Fluids – Lecture 12 Notes

1. Stream Function
2. Velocity Potential

Reading: Anderson 2.14, 2.15

Stream Function

Definition
Consider defining the components of the 2-D mass flux vector $\rho \vec{V}$ as the partial derivatives of a scalar *stream function*, denoted by $\bar{\psi}(x, y)$:

$$\rho u = \frac{\partial \bar{\psi}}{\partial y} \quad , \quad \rho v = -\frac{\partial \bar{\psi}}{\partial x}$$

For low speed flows, ρ is just a known constant, and it is more convenient to work with a scaled stream function

$$\psi(x, y) = \frac{\bar{\psi}}{\rho}$$

which then gives the components of the velocity vector $\vec{V}$.

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x}$$
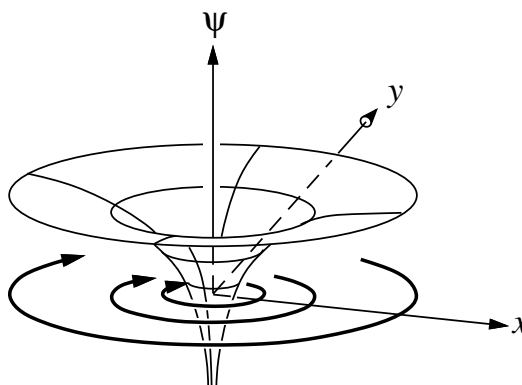
Example
Suppose we specify the constant-density streamfunction to be

$$\psi(x, y) = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$$

which has a circular "funnel" shape as shown in the figure. The implied velocity components are then

$$u = \frac{\partial \psi}{\partial y} = \frac{y}{x^2 + y^2} \quad , \quad v = -\frac{\partial \psi}{\partial x} = \frac{-x}{x^2 + y^2}$$

which corresponds to a vortex flow around the origin.

vortex flow example

Streamline interpretation

The stream function can be interpreted in a number of ways. First we determine the differential of $\bar{\psi}$ as follows.

$$d\bar{\psi} = \frac{\partial \bar{\psi}}{\partial x} dx + \frac{\partial \bar{\psi}}{\partial y} dy$$

$$d\bar{\psi} = \rho u \, dy - \rho v \, dx$$

Now consider a line along which $\bar{\psi}$ is some constant $\bar{\psi}_1$.
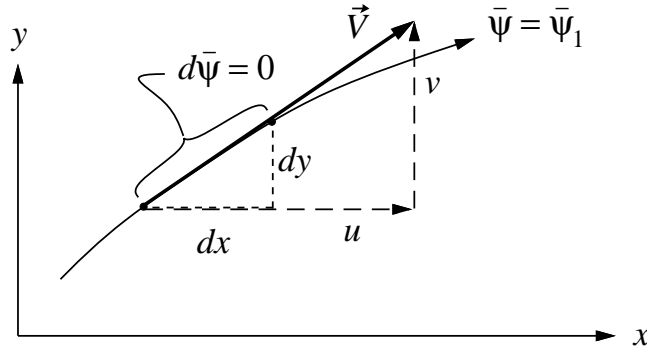
$$\bar{\psi}(x, y) = \bar{\psi}_1$$

Along this line, we can state that $d\bar{\psi} = d\bar{\psi}_1 = d(\text{constant}) = 0$, or

$$\rho u \, dy - \rho v \, dx = 0 \qquad \rightarrow \qquad \frac{dy}{dx} = \frac{v}{u}$$

which is recognized as the equation for a streamline. Hence, lines of constant $\bar{\psi}(x, y)$ are streamlines of the flow. Similarly, for the constant-density case, lines of constant $\psi(x, y)$ are streamlines of the flow. In the example above, the streamline defined by

$$\ln \sqrt{x^2 + y^2} = \psi_1$$

can be seen to be a circle of radius $\exp(\psi_1)$.


Mass flow interpretation

Consider two streamlines along which $\bar{\psi}$ has constant values of $\bar{\psi}_1$ and $\bar{\psi}_2$. The constant mass flow between these streamlines can be computed by integrating the mass flux along any curve AB spanning them. First we note the geometric relation along the curve,

$$\hat{n} \, dA = \hat{\imath} \, dy - \hat{\jmath} \, dx$$

and the mass flow integration then proceeds as follows.

$$\dot{m} = \int_A^B \rho \vec{V} \cdot \hat{n} \, dA = \int_A^B (\rho u \, dy - \rho v \, dx) = \int_A^B d\bar{\psi} = \bar{\psi}_2 - \bar{\psi}_1$$

Hence, the mass flow between any two streamlines is given simply by the difference of their stream function values.
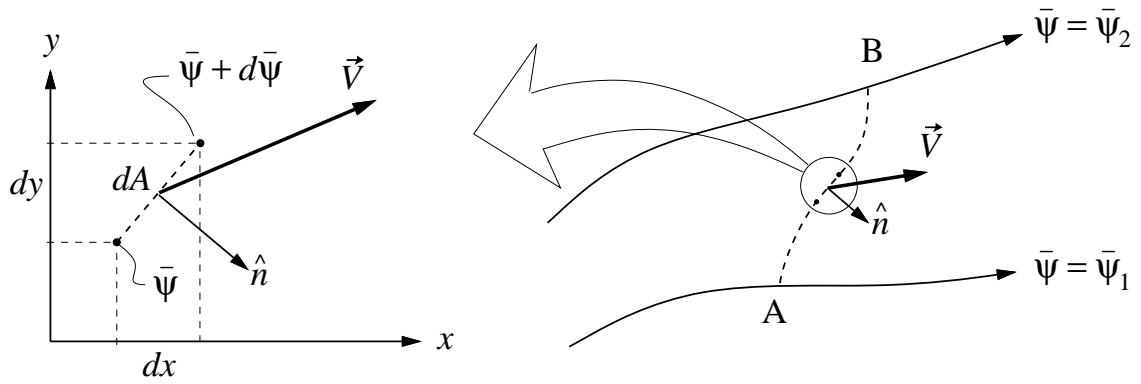
Continuity identity

