Unified Engineering Problem Set
Week 13  Spring, 2008

Solutions

W13.1  Cantilevered Beam:

\[ \phi x^3 \]

\[ \text{P} \]

\[ \text{t} \]

\[ \text{w} = \text{width} \]

\[ L \]

(a) The basic equation for energy per unit volume is:

\[ U = \int_0^L \sigma \varepsilon \text{d} \varepsilon \]

Prior to yielding (and assuming linear behavior):

\[ \sigma = E \varepsilon \]

Combine these two expressions to get:

\[ U = \int_0^L E E \varepsilon \text{d} \varepsilon \]
\[ U = \frac{1}{2} \varepsilon E E^2 \]

Place this in terms of \( \sigma \) and \( \varepsilon \) using the stress-strain relation in the form:

\[ \varepsilon = \frac{\sigma}{E} \]

This gives:

\[ U = \frac{1}{2} \varepsilon (\frac{\sigma}{E})^2 \]

\[ \Rightarrow U = \frac{1}{2} \frac{\sigma^2}{E} \]

(Per unit volume)

For a beam, recall that:

\[ \sigma = -\frac{Mz}{I} \]

So need an expression for the moment \( M \) in the beam.

\[ \Rightarrow \text{Draw the free body diagram of the beam:} \]

\[ \text{Free Body Diagram:} \]

\[ P \]

\[ x_3 (z) \]

\[ x_1 (x) \]
\[ \Sigma F_x = 0 \quad \Rightarrow \quad \sum F_x = 0 \quad \Rightarrow \quad \sum F_x = 0 \quad \Rightarrow \quad N_A = 0 \]

\[ \Sigma F_y = 0 \quad \Rightarrow \quad V_A - P = 0 \quad \Rightarrow \quad V_A = P \]

\[ \Sigma M_A = 0 \quad \Rightarrow \quad M_A - PL = 0 \quad \Rightarrow \quad M_A = PL \]

Now cut the beam:

\[ \begin{align*}
  \Sigma M_x &= 0 \\
  \Rightarrow M(x) + PL - P_L &= 0 \\
  \Rightarrow M(x) &= P(x - L)
\end{align*} \]

Use this in the expression for stress (equation (2)) to get:

\[ \sigma = \frac{-P(x-L)I}{I} \quad (3) \]

and then in the expression for energy of equation (1):

\[ U = \frac{1}{2} \left( \frac{-P(x-L)^2}{I} \right) \]

(per unit volume)
To get to the expression for the total energy, this need to be integrated over the volume of the beam:

$$ U_{\text{total}} = \iiint U \, dV = \iiint U \, dx \, dy \, dz $$

with:
- \( x = 0 \) to \( L \)
- \( y = -w/2 \) to \( w/2 \)
- \( z = -t/2 \) to \( t/2 \)

so:

$$ U_{\text{total}} = \iiint_U \frac{P^2(\ell x^2 - 2xL + L^2)^2}{2EI^2} \, dx \, dy \, dz $$

All items except \( x \) and \( z \) are constant.

So:

$$ U_{\text{total}} = \frac{P^2}{2EI^2} \int x \int y \int [\ell x^2 - 2xL + L^2]^2 \, dx \, dy \, dz $$

Integrate with:

$$ \int_{-t/2}^{t/2} [\ell x^2 - 2xL + L^2] \, dx = \frac{\ell}{3} x^3 \bigg|_{-t/2}^{t/2} = \frac{\ell t^3}{3} $$

This is a constant with respect to \( x \) and \( y \). This comes out of the integral with

$$ \frac{\ell t^3}{3} \bigg|_{-t/2}^{t/2} = 2 \cdot \frac{\ell t^3}{3} = \frac{f^3}{12} $$

Integrate with:

$$ \int_{-w/2}^{w/2} \frac{\ell t^3}{3} \, dy $$
\[ U_{\text{total}} = \frac{P^2 t^3}{24 EI^2} \int_x \left( x^2 - 2xL + L^2 \right) dx \quad \left[ \begin{array}{c} y \frac{w/2}{w} = w \\ y = w \end{array} \right] \]

