Fluids – Lecture 11 Notes

1. Vorticity and Strain Rate
2. Circulation

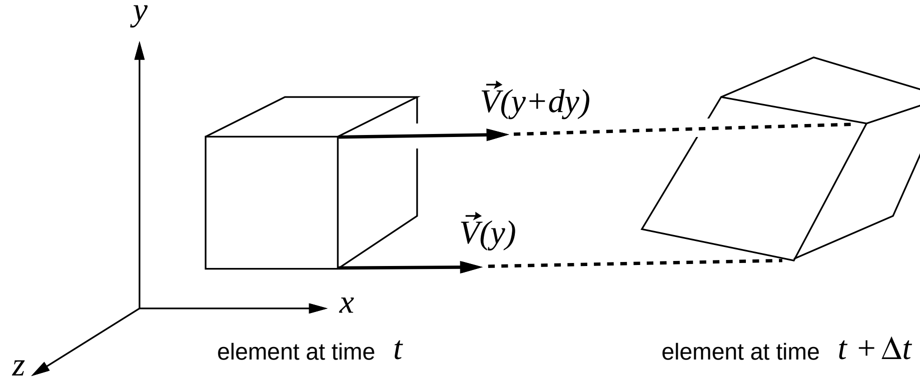
Reading: Anderson 2.12, 2.13

Vorticity and Strain Rate

Fluid element behavior

When previously examining fluid motion, we considered only the changing position and velocity of a fluid element. Now we will take a closer look, and examine the element's changing *shape and orientation*.

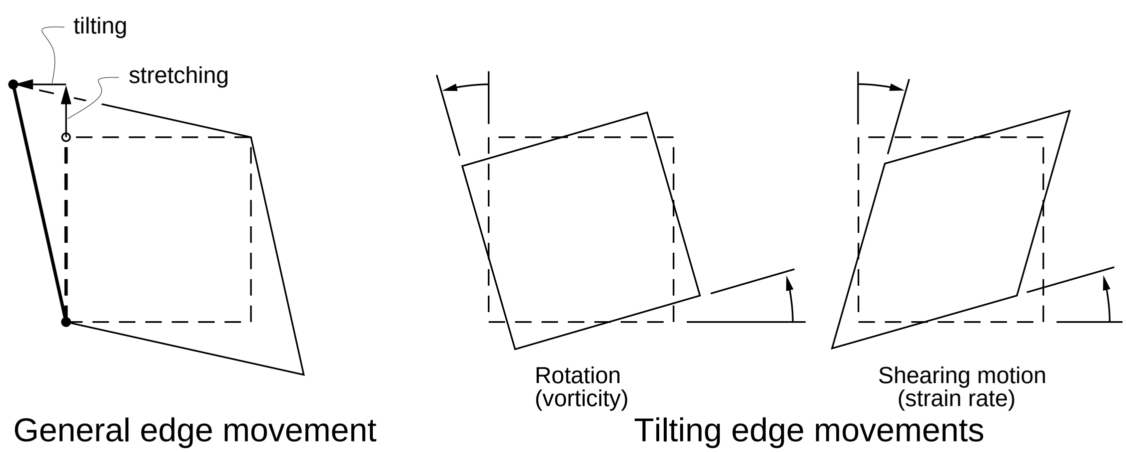
Consider a moving fluid element which is initially rectangular, as shown in the figure. If the velocity varies significantly across the extent of the element, its corners will not move in unison, and the element will rotate and become distorted.



In general, the edges of the element can undergo some combination of *tilting* and *stretching*. For now we will consider only the tilting motions, because this has by far the greatest implications for aerodynamics.

