Signals and Systems Lecture 9

Communication Systems Frequency-Division Multiplexing and Frequency Modulation (FM)

April 11, 2008

Today's Topics

1. Frequency-division multiplexing
2. Frequency modulation of signals
3. Phase-lock loops

Take Away

Many signals can be transmitted simultaneously on carriers at different frequencies. Frequency modulation is an alternative way to encode signals onto a carrier signal.

Required Reading O&W-8.3, 8.7

Frequency-Division Multiplexing

Often it is desirable to transmit many signals over a single communications channel using a wider band signal. Microwave transmission of telephone communications over long distances is the most common example of this approach. In this way the capital investment in communications hardware is reduced because many simultaneous signals are transmitted and received using the same complement of hardware assets. The following diagram depicts the frequency-division multiplexing approach to simultaneous, wideband transmission of multiple signals.

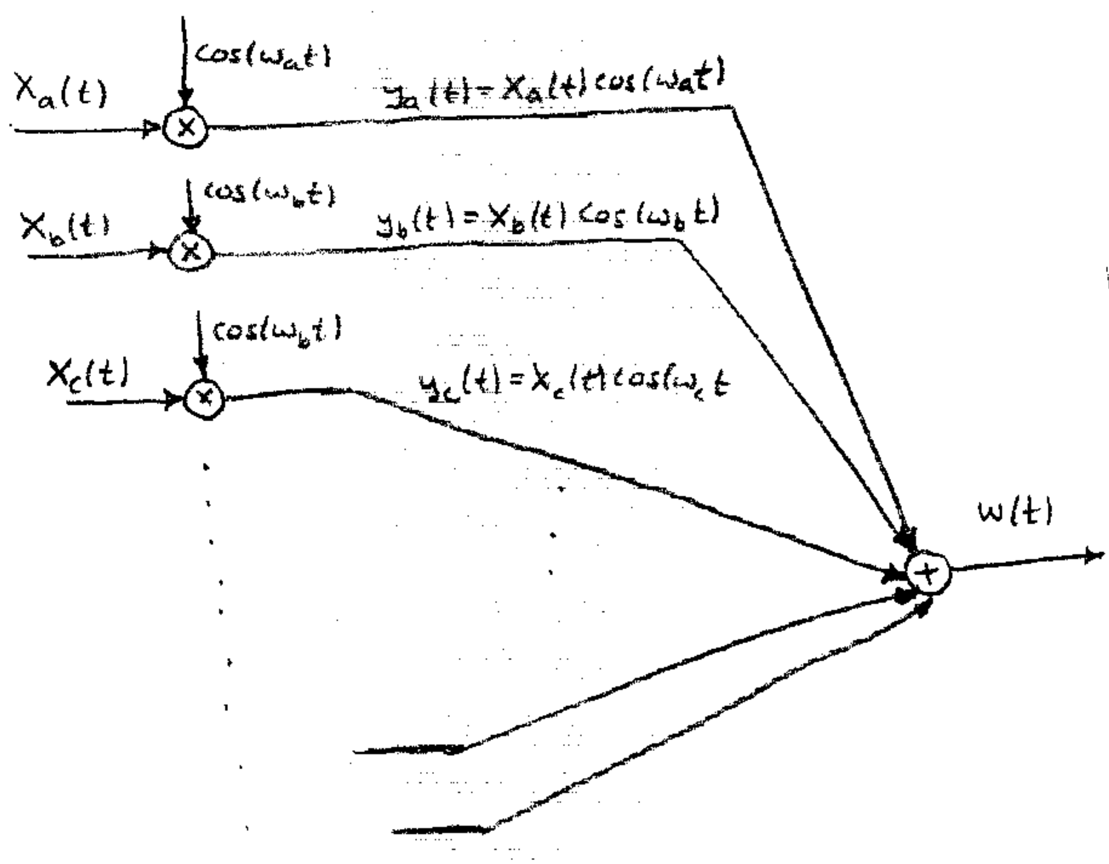
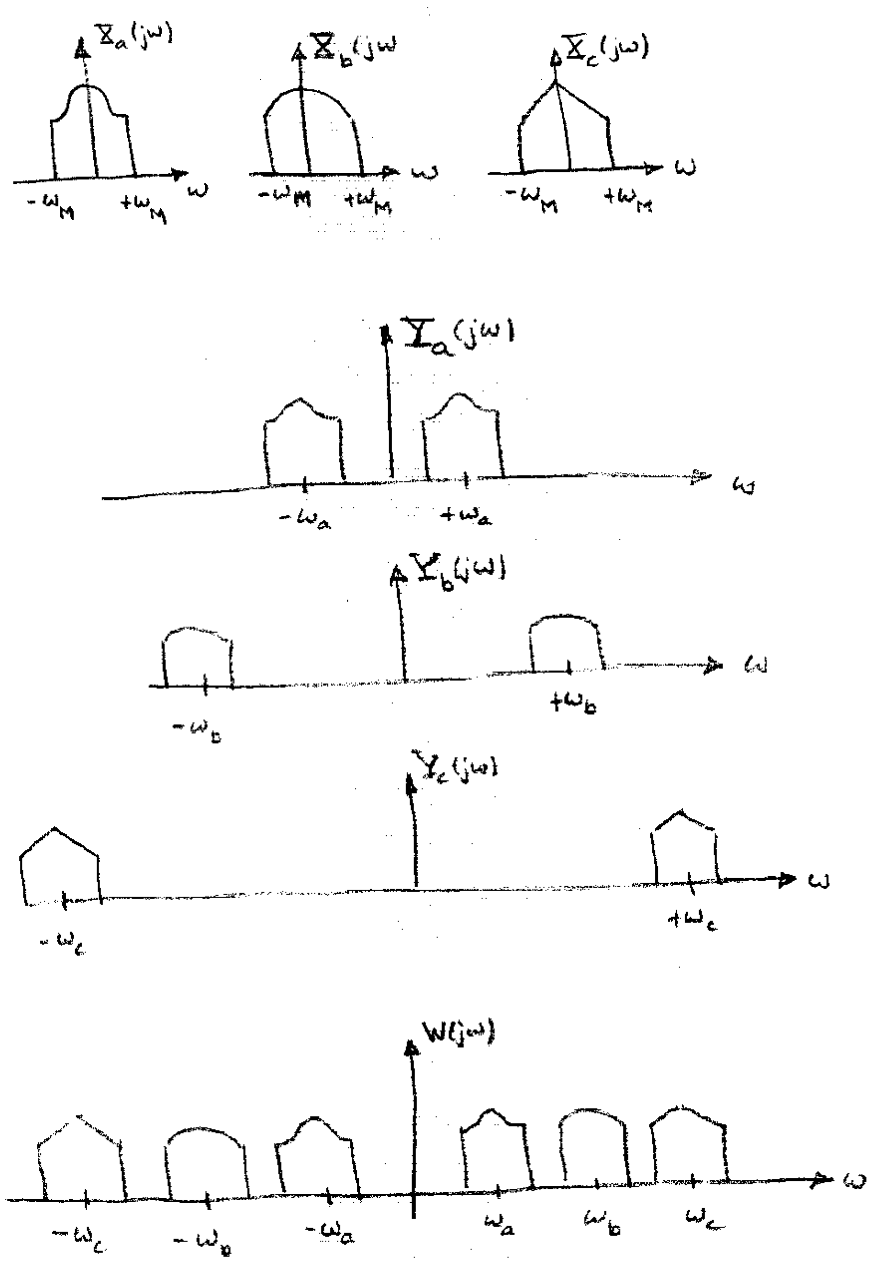
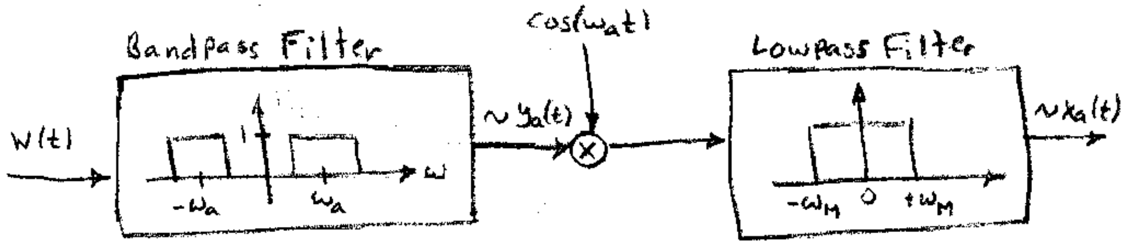