Finally integrating in \( x \):

\[ U_{\text{total}} = \frac{P^2 w t^3}{24 EI^2} \left[ \frac{x^3}{3} - 2 \frac{x^2L}{2} + L^2 x \right]_0 \]

\[ = \frac{P^2 w t^3}{24 EI^2} \left[ \frac{L^3}{3} - L^3 + L^3 \right] \]

\[ = \frac{P^2 w t^3 L^3}{72 EI^2} \]

Now recall that \( I = \frac{bh^3}{12} = \frac{wt^3}{12} \)

Substituting \( \frac{wt^3}{12} \) in the expression for \( U_{\text{total}} \) gives:

\[ U_{\text{total}} = \frac{P^2 w t^3 L^3}{72 E \left( \frac{w t^3}{12} \right)} \]

\[ = 144 \frac{P^2 w t^3 L^3}{72 E w^2 t^6} \]

and finally:

\[ U_{\text{total}} = \frac{2 P^2 L^3}{E w t^3} \quad (4) \]
Check the units!

Total Energy $= [F \cdot L] = \frac{[F]^2 [L]^3}{[F/L^2][L][L^3]} = \frac{[F][L]^3}{[L]^2} = [F \cdot L] \checkmark \, Y = 5$

(b) First consider the issue of yield and work the expression for $U_{\text{total}}$ to include the material parameter of yield stress $= \sigma_Y$

Recall that $\sigma = \frac{M \theta}{I}$

and with the expression for moment that stress is expressed in equation (3) as:

$\sigma = -\frac{P(x-L)^2}{I}$

Find the maximum value of stress in the beam and set this to $\sigma_Y$ to determine the maximum possible value of load (or dependent on the material and $\sigma_Y$).
The maximum value occurs for
\[ x = 0 \]
\[ z = +m - \frac{t}{2} \]

This is generally tensile yielding, so:

\[ \sigma_{\text{max}} = \sigma_y = \frac{P L t}{2 I} \]

Using \( I = \frac{w t^3}{12} \) and solving for \( P_{\text{max}} \):

\[ P_{\text{max}} = \frac{\sigma_y w t^2}{6 L} \]

Use this in the expression of equation (4) for \( U_{\text{total}} \):

\[ U_{\text{total}} = 2 \left( \frac{\sigma_y w t^2}{6 L} \right)^2 L^3 \]

To get:

\[ U_{\text{total}} = \frac{2 \sigma_y^2 w^2 t^4 L^3}{36 L^2 E w t^3} \]

\[ U_{\text{total}} = \frac{\sigma_y^2 w t L}{18 E} \]  \( (5) \)

Given, \( E \times F = \text{const.} \times [W/L] \):

\[ [F \cdot L] = \frac{[F/L^2]^2 [L] [L] [L]}{[F/L^2]} \approx [F \cdot L] \]
Proceeding.....

(i) for a given thickness $t$
$w$ and $l$ are constants
$t$ is for this case

$$\Rightarrow \text{Energy} = \frac{Q^2}{E} \cdot \text{constant}$$

so

$\maximize \frac{Q^2}{E}$

(ii) for a given mass of material

$$\Rightarrow \text{Volume} \cdot \text{Density} = \text{constant} = \text{total mass}$$

so

$wlt \cdot p = \text{constant}$

$w$ and $l$ are also constants giving:

$t \cdot p = \text{constant}$

Look at the expression of equation (5)
for $U_{total}$. All items are constants except for $Q$, $E$, and $t$. Thus:

$$U_{total} = \frac{Q^2 t}{E} \cdot \text{constant}$$
Since \( t \cdot \rho = \text{constant} \)
\[ \Rightarrow t = \frac{\text{constant}}{\rho} \]

and placing this in the derived expression for \( U_{\text{total}} \) gives:

\[ U_{\text{total}} = \frac{\sigma_y^2}{E_p} \cdot \text{constant} \]

\[ \Rightarrow \maximize \frac{\sigma_y^2}{E_p} \]

(iii) for a given cost of material

The key material parameter to include is the cost per mass = \( c \)

Total cost = (cost per mass) (mass) = constant

\[ \Rightarrow (\text{volume}) \cdot (\text{density}) \cdot (\text{cost}) \cdot (\text{mass}) \]

\[ \Rightarrow WLT \cdot \rho \cdot c = \text{constant} \]