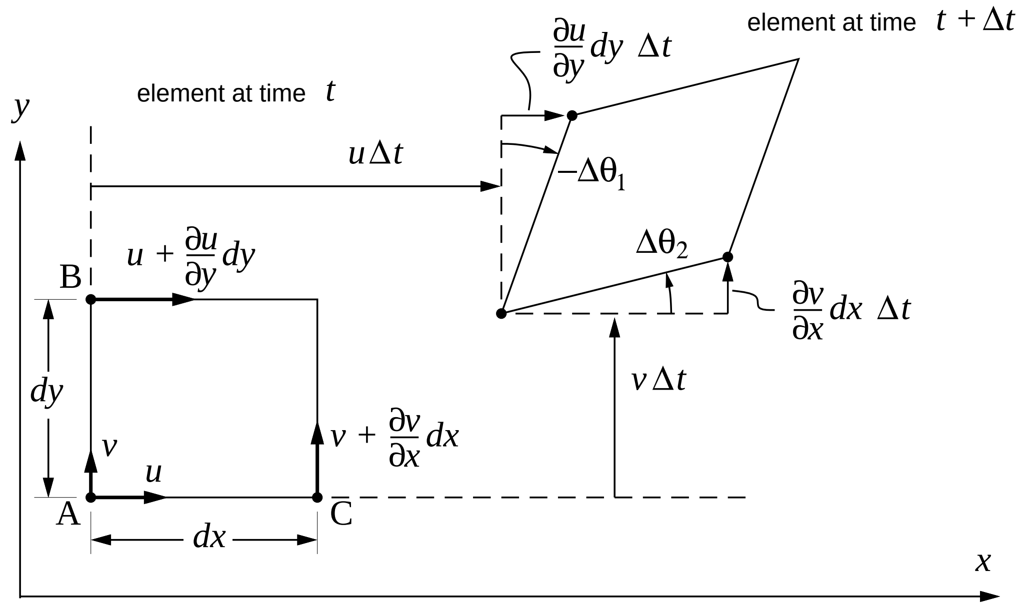
The figure below on the right shows two particular types of element-side tilting motions. If adjacent sides tilt equally and in the same direction, we have pure *rotation*. If the adjacent sides tilt equally and in opposite directions, we have pure *shearing* motion.

Both of these motions have strong implications. The absense of rotation will lead to a great simplification in the equations of fluid motion. Shearing together with fluid viscosity produce shear stresses, which are responsible for phenomena like drag and flow separation.



Side tilting analysis

Consider the 2-D element in the xy plane, at time t, and again at time $t + \Delta t$.



Points A and B have an x-velocity which differs by $\partial u/\partial y\, dy$. Over the time interval Δt they will then have a difference in x-displacements equal to

$$\Delta x_B - \Delta x_A = \frac{\partial u}{\partial y} dy\, \Delta t$$

and the associated angle change of side AB is

$$-\Delta\theta_1 = \frac{\Delta x_B - \Delta x_A}{dy} = \frac{\partial u}{\partial y} \Delta t$$

assuming small angles. A positive angle is defined counterclockwise. We now define a time rate of change of this angle as follows.

$$\frac{d\theta_1}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = -\frac{\partial u}{\partial y}$$

Similar analysis of the angle rate of side AC gives

$$\frac{d\theta_2}{dt} = \frac{\partial v}{\partial x}$$

Vorticity

The angular velocity of the element, about the z axis in this case, is defined as the average angular velocity of sides AB and AC.

$$\omega_z = \frac{1}{2}\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The same analysis in the xz and yz planes will give a 3-D element's angular velocities ω_y and ω_x.

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad , \quad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

These three angular velocities are the components of the *angular velocity vector*.

$$\vec{\omega} \;=\; \omega_x \hat{\imath} \,+\, \omega_y \hat{\jmath} \,+\, \omega_z \hat{k}$$

However, since $2\vec{\omega}$ appears most frequently, it is convenient to define the *vorticity* vector $\vec{\xi}$ as simply twice $\vec{\omega}$.

$$\vec{\xi} \;=\; 2\vec{\omega} \;=\; \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} \,+\, \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} \,+\, \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

The components of the vorticity vector are recognized as the definitions of the curl of $\vec{V}$, hence we have

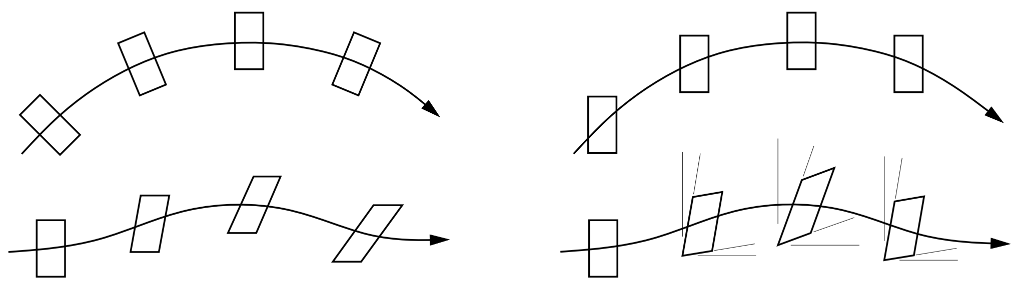
$$\vec{\xi} \;=\; \nabla \times \vec{V}$$