Fig.1

The signals to be transmitted, $x_a(t), x_b(t), ...$ are all band limited in the range $\pm\omega_M$. Each of the signals $x_a(t), x_b(t), ...$ modulates a carrier at its own carrier frequency. For example the carrier $\cos\omega_a(t)$ is amplitude modulated by the signal $x_a(t)$, etc. All of the modulated carriers are then summed to produce the total signal to be transmitted *w(t)*. The following figure illustrates how multiple signals are modulated and summed to produce the transmitted signal *w(t)*.



The transmitted signal is then received and processed to reproduce each of the individual transmitted signals. For example, as illustrated in the following figure, in order to obtain the signal $x_a(t)$ the received signal is bandpass filtered at the carrier ω_a. The result is then synchronously demodulated by multiplying by its carrier and then lowpass filtered to produce a replica of the original signal $x_a(t)$.



In similar fashion the other signals $x_b(t), x_c(t), ...$ can be recovered by bandpass filtering at a single carrier frequency, synchronously demodulating and then lowpass filtering. Thus, by appropriate signal processing of the transmitted and received signals, a multiple array of signals can be simultaneously transmitted by a signal transmitter/receiver pair. This is also the manner in which a radio receiver can be tuned to decode an individual signal from a single radio station, in the presence of many signals from other stations.

Frequency Modulation

Instead of modulating a signal by multiplication, as in amplitude modulation, a carrier signal can also be modulated by altering its frequency. By his method a signal *x(t)* modulates a carrier by altering the carrier frequency in a manner that is proportional to the signal *x(t)*. The instantaneous frequency at time *t* is

$$\omega(t) = \omega_c + k_f x(t)$$

where ω_c is the carrier frequency and k_f is a constant multiplier or gain constant that determines the magnitude of the frequency modulation.

This approach is best understood if we consider phase modulation, which is equivalent to the integral of frequency modulation.

Thus, if *x(t)* is to be transmitted on a carrier with carrier frequency ω_c, then the modulated signal *y(t)* will be

$$y(t) = \cos(\omega_c t + \theta(t))$$

and the phase angle $\theta(t)$ is proportional to the time integral of *x(t)*

$$\theta(t) = \theta_0 + k_f \int_0^t x(\tau)\,d\tau$$

Now suppose, for purposes of understanding, that the signal *x(t)* is a pure tone

$$x(t) = A\cos(\omega_m t)$$

where *A* is the amplitude of the tone and ω_m is its frequency. Then the phase angle is

$$\theta(t) = \theta_0 + k_f \int_0^t A\cos(\omega_m t)\,dt$$

$$= \theta_0 + \frac{k_f A}{\omega_m} \sin(\omega_m t)$$

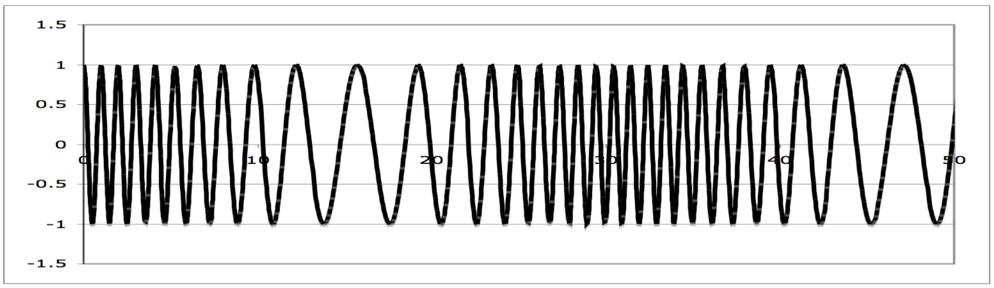
There is no need for a phase bias so we choose the initial condition to be zero and the phase angle becomes

$$\theta(t) = \frac{k_f A}{\omega_m} \sin(\omega_m t)$$

Substituting into the equation for *y(t)* obtains

$$y(t) = \cos\left(\omega_c t + \frac{k_f A}{\omega_m} \sin(\omega_m t)\right)$$

The following diagram illustrates such a frequency modulated signal



The ratio $k_f A / \omega_m$ is called the modulation index and we give it the symbol m.