Again, \( w \) and \( c \) are constants so:

\[ t \cdot \rho \cdot c = \text{constant} \]
As in (ii), all items in the expression of equation (5) for \( U_{\text{total}} \) are constants such that:

\[
U_{\text{total}} = \frac{\sigma_y^2 t}{\epsilon} \cdot \text{constant}
\]

\[
\sin \theta \quad t_{pc} = \text{constant}
\]

\[
\Rightarrow t = \frac{\text{constant}}{pc}
\]

and placing this in the derived expression for \( U_{\text{total}} \) gives:

\[
U_{\text{total}} = \frac{\sigma_y^2}{Epc} \cdot \text{constant}
\]

\[
\Rightarrow \maximize \frac{\sigma_y^2}{Epc}
\]

(c) Looking at these six materials, calculate the pertinent combination of parameters (i.e. the criterion for each case) and compare.
<table>
<thead>
<tr>
<th>Material</th>
<th>Given thickness maximizing $\left( \frac{G_y^2}{E} \right)$</th>
<th>Given Muz: maximizing $\left( \frac{G_y^2}{\rho} \right)$</th>
<th>Given Cost: maximizing $\left( \frac{G_y^2}{E \cdot \rho} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al alloy</td>
<td>$3.57 , [10^6 , Pa]$</td>
<td>$1.32 , [Pa / (g/m^3)]$</td>
<td>$0.53 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
<tr>
<td>Spring steel</td>
<td>$27.4 , [10^6 , Pa]$</td>
<td>$3.43 , [Pa / (g/m^3)]$</td>
<td>$0.98 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
<tr>
<td>Wood</td>
<td>$0.49 , [10^6 , Pa]$</td>
<td>$0.98 , [Pa / (g/m^3)]$</td>
<td>$0.98 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
<tr>
<td>Titanium</td>
<td>$16.9 , [10^6 , Pa]$</td>
<td>$3.76 , [Pa / (g/m^3)]$</td>
<td>$0.29 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
<tr>
<td>Glass/Epoxy</td>
<td>$1.14 , [10^6 , Pa]$</td>
<td>$0.57 , [Pa / (g/m^3)]$</td>
<td>$0.048 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
<tr>
<td>Graphite/Epoxy</td>
<td>$4.23 , [10^6 , Pa]$</td>
<td>$2.82 , [Pa / (g/m^3)]$</td>
<td>$0.014 , \left( \frac{10^3 , N \cdot m}{g} \right)$</td>
</tr>
</tbody>
</table>

**Note:** Being clear and consistent in units is important. Look for each case.

**Efficiency:** $\frac{G_y^2}{\rho} = \frac{[10^6 \, Pa]^2}{[10^3 \, Pa]} = [10^3 \, Pa]$ extra $10^3 \, Pa$ came from numerator.

**Mass:** $\frac{G_y^2}{E \cdot \rho} = \frac{[10^6 \, Pa]^2}{[10^6 \, Pa] \cdot [10^3 \, m^3]} = [Pa / (g/m^3)]$ could also fit to: $[N/m^2] / (g/m^3)$ = $[N \cdot m/g]$

**Cost:** $\frac{Mass \times cost}{g} = \frac{[N \cdot m]}{g} \cdot \frac{1}{[\$ / 10^3 \, g]}$  
  $= \frac{[10^3 \, N \cdot m]}{\$}$  
  could be $[10^3 \, Pa \cdot m^3 / \$]$
Comments

- For the thickness criterion, spring steel is about material by nearly a factor of 2.
- For the mass criterion, titanium just beats out spring steel.
- For the cost criterion, wood ties spring steel.

