Fluids – Lecture 9 Notes

   1. Momentum-Integral Simplifications
   2. Applications

Reading: Anderson 2.6

Simplifications

For steady flow, the momentum integral equation reduces to the following.

$$\oiint \rho \left(\vec{V} \cdot \hat{n} \right) \vec{V} \, dA \; = \; \oiint -p \, \hat{n} \, dA \; + \; \iiint \rho \, \vec{g} \, d\mathcal{V} \; + \; \vec{F}_{\text{viscous}} \tag{1}$$

Defining h as the height above ground, we note that ∇h is a unit vector which points up, so that the gravity acceleration vector can be written as a gradient.

$$\vec{g} \; = \; -g \, \nabla h$$

Using the Gradient Theorem, this then allows the gravity-force volume integral to be converted to a surface integral, provided we make the additional assumption that ρ is nearly constant throughout the flow.

$$\iiint \rho \, \vec{g} \, d\mathcal{V} \; = \; \iiint -\rho \, g \, \nabla h \, d\mathcal{V} \; = \; \oiint -\rho g h \, \hat{n} \, dA$$

We can now combine the pressure and gravity contributions into one surface integral.

$$\oiint -p \, \hat{n} \, dA \; + \; \iiint \rho \, \vec{g} \, d\mathcal{V} \quad \rightarrow \quad \oiint -(p + \rho g h) \, \hat{n} \, dA$$

Defining a corrected pressure $p_c = p + \rho g h$, the Integral Momentum Equation finally becomes

$$\oiint \rho \left(\vec{V} \cdot \hat{n} \right) \vec{V} \, dA \; = \; \oiint -p_c \, \hat{n} \, dA \; + \; \vec{F}_{\text{viscous}} \tag{2}$$

Aerodynamic analyses using (2) do not have to concern themselves with the effects of gravity, since it does not appear explicitly in this equation. In particular, the velocity field $\vec{V}$ will not be affected by gravity. Gravity enters the problem only in a secondary step, when the true pressure field p is constructed from p_c by adding the "tilting" bias $-\rho g h$.

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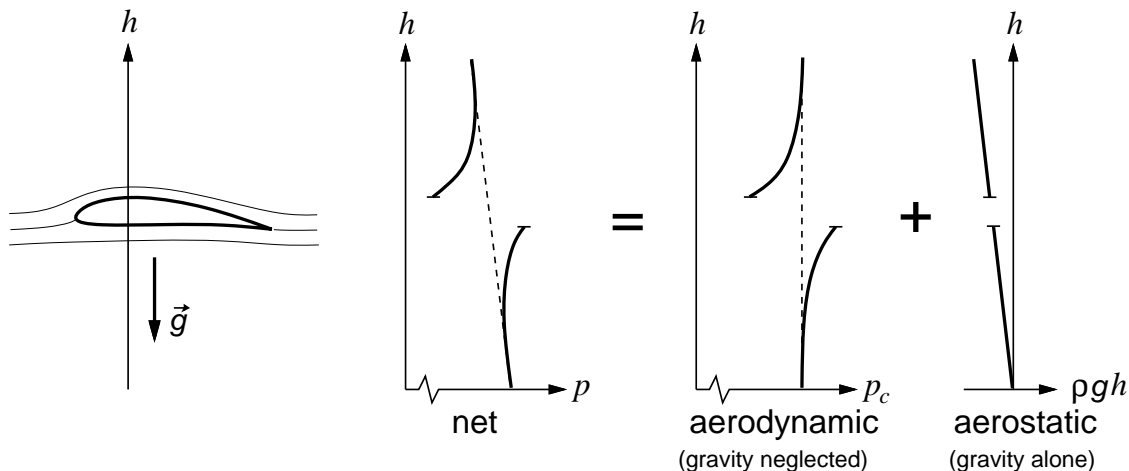
For clarity, we will from now on refer to p_c simply as p. The understanding is that the forces and moments computed using this p will <u>not</u> include the contributions of buoyancy. However, buoyancy is relatively easy to compute, and can be added as a secondary step if appropriate. It should be noted that buoyancy is rarely significant in aerodynamic flow situations, one exception being lighter-than-air vehicles like blimps.