$$m = \frac{k_f A}{\omega_m}$$

Now *y(t)* can be expanded as

$$y(t) = \cos(\omega_c t) \cdot \cos(m \cdot \sin(\omega_m t)) - \sin(\omega_c t) \cdot \sin(m \cdot \sin(\omega_m t))$$

Characterizing frequency modulation in this fashion allows its interpreted as the sum of two amplitude modulated signals. The first term is an amplitude modulation of the cosine of the carrier and the second term is an amplitude modulation of the sine of the carrier.

The modulation index (*m*) defines the type of FM modulation that is used and hence determines the hardware requirements for transmitters and receivers. There are two cases of interest, one called narrowband FM for which m is chosen to be small compared to $\pi/2$, and the other called wideband FM, where m is typically much larger than $\pi/2$.

Narrowband FM

If the gain k_f is chosen small enough then the modulation index will be small (i.e., $m << \pi/2$). As a result the two terms in the equation for *y(t)* can be linearized as

$$\cos(m \cdot \sin(\omega_m t)) \cong 1$$

$$\sin(m \cdot \sin(\omega_m t)) \cong m \sin(\omega_m t)$$

and *y(t)* can be approximated as

$$y(t) \cong \cos(\omega_c t) - m \sin(\omega_m t) \cdot \sin(\omega_c t)$$

The first term in this equation is the unmodulated carrier signal and the second term is an amplitude modulation of the sine of the carrier. If we define the following Fourier transform pairs

$$z_1(t) = \cos(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] = Z_1(j\omega)$$

$$z_2(t) = \sin(\omega_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)] = Z_2(j\omega)$$

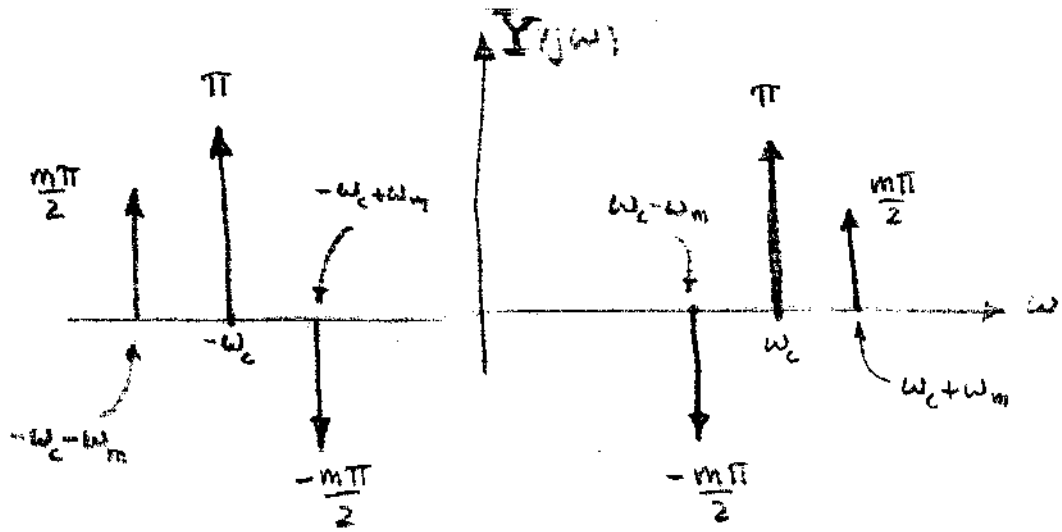
$$z_3(t) = \sin(\omega_m t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_m) - \delta(\omega + \omega_m)] = Z_3(j\omega)$$

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

then the Fourier transform of *y(t)* can be written as

$$Y(j\omega) = Z_1(j\omega) - m \cdot (Z_3(j\omega) * Z_2(j\omega))$$

The following diagram illustrates this transform.