Thought: Does wood fit well in many home applications because of cost issues primarily? Consider a survey board.
14.2 Condition A: \( \sigma_{11} = 49 \) \( \sigma_{22} = -29 \) \( \sigma_{33} = 9 \) \( \sigma_{12} = 0 \) \( \sigma_{13} = 0 \) \( \sigma_{23} = 0 \)

Condition B: \( \sigma_{11} = 9 \) \( \sigma_{12} = 29 \) \( \sigma_{22} = 49 \) \( \sigma_{33} = 0.59 \) \( \sigma_{13} = 0 \) \( \sigma_{23} = 0 \)

Condition C: \( \sigma_{11} = -39 \) \( \sigma_{12} = 0 \) \( \sigma_{22} = -39 \) \( \sigma_{33} = -39 \) \( \sigma_{13} = 0 \) \( \sigma_{23} = 0 \)

Condition D: \( \sigma_{11} = 9 \) \( \sigma_{12} = 0 \) \( \sigma_{22} = 29 \) \( \sigma_{33} = 49 \) \( \sigma_{13} = 0 \) \( \sigma_{23} = 0 \)

\( \sigma_{\text{yield}} = 300 \text{ MPa} \)

(a) Application of the Tresca condition requires knowledge of the principal stresses.
For conditions A, C, and D, there are no applied shear stresses so the applied normal stresses are the principal stresses.

Put these in appropriate order based on magnitude:

<table>
<thead>
<tr>
<th>Condition A</th>
<th>Condition C</th>
<th>Condition D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_I = \sigma_{II} = 4g$</td>
<td>$\sigma_I = \sigma_{II} = -3g$</td>
<td>$\sigma_I = \sigma_{III} = 4g$</td>
</tr>
<tr>
<td>$\sigma_{II} = \sigma_{III} = 2g$</td>
<td>$\sigma_{II} = \sigma_{III} = -3g$</td>
<td>$\sigma_{II} = \sigma_{III} = 2g$</td>
</tr>
<tr>
<td>$\sigma_{III} = g$</td>
<td>$\sigma_{III} = -3g$</td>
<td>$\sigma_{III} = g$</td>
</tr>
</tbody>
</table>

For condition B, there is no applied shear stress in the 3-axis since $\sigma_{13} = \sigma_{23} = 0$ so $\sigma_{33}$ is a principal stress. However, $\sigma_{12}$ is nonzero so the principal stresses in the 1-2 plane need to be determined.

(From last term) for plane stresses, the principal stresses are the roots of the equation:

$$\tau^2 - \tau (\sigma_{II} + \sigma_{III}) + (\sigma_{II} \sigma_{III} - \sigma_{II} \sigma_{III}) = 0$$

For condition B:

$$\tau^2 - \tau (9g + 4g) + (9g^2 + 4g^2 - 2g^2) = 0$$

$$\Rightarrow \tau^2 - 5g \tau + (4g^2 - 4g^2) = 0$$
\[ \Rightarrow \tau (r - 5g) = 0 \]

\text{giving:} \quad \tau = \sigma_1 - 5g \\
\tau = \sigma_2 = 0

\text{Finally we order for Condition B}

\begin{align*}
\sigma_1 &= 5g \\
\sigma_2 &= 0.5g \\
\sigma_3 &= 0
\end{align*}

\[ \Rightarrow \text{Now apply the Tresca criteria where yielding occurs if:} \]

\[ |\sigma_1 - \sigma_2| = \sigma_y \]
\[ \text{or} \]
\[ |\sigma_2 - \sigma_3| = \sigma_y \]
\[ \text{or} \]
\[ |\sigma_3 - \sigma_1| = \sigma_y \]

in addition, the directionality associated with this is that yielding occurs via shear on the plane of maximum shear stress corresponding to the difference in those two principal stresses.

Apply each condition...
Condition A: $|\sigma_1 - \sigma_2| = |4\sigma - (-2\sigma)| = \sigma_y$

$\Rightarrow 6\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 50\text{ MPa}$

$|\sigma_2 - \sigma_3| = |-2\sigma - \sigma| = \sigma_y$

$\Rightarrow 3\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 100\text{ MPa}$

$|\sigma_3 - \sigma_1| = |\sigma - 4\sigma| = \sigma_y$

$\Rightarrow 3\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 100\text{ MPa}$

Critical case is $\sigma_1$

$\Rightarrow$ Yielding at $\sigma = 50\text{ MPa}$ on plane at $45^\circ$ between $\sigma_1$ and $\sigma_2$.