One situation where gravity is crucial is a flow with a free surface, like water waves past a ship. Here, gravity enters not in equation (2), but in the free-surface boundary condition. Free-surface flows are not relevant to most aeronautical problems.

Applications

Basic Procedures
Application of the integral momentum equation (2) uses the same basic techniques as for the integral continuity equation. Both can use the same control volume, and both demand that the integrals are evaluated for the entire surface of the control volume. There are three significant differences, however:

1) Momentum is a vector. Each of the three x, y, and z components of equation (2) is independent, and must be treated separately.

2) A volume boundary lying along a streamline now has a nonzero $-p\hat{n}$ contribution.

3) A volume boundary lying along a solid surface now has a nonzero $-p\hat{n}$ contribution, and possibly also a viscous shear $\vec{\tau}$ contribution if viscous effects are deemed significant.

Effect of body in flow
Equation (2) assumes that there is no internal force acting on the fluid other than gravity. If there is a solid body in the flow, it is therefore necessary to construct the control volume such that the body is on the volume's exterior. This is easily done by placing the body in an indentation $cdefg$ of the volume surface, as shown in the figure for a 2D case.
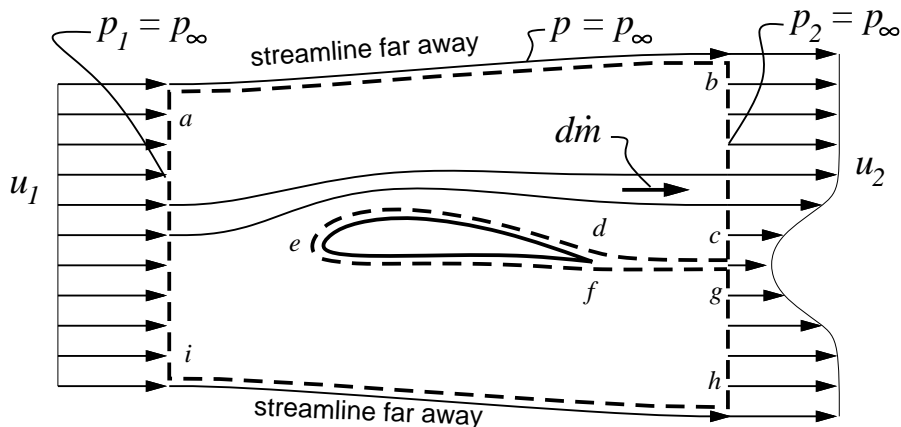

The surface integrals are now broken up into the individual pieces: the outermost part $abhi$, the body surface def, and the "cut" surfaces cd and fg.

$$\oint (\,)\,dA \;=\; \int_{abhi} (\,)\,dA \;+\; \int_{def} (\,)\,dA \;+\; \int_{cd} (\,)\,dA \;+\; \int_{fg} (\,)\,dA$$

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Surfaces cd and fg have the same flow variables, but opposite $\hat{n}$ vectors, so that all their contributions are equal and opposite, and hence cancel.

$$\int\limits_{cd} (\ )\, dA \ + \ \int\limits_{fg} (\ )\, dA \ = \ 0$$

The resultant force/span $\vec{R}'$ on the body is the integrated pressure and shear force distribution on the body surface. Because the body surface and volume surface have opposite normals $\hat{n}$, this $\vec{R}'$ is precisely equal and opposite to all the def surface integrals for the control volume.