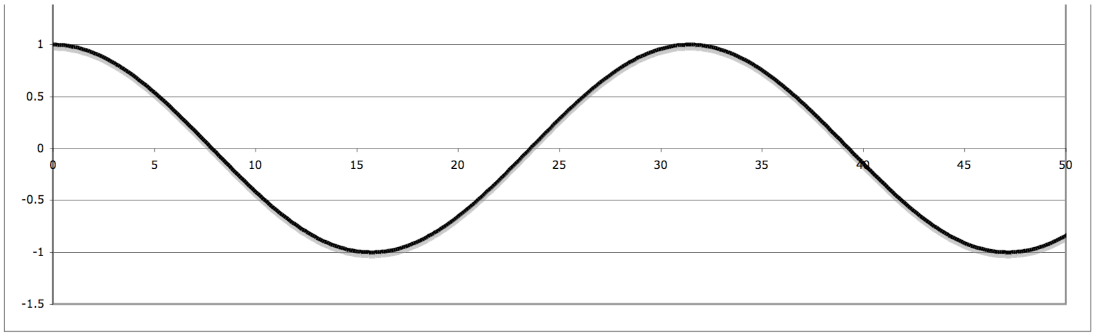


As can be seen in the diagram, the effect of narrowband FM is to create sidebands at $\pm\omega_m$ on each side of the carrier, where ω_m is the frequency of the modulating signal *x(t)*. Typically ω_m will be much lower, by orders of magnitude, than the carrier (e.g., 20 kHz vs. 100 MHz). As a result the sidebands will be quite close to the carrier frequency (i.e., narrowband FM).

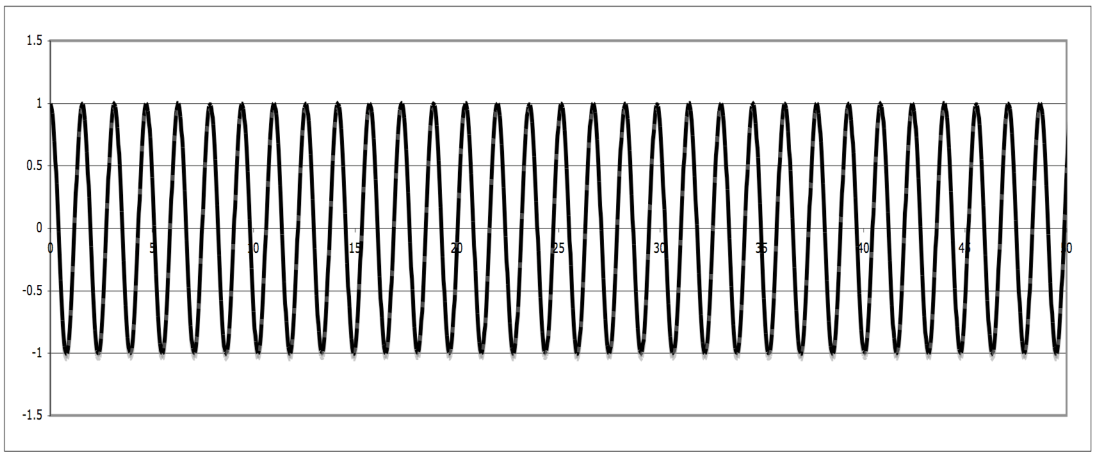
Also, in this development we assumed that the modulating signal is a pure tone of constant amplitude *A* and frequency ω_m. In real applications this is not the case, but we can think of the signal *x(t)* as having an instantaneous amplitude and frequency, both of which can vary with time. Hence, in the diagram above the areas of the sideband delta functions $(m\pi/2)$ and their locations relative to the carrier frequency $(\pm\omega_m)$ will vary with time. The receiver must be able to track these variations in order to reproduce the desired output *x(t)*.