Condition B: $|\sigma_1 - \sigma_2| = |5\sigma - 0.5\sigma| = \sigma_y$

$\Rightarrow 4.5\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 66.7\text{ MPa}$

$|\sigma_2 - \sigma_3| = |0.5\sigma - 0| = \sigma_y$

$\Rightarrow 0.5\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 600\text{ MPa}$

$|\sigma_3 - \sigma_1| = |\sigma - 5\sigma| = \sigma_y$

$\Rightarrow 5\sigma = 300\text{ MPa}$
$\Rightarrow \sigma = 60\text{ MPa}$
critical case is third

$$\Rightarrow \text{yielding at } \sigma = 60 \text{ MPa}$$
in plane at $45^\circ$ to direction of principal stress* in 1-2 plane and $\sigma_{33}$

*Find true angle in 1-2 plane by using:

$$\theta_p = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left( \frac{4\sigma}{\sigma - (-4\sigma)} \right)$$

$$= \frac{1}{2} \tan^{-1} \left( \frac{4}{5} \right)$$

$$\Rightarrow \theta_p = \frac{1}{2} (38.7^\circ) = 19.3^\circ$$

Condition C: $|\sigma_1 - \sigma_2| = 0$ ....

All differences are 0 since this is by most static stress

$$\Rightarrow \text{No yielding}$$

Condition D: $|\sigma_1 - \sigma_2| = 14\sigma - 2\sigma = \sigma_3$

$$\Rightarrow 2\sigma = 350 \text{ MPa}$$

$$\Rightarrow \sigma = 175 \text{ MPa}$$
\[ |\sigma_{I} - \sigma_{III}| = |2\sigma - \sigma| = \sigma_{y} \]
\[ \Rightarrow \sigma = 300 \text{ MPa} \]
\[ |\sigma_{II} - \sigma_{I}| = |\sigma - 4\sigma| = \sigma_{y} \]
\[ \Rightarrow 3\sigma = 300 \text{ MPa} \]
\[ \Rightarrow \sigma = 100 \text{ MPa} \]

Critical case is third

\[ \Rightarrow \text{yielding at } \sigma = 100 \text{ MPa on plane at } 45^\circ \text{ between } \sigma_{II}, \sigma_{III} \]

(b) The von Mises criterion is:

\[
(\sigma_{I} - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_{I})^2 = 2\sigma_{y}^2
\]

Look at each condition again:

**Condition A:**

\[
(\sigma_{I} - (-2\sigma))^2 + ((-2\sigma) - \sigma)^2 + (\sigma - 4\sigma)^2 = 2\sigma_{y}^2
\]
\[ \Rightarrow 36\sigma^2 + 9\sigma^2 + 9\sigma^2 = 2\sigma_{y}^2 \]
\[ \Rightarrow 54\sigma^2 = 2\sigma_{y}^2 \]
\[ \sigma^2 = \frac{\sigma_{y}^2}{54} \]
\[ \sigma = \sqrt{\frac{\sigma_{y}^2}{54}} \]
\[ \Rightarrow \text{ for } \sigma = 57.7 \text{ MPa} \]
**Condition B:**

\[
(5y - 0.5y)^2 + (0.5y - 0)^2 + (0 - 5y)^2 = 20y^2
\]

\[
= 20.25y^2 + 0.25y^2 + 25y^2, 20y^2
\]

\[
= 45.5y^2 = 20y^2
\]

\[
\Rightarrow y = \sqrt{\frac{2}{45.5}} \sigma_y
\]

\[
\text{for } D: \quad y = 62.9\text{ MPa}
\]

**Condition C:**

\[
(-3y - 3y)^2 + (-3y - (-3y))^2 + (-3y - (3y))^2 = 20y^2
\]

\[
\text{WAT! Again all the differences are } 0 \text{ or this is hydrostatic stress}
\]

\[
\Rightarrow \text{ NO Yielding}
\]

**Condition D:**

\[
(4q - 2q)^2 + (2q - q)^2 + (q - 4q)^2 = 20q^2
\]

\[
= 4q^2 + q^2 + 9q^2 = 20q^2
\]

\[
\Rightarrow q = \sqrt{\frac{2}{14}} \sigma_y
\]

\[
\text{for } D: \quad q = 113.4\text{ MPa}
\]
(c) Summarize the results of (b):

<table>
<thead>
<tr>
<th>Condition</th>
<th>Tresca [MPa]</th>
<th>Von Mises</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>57.7</td>
</tr>
<tr>
<td>B</td>
<td>60</td>
<td>62.9</td>
</tr>
<tr>
<td>C</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>D</td>
<td>100</td>
<td>113.4</td>
</tr>
</tbody>
</table>