Consider the mass flux field $\rho\vec{V}(x,y)$ specified by some $\bar{\psi}(x,y)$. Computing the divergence of this field we have

$$\nabla \cdot \left(\rho\vec{V}\right) \;=\; \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \;=\; \frac{\partial}{\partial x}\left(\frac{\partial\bar{\psi}}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\bar{\psi}}{\partial x}\right) \;=\; \frac{\partial^2\bar{\psi}}{\partial y\,\partial x} - \frac{\partial^2\bar{\psi}}{\partial x\,\partial y} \;=\; 0$$

so that any mass flux field specified via $\bar{\psi}(x,y)$ will automatically satisfy the steady mass continuity equation. In low speed flow, a similar computation shows that any velocity field specified via $\psi(x,y)$ will automatically satisfy

$$\nabla \cdot \vec{V} \;=\; 0$$

which is the constant-density mass continuity equation. Because of these properties, using the stream function to define the velocity field can give mathematical simplification in many fluid flow problems, since the continuity equation then no longer needs to be addressed.

Velocity Potential

Definition

Consider defining the components of the velocity vector $\vec{V}$ as the gradient of a scalar *velocity potential function*, denoted by $\phi(x,y,z)$.

$$\vec{V} \;=\; \nabla\phi \;=\; \hat{i}\,\frac{\partial\phi}{\partial x} \;+\; \hat{j}\,\frac{\partial\phi}{\partial y} \;+\; \hat{k}\,\frac{\partial\phi}{\partial z}$$

If we set the corresponding x, y, z components equal, we have the equivalent definitions

$$u \;=\; \frac{\partial\phi}{\partial x} \quad, \qquad v \;=\; \frac{\partial\phi}{\partial y} \quad, \qquad w \;=\; \frac{\partial\phi}{\partial z}$$
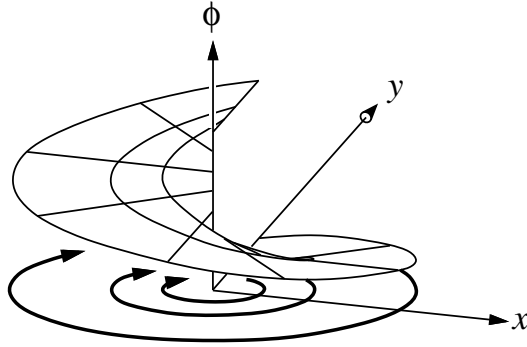
Example

For example, suppose we specify the potential function to be

$$\phi(x,y) \;=\; \arctan\left(\frac{x}{y}\right)$$

which has a corkscrew shape as shown in the figure. The implied velocity components are then

$$u \;=\; \frac{\partial\phi}{\partial x} \;=\; \frac{y}{x^2+y^2} \quad, \qquad v \;=\; \frac{\partial\phi}{\partial y} \;=\; \frac{-x}{x^2+y^2}$$

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vortex flow example

which corresponds to a vortex flow around the origin. Note that this is exactly the same velocity field as in the previous example using the stream function.

Irrotationality

If we attempt to compute the vorticity of the potential-derived velocity field by taking its curl, we find that the vorticity vector is identically zero. For example, for the vorticity x-component we find

$$\xi_x \ \equiv \ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \ = \ \frac{\partial}{\partial y}\frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial y} \ = \ \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \ = \ 0$$

and similarly we can also show that $\xi_y = 0$ and $\xi_z = 0$. This is of course just a manifestation of the general vector identity   curl(grad) $= 0$. Hence, any velocity field defined in terms of a velocity potential is automatically an *irrotational flow*. Often the synonymous term *potential flow* is also used.

Directional Derivative

In many situations, only one particular component of the velocity is required. For example, for computing the mass flow across a surface, we only require the normal velocity component. This is typically computed via the dot product $\vec{V} \cdot \hat{n}$. In terms of the velocity potential, we have

$$\vec{V} \cdot \hat{n} \ = \ \nabla \phi \cdot \hat{n} \ = \ \frac{\partial \phi}{\partial n}$$

where the final partial derivative $\partial \phi / \partial n$ is called the *directional derivative* of the potential along the normal coordinate n. The figure illustrates the relations.
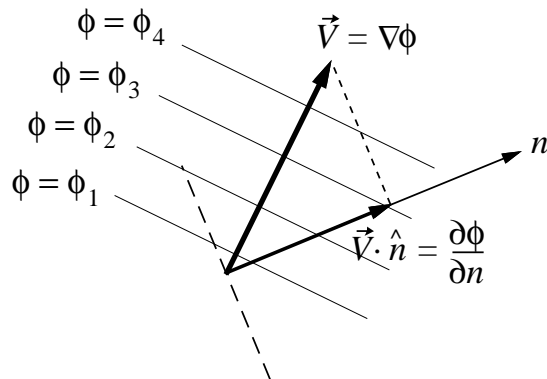
In general, the component of the velocity along any direction can be obtained simply by taking the directional derivative of the potential along that same direction.