Unified Engineering Problem Set
Week 7  Spring, 2008

Solutions

7.1 Aluminum rod

(a) Begin by determining equations for the distributed force.
   - It is negative
   - It equals 0 at \( x_1 = 0 \)
   - It has intensity of 100 N·m/m at \( x_1 = 3 \) m

PAL
This results in
\[ t(x) = - \frac{100 \text{ N}\cdot\text{m/m}}{3 \text{ m}} \cdot x = -33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} x. \]

(Note: units are still torque intensity => correct)

→ Now draw Free Body Diagram

Take equilibrium of moments/torques about \( x \). Use right hand rule (RHR) for +:

\[ \sum T_{x} = 0 \quad \Rightarrow -T_{A} + 200 \text{ N} \cdot \text{m} = \int_{0}^{3 \text{ m}} 33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} x \, dx. \]

\[ \Rightarrow T_{A} = 200 \text{ N} \cdot \text{m} - 33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} \left[ \frac{3}{2} \left( \frac{x^2}{3} \right) \right]_{0}^{3 \text{ m}} \]

\[ = 200 \text{ N} \cdot \text{m} - 150 \text{ N} \cdot \text{m} \]
Resulting in: $T_4 = 50 \text{ N} \cdot \text{m}$

To determine the torque distribution $T(x_i)$, make cuts in the shaft/rod in the regions of different loading.

**First region** $0 < x_i < 1.5 \text{ m}$

\[
33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} x_i
\]

\[
50 \text{ N} \cdot \text{m}
\]

\[\mathcal{T}(x_i) = 50 \text{ N} \cdot \text{m} + 33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} \left( \frac{x_i^2}{2} \right) \bigg|_0 \]

\[
\Rightarrow T(x_i) = 50 \text{ N} \cdot \text{m} + 16.7 \frac{\text{N} \cdot \text{m}}{\text{m}^2} x_i^2
\]

\[0 < x_i < 1.5 \text{ m}\]

\[\text{OR} \quad \frac{dT}{dx_i} = -T(x_i)
\]

\[= 33.3 \frac{\text{N} \cdot \text{m}}{\text{m}^2} x_i\]
\[ T(x_i) = \int 33.3 \frac{N \cdot m}{m^2} x_i \, dx_i, \]
\[ = 33.3 \frac{N \cdot m}{m^2} \left( \frac{x_i^2}{2} \right) + C, \]

Use boundary condition that at \( x_i = 0 \) \( T \) opposes the reaction torque \( \Rightarrow T(0) = 50 \, N \cdot m \)

This gives \( C = 50 \, N \cdot m \)

\[ \Rightarrow T(x_i) = 16.7 \frac{N \cdot m}{m^2} x_i^2 + 50 \, N \cdot m \]
\[ 0 < x_i < 1.5 \, m \]

Same result (as it must be)

Proceed to ....

Second region \[ 1.5 \, m < x_i < 3 \, m \]

Take a cut ....

\[ \sum T_{x_i} = 0 \quad \Rightarrow -50 \, N \cdot m + 200 \, N \cdot m - \int_{0}^{x_i} 33.3 \frac{N \cdot m}{m^2} x_i \, dx_i, \]
\[ + T(x_i) = 0 \]
\[ T(x) = -150 \text{ N} \cdot \text{m} + 33.3 \frac{N \cdot m}{m^2} \left( \frac{x^2}{2} \right) \]

if \( 1.5 \text{ m} < x < 3 \text{ m} \)

or

use \( \frac{dT}{dx} = -t(x) \)

\[ = 33.3 \frac{N \cdot m}{m^2} x \]

after integrating: \( T(x) = 33.3 \frac{N \cdot m}{m^2} \left( \frac{x^2}{2} \right) + C_2 \)

Now need a boundary condition within this region. Could use the interface with region 1 or use the earlier condition at the free end \( (x_i = 3 \text{ m}) \) that there is no torque resultant: \( T(3 \text{ m}) = 0 \)

\[ \Rightarrow T(3 \text{ m}) = 0 = -150 \text{ N} \cdot \text{m} + C_2 \]

\[ \Rightarrow C_2 = -150 \text{ N} \cdot \text{m} \]

