Signals and Systems
Lecture (S2)

Orthogonal Functions
and
Fourier Series

March 17, 2008

Today's Topics

1. Analogy between functions of time and vectors
2. Fourier series

Take Away

Periodic complex exponentials have properties analogous to vectors in n dimensional spaces. Periodic signals can be represented as a sum of sinusoidal functions.

Required Reading

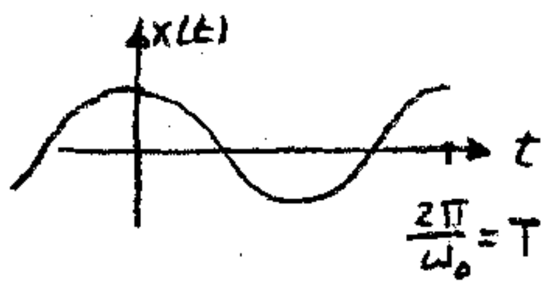
O&W-3.3.1, 3.3.2 (through Example 3.4)

Periodic Functions

In our previous lecture we saw how sinusoidal functions can be usefully represented as complex exponentials. Today we will first show that complex exponentials behave, in many ways, like vectors in a linear space. We will than show how a broad class of signals can be represented by sums of complex exponentials. Then we will be able to apply many powerful tools, all of which are developed from complex exponentials, to the analysis and design of systems.

We begin by defining periodic signals that repeat themselves at some period *T*. One example is the cosine function

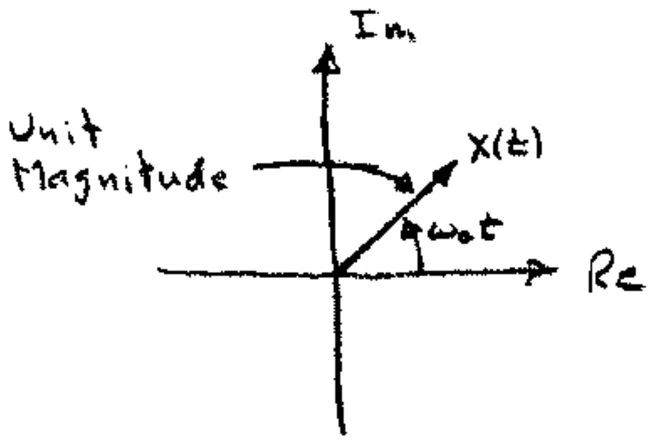
$$X(t) = \cos \omega_0 t$$



which repeats itself at multiples of $T = 2\pi/\omega_0$.

Another is the complex exponential

$$x(t) = e^{j\omega_0 t}$$



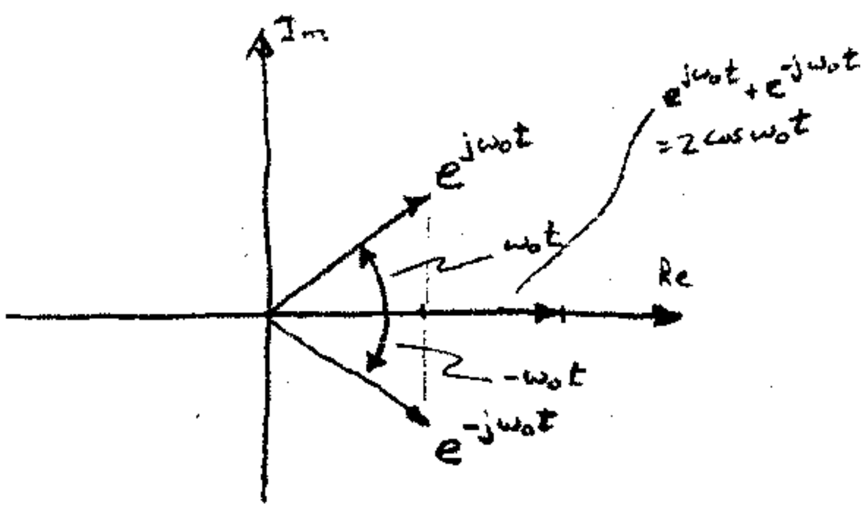
and using Euler's Equation

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

we can readily obtain two expressions for the <u>real</u> periodic function $\cos\omega_0 t$ in terms of complex exponentials