For each of the conditions, the Tresca criterion gives a more conservative estimate of the yielding characteristic. The one case where this is not true is condition C, where there is no yielding since there is a state of hydrostatic stress.
14.3 Airplane wing skin

\[ \sigma_{11} = 15 \text{ ksi} \]
\[ \sigma_{12} = 10 \text{ ksi} \]
\[ \sigma_{22} = 6 \text{ ksi} \]

\[ \sigma_{\text{yield}} = 50 \text{ ksi} \]

(a) The Tresca criterion is that:

\[
\left| \sigma_1 - \sigma_2 \right| = \sigma_y \\
\text{or} \\
\left| \sigma_2 - \sigma_3 \right| = \sigma_y \\
\text{or} \\
\left| \sigma_3 - \sigma_1 \right| = \sigma_y
\]

Here we have plane stress with \( \sigma_{11} = 0 \), so this becomes:

\[
\left| \sigma_1 - \sigma_2 \right| = \sigma_y \\
\text{or} \\
\left| \sigma_2 \right| = \sigma_y \text{ or } \left| \sigma_1 \right| = \sigma_y
\]

It is necessary to find the principal stresses for the plane stress case.
Use:
\[ \tau^2 - \tau (\sigma_1 + \sigma_2) + (\sigma_1 \sigma_2 - \sigma_{12}^2) = 0 \]

\[ \Rightarrow \tau^2 - \tau (15n + 6n) + [(15n)(6n) - (10m)^2] = 0 \]

Stress in [ksi]

Subj:
\[ \tau^2 - 21n \tau + 90n^2 - 100m^2 = 0 \]

To find the root, use the quadratic solution:
\[ \tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ \sigma_{I} = \frac{1}{2} \left( 21n \pm \left[ (21n)^2 - 360n^2 + 400m^2 \right] \right)^{1/2} \]

in [ksi]

\[ = 10.5n \pm \frac{1}{2} \left[ 81n^2 + 400m^2 \right]^{1/2} \]

\[ \Rightarrow \sigma_{I} = 10.5n + \frac{1}{2} \left( 81n^2 + 400m^2 \right)^{1/2} \]

\[ \sigma_{II} = 10.5n - \frac{1}{2} \left( 81n^2 + 400m^2 \right)^{1/2} \]

The Tresca condition can be rewritten using these expressions:

\[ \sigma_{III} = |\sigma_{I} - \sigma_{II}| = \left| \left( 81n^2 + 400m^2 \right)^{1/2} \right| \quad (1) \]

\[ \sigma_{III} = \left| \sigma_{I} \right| = \left| 10.5n + \frac{1}{2} \left( 81n^2 + 400m^2 \right)^{1/2} \right| \quad (2) \]
\[ 50 \leq \frac{1}{\sigma_1} \leq \frac{1}{10.5n - \frac{1}{2}(\delta/n^2 + 400m^2)^{1/2}} \] (3)

for \( n \) and \( m \geq 0 \), it is always the case that:

\[ \sigma_1 > \sigma_1^{II} \]
\[ \sigma_1 > 0 \]

so only (1) and (2) are operative and without need for (1)

For (1) or (2), express \( \sigma_1 \) in terms of \( n \): (Note that \( \delta/n^2 \) is included in all terms)

for (1):

\[ 50 = (\delta/n^2 + 400m^2)^{1/2} \]

\[ \Rightarrow 2500 - \delta/n^2 = 400m^2 \]

\[ \Rightarrow m = \sqrt{6.25 - 0.2025n^2} \] (1)

This limit on \( n \) is where

\[ 6.25 - 0.2025n^2 \geq 0 \]

so must have:

\[ 0.2025n^2 \leq 6.25 \]

\[ \Rightarrow n \leq 5.55 \]
for (2) ....