This results in:

\[ T(x) = 16.7 \frac{N \cdot m}{m^2} x^2 - 150 \text{ N} \cdot \text{m} \]

\( 1.5 \text{ m} < x < 3 \text{ m} \)

Agrees same result

Now sketch the result...
Note key points:
- \( x_1 = 0 \), \( T(x_1) = 50 \text{ N.m} \)
- \( x_1 = 1.5 \text{ m} \), \( T(x_1) = 87.5 \text{ N.m} \)
- \( x_1 = 1.5 + x \), \( T(x_1) = -112.5 \text{ N.m} \)
  \( \text{(Note: Jump of } -200 \text{ N.m... equal}
  \text{ and opposite to applied point-torque)} \)
- \( x_1 = 3 \text{ m} \), \( T(x_1) = 0 \)
and \( T(x_1) \) varies quadratic and positive
in the two segments.
(b) To determine the twist of the rod, use the equation for the twist angle:

\[ \frac{d\phi}{dx_1} = \frac{T}{GJ} \]

Consider each region starting at the point of the boundary condition on the twist angle: \( \phi(x_1 = 0) = 0 \)

So for Region 1:

\[ \phi = \frac{1}{GJ} \int \left( 50 \text{N}\cdot\text{m} + 16.7 \frac{\text{N}\cdot\text{m}^2}{\text{m}^2} x_1^2 \right) dx_1 \]

\[ \Rightarrow \phi(x_1) = \frac{1}{GJ} \left( 50 \text{N}\cdot\text{m} x_1 + \frac{16.7}{3} \frac{\text{N}\cdot\text{m}^2}{\text{m}^2} x_1^3 \right) + C_3 \]

\[ 0 < x_1 < 1.5 \text{m} \]

using the boundary condition gives:

\[ C_3 = 0 \]

So:

\[ \phi(x_1) = \frac{1}{GJ} \left( 50 \text{N}\cdot\text{m} x_1 + 5.56 \frac{\text{N}\cdot\text{m}^2}{\text{m}^2} x_1^3 \right) \]

\[ 0 < x_1 < 1.5 \text{m} \]

Proceed to Region 2 \ldots \ldots \ldots
Again using:
\[
\frac{d\Phi}{dx_i} = \frac{1}{GJ} f_i
\]
with \( f_i(x_i) \) for this region:
\[
\Rightarrow \frac{d\Phi}{dx_i} = \frac{1}{GJ} \left( -150 \text{ N.m} + 16.7 \frac{N.m}{m^2} x_i^2 \right)
\]
So:
\[
\Phi = \frac{1}{GJ} \int \left( -150 \text{ N.m} + 16.7 \frac{N.m}{m^2} x_i^2 \right) dx_i
\]
Further:
\[
\Phi(x_i) = \frac{1}{GJ} \left( -150 \text{ N.m} x_i + 16.7 \frac{N.m}{m^2} x_i^3 \right) + C_4
\]
Get a value to determine \( C_4 \) by matching the twist angle at the interface (boundary) with Region I. From that result:
\[
\Phi(1.5 \text{ m}) = \frac{1}{GJ} (93.8 \text{ N.m}^2)
\]
Matching gives:
\[
\Phi(1.5 \text{ m}) = \frac{1}{GJ} (93.8 \text{ N.m}^2) = \frac{1}{GJ} (-206.2 \text{ N.m}^2) + C_4
\]
\[
\Rightarrow C_4 = \frac{1}{GJ} (300 \text{ N.m}^2)
\]
So:
\[
\Phi(x_i) = \frac{1}{GJ} \left( -150 \text{ N.m} x_i + 5.56 \frac{N.m}{m^2} x_i^3 + 300 \text{ N.m}^2 \right)
\]
\[1.5 \text{ m} < x_i < 3 \text{ m}\]
So at the top:
\[ \phi(3\,\text{m}) = \frac{1}{GJ} \left(-450\,\text{N.m}^2 + 150\,\text{N.m}^2 + 350\,\text{N.m}^2 \right) \]

\[ \Rightarrow \phi(3\,\text{m}) = 0 \]

Could also determine GJ to get the torsional stiffness of the rod and subdue for the overall expression of \( \phi \).