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

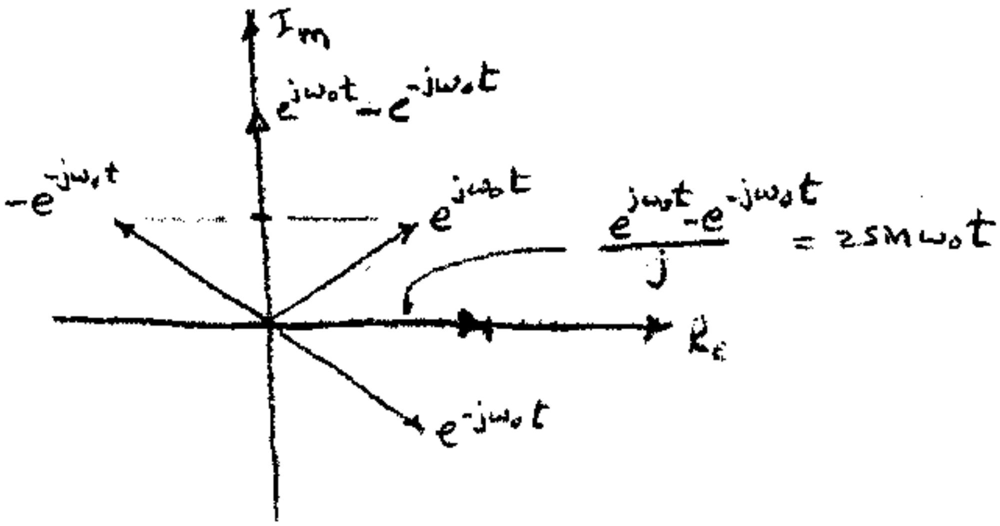
$$= \mathrm{Re}\{e^{j\omega_0 t}\}$$



Similarly, we have

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$= \mathrm{Im}\{e^{j\omega_0 t}\}$$



Thus sines, cosines, and complex exponentials are readily interchangeable in terms of their representations.

Because it will turn out to be so useful we are going to concentrate on the complex exponential. In particular, we will consider the set of harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} \qquad k = 0, \pm 1, \pm 2, \ldots$$

which can be represented as an infinite set of vectors in the complex plain, rotating at angular velocities that are integer multiples of $\omega_0 = \frac{2\pi}{T}$.

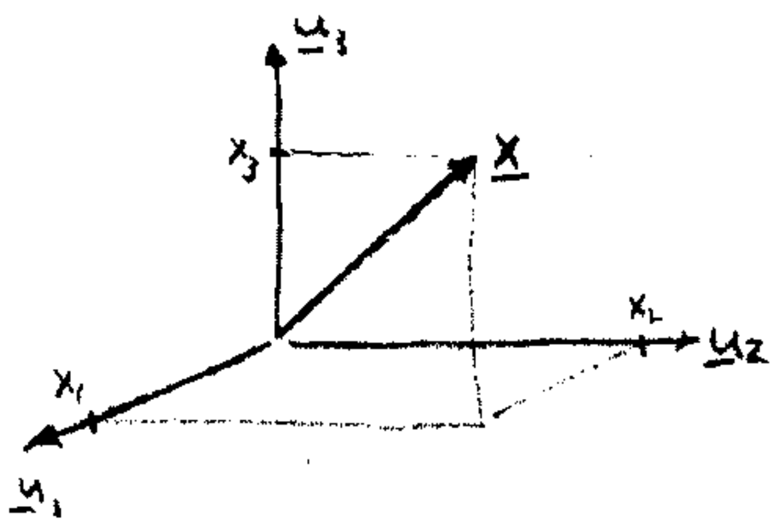
This picture of vectors rotating in the two dimensional space at harmonic frequencies, so that they all pass through the point +1 simultaneously, is quite useful. However, there is an even more important picture and analogy that we will also explore.

Orthogonal Vectors and Functions

It turns out that the harmonically related complex exponential functions have an important set of properties that are analogous to the properties of vectors in an *n* dimensional Euclidian space. Consider a linear vector space of dimension *n*, with othonormal basis vectors $\underline{u}_k$, so

$$\underline{u}_i \cdot \underline{u}_i = 1 \qquad \underline{u}_i \cdot \underline{u}_j = 0 \qquad i \neq j$$

and if $n=3$ we have the following picture.



