The signal \( x(t) \) is periodic and

\[
x(t) = \begin{cases} 
  t+1 & -1 < t < 0 \\
  1-t & 0 < t < 1 
\end{cases}
\]

with period \( T = 2 \) \( \omega_0 = \frac{2\pi}{2} = \pi \)

This signal has the following derivative

\[
\frac{dx(t)}{dt} = y(t)
\]

We can readily obtain the Fourier coefficients for \( \frac{dx}{dt} = y(t) \). It has zero average value so

\( a_0 = 0 \)
In class we obtained Fourier coefficients for the following square wave with period $T = 4$

Its Fourier coefficients are

$$a_k = \begin{cases} \frac{1}{\pi j k} & k \text{ odd} \\ 0 & k \text{ even, } k \neq 0 \end{cases}$$

$$a_0 = \frac{1}{2}$$

The square wave $y(t)$ has zero average value, it is twice the magnitude of $z(t)$ and it is shifted forward in time by $\frac{1}{2}$ period. Hence, if $b_k$ are the Fourier coefficients for $y(t)$ and since $T = 2$ for $y(t)$

$$b_0 = 0$$

$$b_k = 2 e^{j \omega_0 T} e^{j \frac{2 \pi k}{T}} a_k = 2 e^{j \frac{2 \pi k}{4}} a_k = -2 a_k$$

$$b_k = \begin{cases} -\frac{2}{\pi j k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$
Now the signal we want is the integral of \( y(t) \). Thus if \( c_k \) are the Fourier coefficients of \( x(t) \)

\[
c_k = \frac{1}{jk\omega_0} b_k = \frac{1}{jk\pi} b_k = \left\{ \begin{array}{ll}
-\frac{2}{(jk\pi)^2} & \text{k odd} \\
0 & \text{k even}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\frac{2}{(k\pi)^2} & \text{k odd} \\
0 & \text{k even, k \neq 0}
\end{array} \right.
\]

For \( k=0 \) the coefficient is indeterminate with this equation. However, we know that \( c_0 \) must be the average value of \( x(t) \), which is \( \frac{1}{2} \), so

\[
c_0 = \frac{1}{2}
\]
\( a_k = \begin{cases} 
1 & k=0 \\
-j(\frac{1}{2})|k| & k \neq 0 
\end{cases} \)

\( a^*_k = \begin{cases} 
1 & k=0 \\
-j(\frac{1}{2})|k| & k \neq 0 
\end{cases} \)

\( a_{-k} = \begin{cases} 
1 & k=0 \\
j(\frac{1}{2})|k| & k \neq 0 
\end{cases} \)

Since \( a^*_k \neq a_{-k} \) then \( x(t) \) is not real

\( ii) \)

if

\[ \begin{align*}
X(t) & \underset{\text{Fs}}{\mapsto} a_k \\
\text{and} \\
x(-t) & \underset{\text{Fs}}{\mapsto} b_k
\end{align*} \]

\( b_k = a_{-k} = \begin{cases} 
1 & k=0 \\
j(\frac{1}{2})^k & k \neq 0 
\end{cases} \)

Thus \( b_k = a_k \)

So \( x(-t) = x(t) \) and \( x(t) \) is even
if $F_S$,

$$x(t) \rightarrow a_k$$

and

$F_S$

$$y(t) = \frac{dx(t)}{dt} \rightarrow b_k$$

then

$$b_k = jk\omega_0 a_k = \begin{cases} 0 & k = 0 \\ -k\omega_0 \left( \frac{1}{2} \right)^{|k|} & k \neq 0 \end{cases}$$

and

$F_S$

$$y(-t) \rightarrow b_{-k} = \begin{cases} 0 & k = 0 \\ k\omega_0 \left( \frac{1}{2} \right)^{|k|} & k \neq 0 \end{cases}$$

Thus, since $b_{-k} \neq b_k \Rightarrow y(-t) \neq y(t)$

and hence $y(t) = \frac{dx(t)}{dt}$ is not even.
i) \[ x(t) = \cos(\pi t) = \frac{e^{j\pi t} + e^{-j\pi t}}{2} = \frac{1}{2} e^{j\pi t} + \frac{1}{2} e^{-j\pi t} \]

\[ \omega_0 = \pi \]

Fourier coefficients are

\[ a_0 = 0, \quad a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ otherwise} \]

\[ X(t) = \sin(\pi t) = \frac{e^{j\pi t} - e^{-j\pi t}}{2j} = \frac{1}{2j} e^{j\pi t} - \frac{1}{2j} e^{-j\pi t} \]

\[ \omega_0 = \pi \]

Fourier coefficients are

\[ b_0 = 0, \quad b_1 = \frac{1}{2j}, \quad b_{-1} = -\frac{1}{2j}, \quad b_k = 0 \text{ otherwise} \]

ii) If

\[ Z(t) = x(t) \cdot y(t) \xrightarrow{F.S.} C_k \]

Then

\[ C_k = \frac{2}{\pi} a_l b_{k-l} \quad \text{for} \quad k-l = -\omega \]

Since only \( b_1 \) and \( b_{-1} \) are non-zero, the only non-zero terms in the sum are for \( k-l = 1 \) and \( k-l = -1 \).
Also, since only \( a_1 \) and \( a_{-1} \) are non-zero, the only terms in the sum that will be non-zero are \( l = +1 \) and \( l = -1 \).

Start at \( k = 0 \):

\[
c_0 = a_1 b_1 + a_{-1} b_{-1} = \frac{1}{2} (-\frac{1}{2j}) + \frac{1}{2} (\frac{1}{2j}) = 0
\]

\[
c_1 = a_1 b_0 + a_{-1} b_{-2} = 0
\]

\[
c_2 = a_1 b_1 + a_{-1} b_{-1} = \frac{1}{2} (\frac{1}{2j}) = \frac{1}{4j}
\]

\[
c_{-1} = a_1 b_{-2} + a_{-1} b_0 = 0
\]

\[
c_{-2} = a_1 b_{-3} + a_{-1} b_{-1} = \frac{1}{2} (-\frac{1}{2j}) = -\frac{1}{4j}
\]

All other \( c_k \)s = 0

\[
2 \langle t \rangle = \frac{1}{4j} e^{j2\pi t} - \frac{1}{4j} e^{-j2\pi t} = \frac{1}{2} \sin (2\pi t)
\]