$$\int\limits_{def} (\ )\, dA \ = \ -\vec{R}'$$
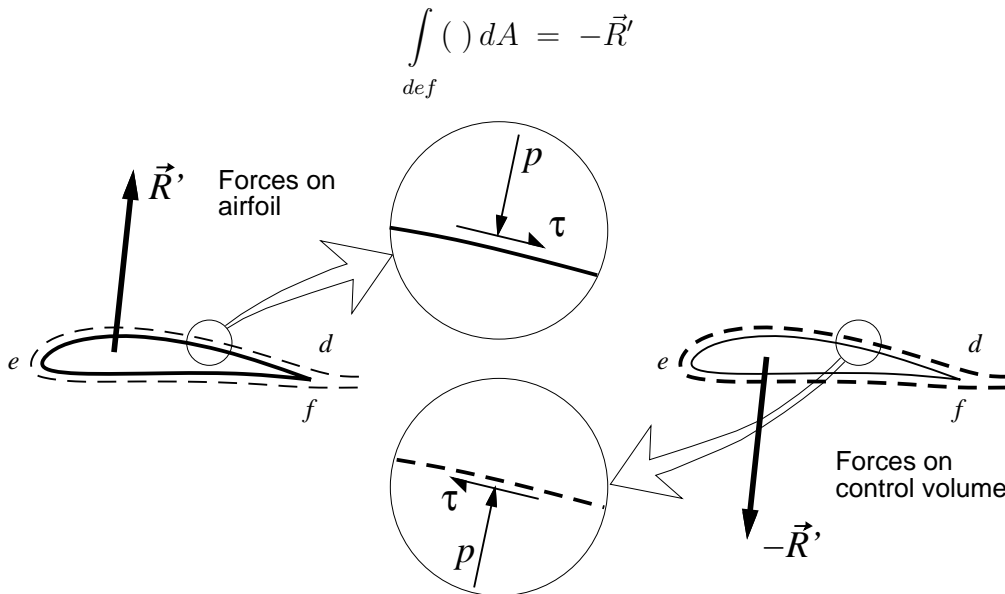
An intuitive explanation is that the aerodynamic force exerted by the fluid on the body is exactly equal and opposite to the force which the body exerts on the fluid.

Collecting the remaining integrals produces the very general result

$$\oint\limits_{abhi} \rho \left(\vec{V}\cdot\hat{n}\right) \vec{V}\, dA \ + \ \oint\limits_{abhi} p\,\hat{n}\, dA \ = \ -\vec{R}' \tag{3}$$

which gives the resultant force on the body only in terms of the surface integrals on the entire outer surface of the volume.

Drag relations
To obtain the drag/span, we define the x direction to be aligned with $\vec{V}_\infty$, and take the x-component of equation (3).

$$\oint\limits_{abhi} \rho \left(\vec{V}\cdot\hat{n}\right) u\, dA \ + \ \oint\limits_{abhi} p\,\hat{n}\cdot\hat{i}\, dA \ = \ -D' \tag{4}$$

The pressure on the outer surface is uniform and equal to p_∞, and hence the pressure integral is zero. The momentum-flux integral is zero on the top and bottom boundaries, since

these are defined to be along streamlines, and hence have zero momentum flux. Only the momentum flux on the inflow and outflow planes remain.

$$\int_{bh} \rho_2 u_2^2 \, dA_2 \; - \; \int_{ia} \rho_1 u_1^2 \, dA_1 \; = \; -D' \tag{5}$$

Using the concept of a streamtube, we see that planes 1 and 2 have the same mass flows, with

$$\rho_1 u_1 \, dA_1 \; = \; \rho_2 u_2 \, dA_2 \; = \; d\dot{m}$$

as shown in the figure. Therefore both the plane 1 and plane 2 integrals can be lumped into one single plane 2 integral.

$$D' \; = \; \int_h^b \rho_2 u_2 \, (u_1 - u_2) \, dA_2 \tag{6}$$

Equation (6) gives the drag only in terms of the freestream speed u_1, and the downstream wake "profiles" ρ_2 and u_2. These downstream profiles can be measured in a wind tunnel, and numerically integrated in equation (6) to obtain the drag.

A physical interpretation of (6) can be obtained by writing it as

$$D' \; = \; \int_{\dot{m}} (u_1 - u_2) \, d\dot{m}$$

This is simply the summation of the momentum flow "lost" by all the infinitesimal streamtubes in the flow. The lost momentum flow appears as drag on the body.