Wideband FM

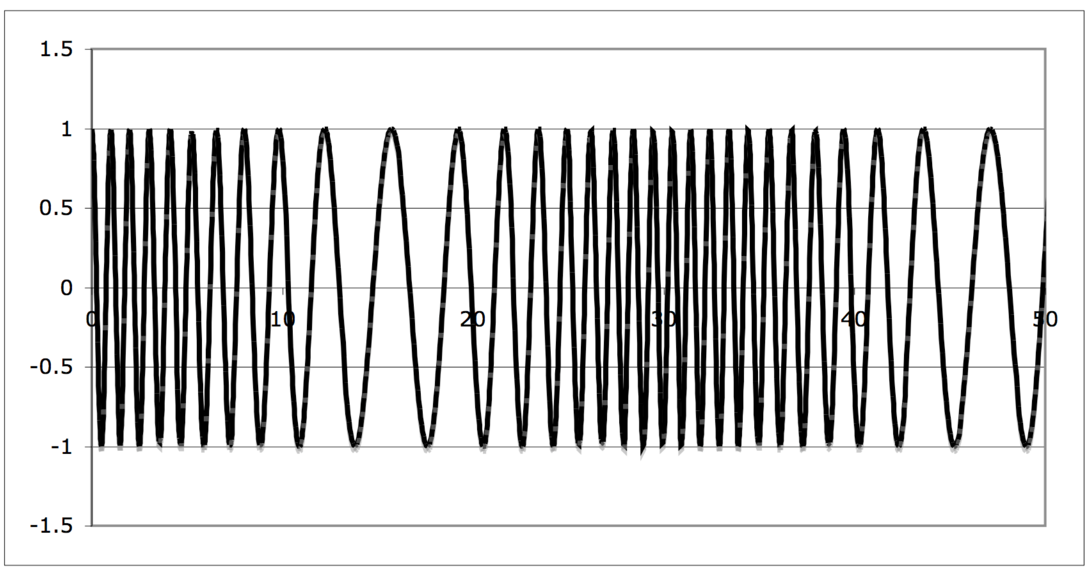
In contrast to narrowband FM the modulation index for wideband FM is typically much larger than $\pi/2$ and the equation for *y(t)* cannot be linearized. The following three diagrams illustrate a modulating signal *x(t),* at the modulating frequency $\omega_m = 0.2$, a carrier at twenty times the frequency of the modulating signal, so $\omega_c = 4.0$, and a frequency modulated signal with modulation index *m=12.0.*



. Modulating Signal- $x(t)$



Carrier Signal- $\cos(\omega_c t)$



Frequency Modulated Signal- $y(t) = \cos(\omega_c t + \theta(t))$

Note that when the value of the modulating signal is high the frequency of the modulated signal is increased, and conversely.

Earlier we showed that the modulated signal *y(t)* can be written as

$$y(t) = \cos(\omega_c t)\cdot\cos(m\cdot\sin(\omega_m t))$$

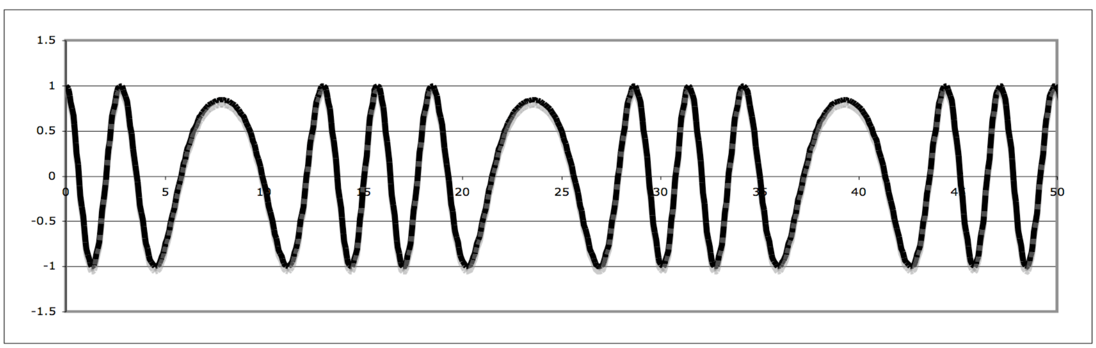
$$-\sin(\omega_c t)\cdot\sin(m\cdot\sin(\omega_m t))$$