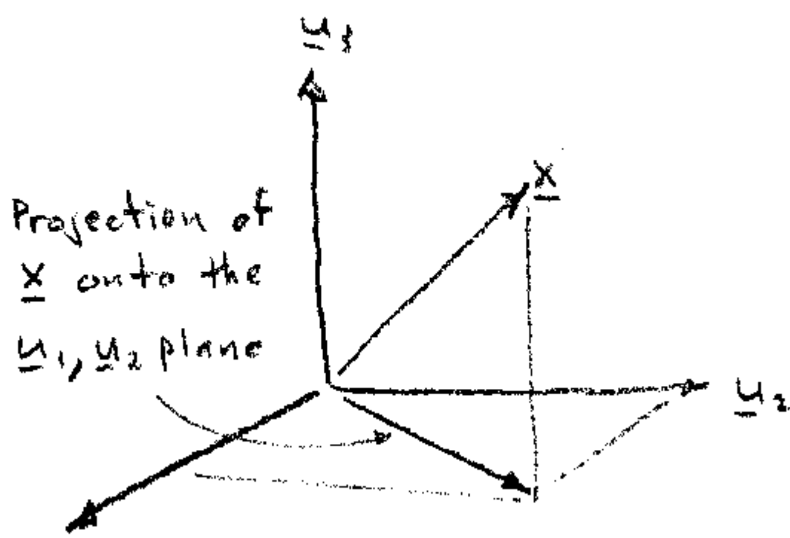
Then any vector $\underline{x}$ in the space can be represented in terms of its components as

$$\underline{x} = \sum_{i=1}^{n} x_i \underline{u}_i$$

and the components of $\underline{x}$, which are projections onto the basis vectors, are obtained using inner (dot) products

$$x_i = \underline{x} \cdot \underline{u}_i$$

Also, if we wish to represent a vector in a space of lower dimension then its projection onto the lower dimensional space is the vector of smallest distance from the original vector. For example, the projection of the 3D vector depicted above onto a 2D space is the closest vector in the 2D space to the original 3D vector.



The harmonic complex exponentials have analogous properties to basis vectors if we define the inner product operation as an integral over the period *T*. In particular, we define the inner product operation on two harmonic complex exponentials as

$$\phi_k(t) \cdot \phi_\ell(t) = \frac{1}{T}\int_0^T \phi_k(t)\, \phi_\ell^*(t)\, dt$$

where the * symbol means complex conjugate.
With this definition the harmonic complex exponentials have properties analogous to orthogonal basis vectors. In particular for $l = k$

$$\phi_k(t) \cdot \phi_k(t) = \frac{1}{T}\int_0^T e^{jk\omega_0 t} e^{-jk\omega_0 t} dt = \frac{1}{T}\int_0^T dt = 1$$

and for $l \neq k$

$$\phi_k(t) \cdot \phi_l(t) = \frac{1}{T}\int_0^T e^{jk\omega_o t} e^{-jl\omega_o t} dt = \frac{1}{T}\int_0^T e^{j(k-l)\omega_o t} dt$$

$$= \frac{1}{T}\left[\frac{e^{j(k-l)\omega_o t}}{j(k-l)\omega_o}\right]_0^T = \frac{1}{T}\left[\frac{e^{j(k-l)\cdot 2\pi} - 1}{j(k-l)\omega_o}\right] = 0$$

In fact these kinds of functions are given the name orthogonal basis functions. Furthermore, because of the periodic nature of these basis functions, it is straightforward to show that the integral can be taken over any period of length T so we write the inner product as

$$\phi_k(t) \cdot \phi_l(t) = \frac{1}{T}\int_T e^{jk\omega_o t} e^{-jl\omega_o t} dt = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

any interval T

Fourier Series

With this analogy in mind we define the real valued, periodic, time function *x(t)* as a sum of the basis functions times complex coefficients

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

which is periodic at intervals *T*. This sum is given the name Fourier series, in recognition of the marvelous French mathematician Jean Baptiste Joseph Fourier. Fourier did not, in fact, invent this series but he was the first person to recognize its usefulness and versatility.

The exponent of each term is an integer multiple of the frequency $\omega_0 = 2\pi / T$. This frequency is called the <u>fundamental frequency</u> of the series and the integer multiples are called <u>harmonic frequencies</u>.

Since we defined *x(t)* as a real function of time it will not have an imaginary part so it must be equal to its complex conjugate

$$x(t) = x^*(t)$$

and hence we must have

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{l=\infty}^{-\infty} a_{-l}^* e^{jl\omega_0 t} = \sum_{l=-\infty}^{\infty} a_{-l}^* e^{jk\omega_0 t}$$

$$l = -k$$

Since this equation must hold for all time

$$a_k = a_{-k}^* \quad \text{or} \quad a_{-k} = a_k^*$$

so a_k and a_{-k} are complex conjugates and if we know a_k then we immediately know a_{-k}.