Two types of flow can now be defined:

1) Rotational flow. Here $\nabla \times \vec{V} \neq 0$ at every point in the flow. The fluid elements move and deform, and also rotate.

2) Irrotational flow. Here $\nabla \times \vec{V} = 0$ at every point in the flow. The fluid elements move and deform, but do not rotate.

The figure contrasts the two types of flow.



Rotational flows　　　　Irrotational flows

Strain rate

Using the same element-side angles $\Delta\theta_1$, $\Delta\theta_2$, we can define the *strain* of the fluid element.

$$\text{strain} \;=\; \Delta\theta_2 - \Delta\theta_1$$

This is the same as the strain used in solid mechanics. Here, we are more interested in the *strain rate*, which is then simply

$$\frac{d(\text{strain})}{dt} \;\equiv\; \varepsilon_{xy} \;=\; \frac{d\Delta\theta_2}{dt} - \frac{d\Delta\theta_1}{dt} \;=\; \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Similarly, the strain rates in the yz and zx planes are

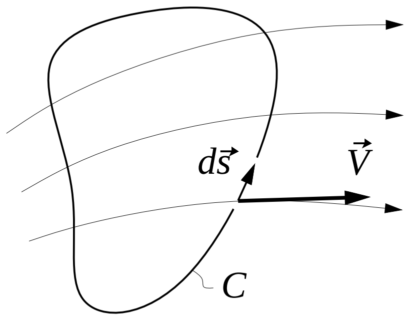
$$\varepsilon_{yz} \;=\; \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad , \qquad \varepsilon_{zx} \;=\; \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

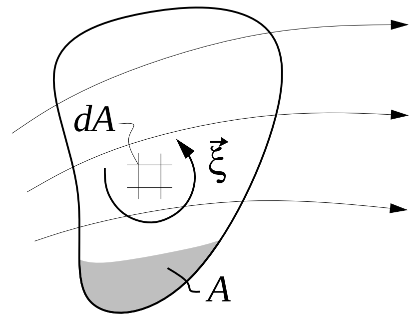
Circulation

Consider a closed curve C in a velocity field as shown in the figure on the left. The instantaneous *circulation* around curve C is defined by

$$\Gamma \;\equiv\; -\oint_C \vec{V} \cdot d\vec{s}$$

In 2-D, a line integral is counterclockwise by convention. But aerodynamicists like to define circulation as positive clockwise, hence the minus sign.





Circulation is closely linked to the vorticity in the flowfield. By Stokes's Theorem,

$$\Gamma \;\equiv\; -\oint_C \vec{V} \cdot d\vec{s} \;\;=\;\; -\iint_A \left(\nabla \times \vec{V}\right) \cdot \hat{n}\, dA \;=\; -\iint_A \vec{\xi} \cdot \hat{n}\, dA$$

where the integral is over the area A in the interior of C, shown in the above figure on the right, and $\hat{n}$ is the unit vector normal to this area. In the 2-D xy plane, we have $\vec{\xi} = \xi \hat{k}$ and $\hat{n} = \hat{k}$, in which case we have a simpler scalar form of the area integral.

$$\Gamma \;=\; -\iint_A \xi\, dA \qquad\qquad \text{(in 2-D)}$$

From this integral one can interpret the vorticity as –circulation per area, or

$$\xi \;=\; -\frac{d\Gamma}{dA}$$

Irrotational flows, for which $\xi = 0$ by definition, therefore have $\Gamma = 0$ about any contour inside the flowfield. Aerodynamic flows are typically of this type. The only restriction on this general principle is that the contour must be reducible to a point while staying inside the flowfield. A contours which contains a lifting airfoil, for example, is not reducible, and will in general have a nonzero circulation.