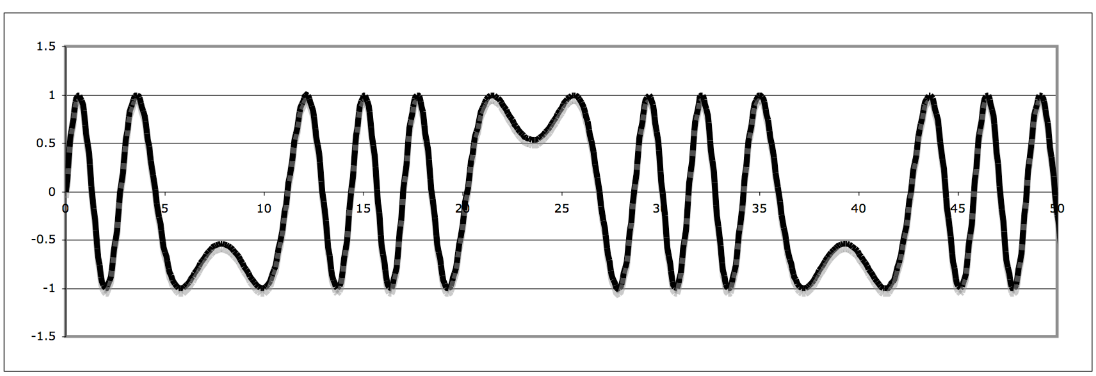
and, as earlier, this expression allows us to interpret frequency modulation as a kind of amplitude modulated signal. However, now *m* is large and we cannot linearize. The first term in this equation is an amplitude modulation of the cosine of the carrier and the second term is an amplitude modulation of the sine of the carrier. Also, since $\sin(\omega_m t)$ is periodic with period $T = (2\pi/\omega_m)$, then the two modulating terms

$$\cos(m\cdot\sin(\omega_m t))$$

$$\sin(m\cdot\sin(\omega_m t))$$

must also be periodic, at that same period. The following figures illustrate these two terms with m=12.0 and $\omega_m = 0.2$.

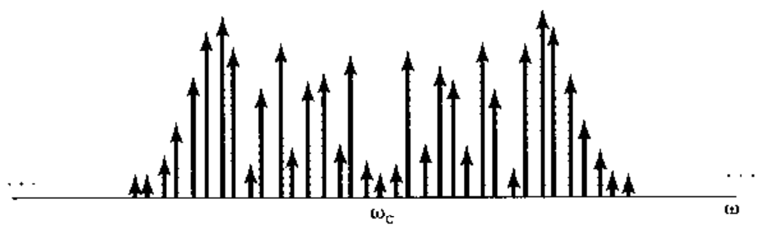




As can be seen in the second diagram the signal repeats itself at $T = (2\quad /\#_m) = 31.4$ units of time.

In effect the sine and cosine carriers are modulated by periodic signals. Since these modulating signals are periodic they can be represented as Fourier series. Hence their Fourier transforms are impulse trains, with the areas of the impulses determined by the Fourier series coefficients.

Determination of these coefficients requires Bessel functions, which are beyond the scope of mathematics that we will use in this course. However, the following diagram illustrates the nature of the pulse train magnitudes when then modulation index is *m=12.0*. The impulse train is spread symmetrically about $\pm \ {}_c$ and the impulses decay to negligible magnitude as the number of harmonics increases.



(c)

The Bessel function development yields useful information about this impulse spectrum. In particular the impulse train decreases rapidly at frequencies larger than a bandwidth equal to twice the modulation index times the modulating frequency ($B = 2m \ {}_m$). Thus, if the modulating frequency is at the limit for voice and/or music, namely 20 kHz, then with *m=12.0*, the bandwidth around the carrier is *B=480 kHz=0.48 MHz.* On your FM receiver, the minimum incremental step in frequency, from one station to another, is *0.5 MHz.*

Frequency Demodulation/Phase-Lock Loop

The frequency modulated signal *y(t)* is transmitted and then received by an FM receiver. The received signal must be decoded by the FM receiver, which typically employs a phase-lock loop. This device serves to track the phase of the received signal and in the process performs a time differentiation of the phase to determine the desired signal *x(t).*