In general:
\[ J = \int (x_2^2 + x_3^2)\,dA \]

We know that for a circular cross-section:
\[ J = \frac{\pi R^4}{2} \]

Here: \( R = 10\,\text{cm} = 0.1\,\text{m} \)
\[ \Rightarrow J = 1.57\times10^{-7}\,\text{m}^4 \]

Now find the shear modulus. We are given \( E = 67\,\text{GPa} \), \( v = 0.3 \) for aluminum.

For isotropic materials:
\[ G = \frac{E}{2(1+v)} \Rightarrow G = \frac{67\,\text{GPa}}{2(1+0.3)} = 25.8\,\text{GPa} \]

\[ G\,J = (25.8\times10^9\,\text{N/m}^2)(1.57\times10^{-7}\,\text{m}^4) = 4.05\times10^6\,\text{N.m}^2 \]
(c) To find the shear stress use:

\[ \tau_{res} = \frac{T_r}{J} \]

To determine the maximum magnitude and the maximum magnitude of \( T(x_r) \) and of \( r \):

- The value of \( r \) is maximized along the outer surface of the rod (\( r = 10 \text{ cm} \))
- \( T(x_r) \) has a maximum magnitude of \(-112.5 \text{ N.m} \) at \( x_r = 1.5 \text{ m} \)

Converting to consistent units:

\[ \tau_{res} = \frac{(-112.5 \text{ N.m}) (10 \times 10^{-2} \text{ m})}{J} \]

From (6) calculated \( J = 0.57 \times 10^{-7} \text{ m}^4 \)

So:

\[ \tau_{res} = \frac{(-112.5 \text{ N.m}) (10 \times 10^{-2} \text{ m})}{1.57 \times 10^{-7} \text{ m}^4} \]

The maximum magnitude is: (and is negative in direction)

\[ \tau_{res} = 76.7 \times 10^3 \frac{\text{N}}{\text{m}^2} = 0.072 \text{ MPa} \]

at \( r = 0.1 \text{ m} \)

\( x_r = 1.5 \text{ m} \)
(d) Rod is now a hollow tube with the same outer radius and a wall thickness of 20 mm:

\[ R_o = 10 \text{ cm} \]
\[ R_i = 10 \text{ cm} - 20 \text{ mm} = 8 \text{ cm} \]

The only thing that changes from the solid rod case is the cross-sectional polar moment of inertia (Shear Modular stays the same. So ...)

\[ \rightarrow (a) \quad \text{Torque distribution} \quad \begin{array}{c} \text{does not} \\ \text{change} \end{array} \]

This configuration is statically determinate and does not depend upon the cross-sectional properties.

\[ \rightarrow (b) \quad \text{All remain the same except if changes} \]
Can calculate the new $J$ via superposition (remove inner section):

$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} - \frac{\pi}{2} (R_o^4 - R_i^4)$$

$$= \frac{\pi}{2} (1^4 - 0.8^4) \text{ [m}^4\text{]} (\times 10^{-9})$$

$$\Rightarrow J = 0.927 \times 10^{-4} \text{ m}^4$$

This gives change for $\phi$ in general

$$\frac{\phi_{\text{new}}}{\phi_{\text{old}}} = \frac{J_{\text{old}}}{J_{\text{new}}} \quad \text{since } \phi \propto \frac{1}{J}$$

$$= \frac{1.67 \times 10^{-4} \text{ m}^4}{0.927 \times 10^{-4} \text{ m}^4}$$

$$\Rightarrow \frac{\phi_{\text{new}}}{\phi_{\text{old}}} = 1.69$$

$\phi$ generally increase by 69%.

However $\phi_{\text{tip}} = 0$ so that value does not change.
(c) The shear stress is \( \tau_{\text{res}} = \frac{T_r}{J} \).

So only the polar moment of inertia \( J \) changes, as \( T_{\text{max}} \) stays the same (via (a)) and the maximum value of \( r \) does not change.

So the change is via the inverse of \( J \).

As shown in part (b):

- Maximum magnitude
  \[ \tau_{\text{res}} \text{ increased by 69\%} \]

  Location stays at
  \[ r = 0.1 \text{ m} \]
  \[ x_t = 1.5 \text{ m} \]