac{aA}{2j} \left[\frac{(-j\omega_o + a) e^{j\omega_o t} - (j\omega_o + a) e^{-j\omega_o t}}{(a^2 + \omega_o^2)} \right]$$

The terms in the numerator can be put into polar form as

$$j\omega_o + a = \sqrt{\omega_o^2 + a^2}\; e^{j\phi} \qquad \phi = \tan^{-1}\left(\frac{\omega_o}{a}\right)$$

$$-j\omega_o + a = \sqrt{\omega_o^2 + a^2}\; e^{-j\phi}$$

so

$$u_1(t) = A \frac{a}{\sqrt{\omega_0^2+a^2}} \left[\frac{e^{-j\phi} e^{j\omega_0 t} - e^{j\phi} e^{-j\omega_0 t}}{2j} \right]$$

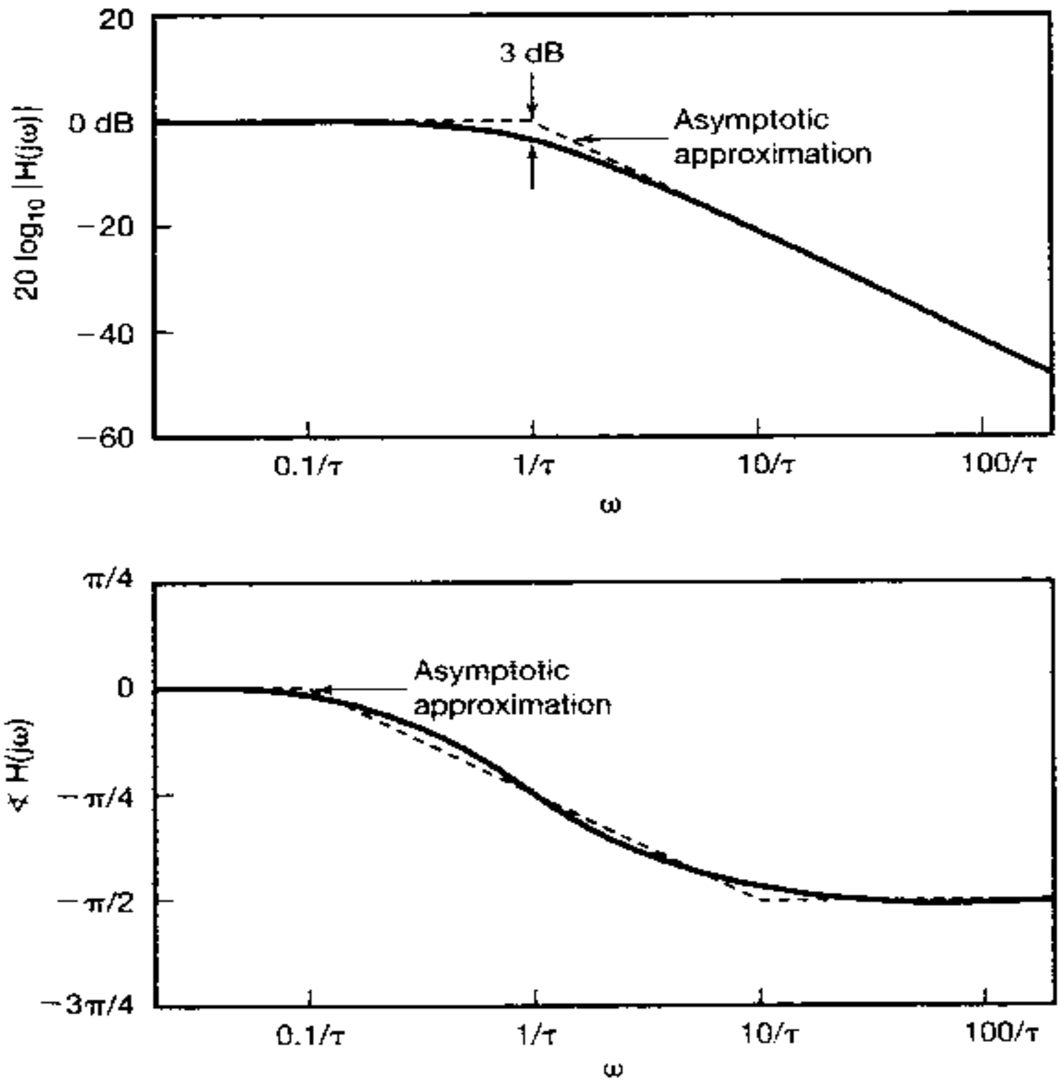
$$= A \frac{a}{\sqrt{\omega_0^2+a^2}} \left[\frac{e^{j(\omega_0 t-\phi)} - e^{-j(\omega_0 t-\phi)}}{2j} \right]$$

$$= A \frac{a}{\sqrt{\omega_0^2+a^2}} \sin(\omega_0 t - \phi)$$

Hence the transmitter still transmits a sine function at the desired frequency ω_0 but its amplitude is reduced by an input/output amplitude ratio factor *(M)* and the phase of the signal lags by the phase angle (ϕ).

$$M(\omega_0) = \frac{a}{\sqrt{\omega_0^2+a^2}} \qquad \phi = \mathrm{Tan}^{-1}\left(\frac{\omega_0}{a}\right)$$

Note in particular that both *M* and ϕ are functions of the input frequency ω_0. These functions are plotted as Fig. 6.20 on page 499 in the text, as follows



where $\tau = 1/a$ is the system time constant.

As can be seen in the diagram, for frequencies well below $a = 1/\tau$ the transmitter passes the signal virtually unchanged but for frequencies much higher than a the signal is attenuated and there is also a phase lag. The amplitude reduction goes inversely with frequency while the phase lag approaches $-\pi/2$ asymptotically.

It is important to point out that we have tacitly made an important assumption in this development. In fact we have only obtained the steady state solution to our problem. By setting $s = \pm j\omega_0$ in the transfer function we have ignored the real part of the complex variable s. In doing so we have eliminated the transient (homogeneous) part of the solution. When we do this we are ignoring any startup process. Physically this means that the stable system has been operating long enough with the sinusoidal input so that all effects of the startup process have disappeared.

In general, when we do this substitution into the system transfer function, we obtain the <u>system frequency response</u>.

$$H(s)\Big|_{s=j\omega} = H(j\omega) = \text{Frequency Response}$$

$$|H(j\omega)| = M(\omega)$$

$$\measuredangle H(j\omega) = \phi(\omega)$$