Example 1

Suppose *x(t)* is a real valued function of time and has the following Fourier coefficient for *k=2*

$$a_2 = 2 + 3j$$

then

$$a_{-2} = a_2^* = 2 - 3j$$

If, in addition, all of the other a_k are zero and *T=1* then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = a_2 e^{j2\left(\frac{2\pi}{T}\right)t} + a_{-2} e^{-j2\left(\frac{2\pi}{T}\right)t}$$

$$= (2+3j)e^{j4\pi t} + (2-3j)e^{-j4\pi t}$$

$$= 2\left(e^{j4\pi t} + e^{-j4\pi t}\right) + 3j\left(e^{j4\pi t} - e^{-j4\pi t}\right)$$

$$= 4\cos 4\pi t - 6\sin 4\pi t$$

Example 2
Suppose the function *x(t)* is

$$x(t) = 2\sin t + 4\cos 3t$$

then using Euler's Equation

$$x(t) = 2\left(\frac{e^{jt} - e^{-jt}}{2j}\right) + 4\left(\frac{e^{j3t} + e^{-j3t}}{2}\right)$$

$$= \frac{1}{j}\left(e^{jt} - e^{-jt}\right) + 2\left(e^{j3t} + e^{-j3t}\right)$$

The Fourier coefficients are readily determined as

$$a_1 = \frac{1}{j} \qquad a_{-1} = a_1^* = -\frac{1}{j} \qquad a_3 = 2 \qquad a_{-3} = a_3^* = 2$$

where all other coefficients are zero and the Fourier series is

$$x(t) = 2e^{-j3t} - \frac{1}{j}e^{-jt} + \frac{1}{j}e^{jt} + 2e^{j3t}$$

Some more common forms of Fourier series, <u>for real functions of time</u>, can be obtained by rewriting the sum using the identity

$$a_k = a_{-k}^*$$

so

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

$$= a_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}$$

Then writing the complex variable a_k as the sum of real and imaginary parts

$$a_k = B_k + j\,C_k$$

and substituting above

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}\left\{(B_k + jC_k)\, e^{jk\omega_0 t}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}\left\{(B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega_0 t)\right\}$$

$$= a_0 + 2\sum_{c=1}^{\infty} (B_k \cos k\omega_0 t - C_k \sin k\omega_0 t)$$

If the B_k are all zero then we get the sine form of Fourier series and if the C_k are all zero we get the cosine form. If *x(t)* is an even function of *t,* so that *x(-t)=x(t)* then only the cosine form of the Fourier series is possible because any sine terms will produce odd functions in the Fourier sum. Conversely, if *x(t)* is an odd function of *t*, so that *x(-t)=-x(t),* then only the sine form of the Fourier series is possible

The importance of the Fourier series is that it can represent a very broad class of functions as <u>sums of sinusoidal functions</u>.

Fourier Series Representations of Continuous Periodic Functions

Suppose *x(t)* is a continuous signal with period T and we <u>assume</u> it can be represented as a Fourier series, so

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = 2\pi / T$. Now take the inner product with another complex exponential, which is any harmonic of the fundamental frequency, and apply the orthogonal properties of the basis functions $\phi_k(t)$

$$x(t)\cdot\phi_n(t)=\frac{1}{T}\int_0^T x(t)\,\phi_n^*(t)\,dt$$

$$=\frac{1}{T}\int_0^T\left(\sum_{k=-\infty}^{\infty}a_k\,\phi_k(t)\right)\phi_n^*(t)\,dt$$

$$=\sum_{k=-\infty}^{\infty}a_k\cdot\frac{1}{T}\int_0^T\phi_k(t)\,\phi_n^*(t)\,dt=a_n$$

so for each value of *n* we have

$$a_n=\frac{1}{T}\int_0^T x(t)\,\phi_n^*(t)\,dt=\frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt$$

Furthermore, the integral can be taken over <u>any</u> interval of length *T* so we write.

$$a_n=\frac{1}{T}\int_T x(t)\,e^{-jn\omega_0 t}\,dt$$

By analogy with vector space methods the coefficient a_k is analogous to the projection of the function *x(t)* onto the basis function $\phi_k(t)$.

Thus we have a method to determine the coefficients of the Fourier series for the function *x(t).* More importantly, as we will see, we have a way to find the coefficients of Fourier series representations of a broad class of signals *x(t).*

In summary, a real periodic function of time *x(t)* , of period *T,* can be represented as a sum of complex exponentials as follows

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} = \frac{1}{T}\int_T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt$$