$50 = 10.5n + \frac{1}{2} \left( \frac{8}{n^2} + 400 \text{ m}^2 \right)^{1/2}$

$\Rightarrow \left( 1000 - 21n \right)^2 = \frac{8}{n^2} + 400 \text{ m}^2$

$m = \frac{1}{20} \sqrt{3600n^2 - 4200n + 10000}$

$\Rightarrow m = \sqrt{0.9n^2 - 10.5n + 25}$ \hspace{1cm} (2')

Find point where $0.9n^2 - 10.5n + 25 = 0$

Lowest value is maximum allowable

Solve via quadratic: \[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

$\Rightarrow 0.5 \pm \sqrt{(10.5)^2 - 4(0.9)(25)}$

$2(0.9)$

$\Rightarrow 10.5 \pm \sqrt{110.25 - 90}$

$1.8$

$\Rightarrow 10.5 \pm 4.5$

Lowest value = 3.33

$\Rightarrow n \leq 3.33$
Now use values of $n$ to determine values of $m$ for the two conditions:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$ via (1')</th>
<th>$m$ via (2')</th>
<th>(3) always less</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.5</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>2.5</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>2.5</td>
<td>3.9</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>2.4</td>
<td>3.4</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>2.3</td>
<td>2.8</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>2.2</td>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>2.1</td>
<td>1.3</td>
<td></td>
</tr>
<tr>
<td>3.33</td>
<td>2.0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>1.9</td>
<td>&lt; 0</td>
<td></td>
</tr>
<tr>
<td>5.55</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Plot: "operating envelope" via Trace condition (next page)
Operating Envelope via Trisection Criterion

\[ \equiv \text{operating} \]
(b) With the "damage tolerant" approach, use the basic fracture mechanics equation:

\[ J_f = \frac{K_c}{\sqrt{\pi a}} \]

Here, \( 2a = 0.40\text{ in} \Rightarrow a = 0.20\text{ in} \)

\( K_c = 31\text{ ksi} \sqrt{\text{in}} \) for the 2024 aluminum.

\[ \sigma_{nf} = \frac{31\text{ ksi} \sqrt{\text{in}}}{\sqrt{\pi (0.20\text{ in})}} \]

\[ \Rightarrow \sigma_f = 39.1\text{ ksi} \]

Thus, if the stress perpendicular to the crack exceeds 39.1 ksi, there is failure. However, the crack could be oriented in any direction so one must find the principal stresses (i.e., the maximum extensional stressing) and then the related direction for the worst case.

In part (c), found the principal stresses to be:
\[ \sigma_I = 10.5n + \frac{1}{2} (81n^2 + 400 m^2)^{1/2} \]
\[ \sigma_{II} = 10.5n - \frac{1}{2} (81n^2 + 400 m^2)^{1/2} \]

with \( n \) and \( m \geq 0 \), it was also determined that \( \sigma_I > \sigma_{II} \). So consider only \( \sigma_I \).

The limit condition gives:
\[ \sigma_I = 10.5n + \frac{1}{2} (81n^2 + 400 m^2)^{1/2} < 39.1 \text{ ksi} \]
[ksi] included hall term

so boundary determined via:
\[ 10.5n + \frac{1}{2} (81n^2 + 400 m^2)^{1/2} = 39.1 \text{ ksi} \]
\[ 81n^2 + 400 m^2 = (78.2 - 21n)^2 \]
\[ \Rightarrow m = \frac{1}{20} \sqrt{360n^2 - 328n + 615} \]
\[ \Rightarrow m = \sqrt{0.9 n^2 - 8.21n + 10.3} \]

A table gives:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>3.9</td>
<td>3.4</td>
<td>2.8</td>
<td>2.2</td>
<td>1.6</td>
<td>0.6</td>
</tr>
</tbody>
</table>

limit occurs for \( 6.21 \pm \sqrt{(78.2)^2 - 4(0.9)(10.3)} \)
\[ \Rightarrow 2(0.9) \]
\[ \Rightarrow \]
\[ \Rightarrow 2.1 \pm \sqrt{67.5 - 55.1} \]
\[ \Rightarrow 2.6 \]
Operating Envelope w/ Percolation Tolerance Approach
(c) Each of these approaches is different in the criteria used and thus the plots are substantially different. For this particular application, the true condition and the damage tolerant condition are both dependent on the same value of the principal strain and therefore have similar shape through different values.

In general, there will be no overall similarity between two such different criteria.
m 14.4

1. D
2. G
3. A
4. H
5. B
6. F
7. E
8. C