Fluids – Lecture 16 Notes

1. Vortex
2. Lifting flow about circular cylinder

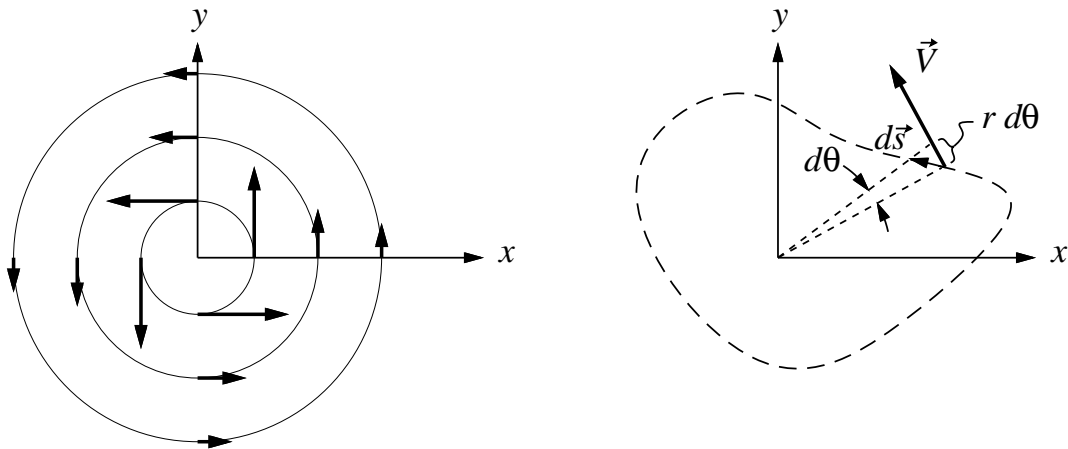Reading: Anderson 3.14 – 3.16

Vortex

Flowfield Definition

A *vortex flow* has the following radial and tangential velocity components

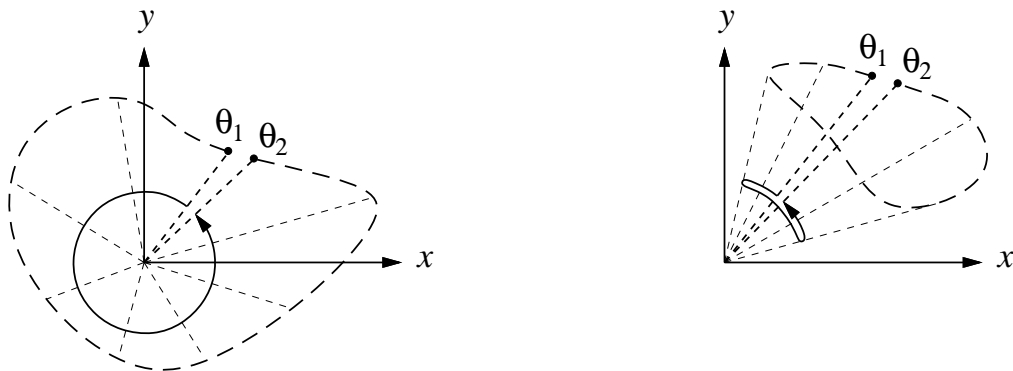$$V_r = 0 \qquad , \qquad V_\theta = \frac{C}{r}$$

where C is a scaling constant. The circulation around any closed circuit is computed as

$$\Gamma \equiv -\oint \vec{V} \cdot d\vec{s} = -\oint V_\theta \, r \, d\theta = -\int_{\theta_1}^{\theta_2} \frac{C}{r} \, r \, d\theta = -C \, (\theta_2 - \theta_1)$$

The integration range $\theta_2 - \theta_1 = 2\pi$ if the circuit encircles the origin, but is zero otherwise.

$$\Gamma = \begin{cases} -2\pi C & , \quad \text{(circuit encircles origin)} \\ 0 & , \quad \text{(circuit doesn't encircle origin)} \end{cases}$$

In lieu of C, it is convenient to redefine the vortex velocity field directly in terms of the circulation of any circuit which encloses the vortex origin.

$$V_\theta = -\frac{\Gamma}{2\pi \, r}$$

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A positive Γ corresponds to clockwise flow, while a negative Γ corresponds to counterclockwise flow.

Cartesian representation
The cartesian velocity components of the vortex are

$$u(x,y) \;=\; \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$$

$$v(x,y) \;=\; -\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$$

and the corresponding potential and stream functions are as follows.

$$\phi(x,y) \;=\; -\frac{\Gamma}{2\pi}\arctan(y/x) \;=\; -\frac{\Gamma}{2\pi}\,\theta$$

$$\psi(x,y) \;=\; \frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2} \;=\; \frac{\Gamma}{2\pi}\ln r$$

Singularity
As with the source and doublet, the origin location $(0,0)$ is called a singular point of the vortex flow. The magnitude of the tangential velocity tends to infinity as

$$V_\theta \;\sim\; \frac{1}{r}$$

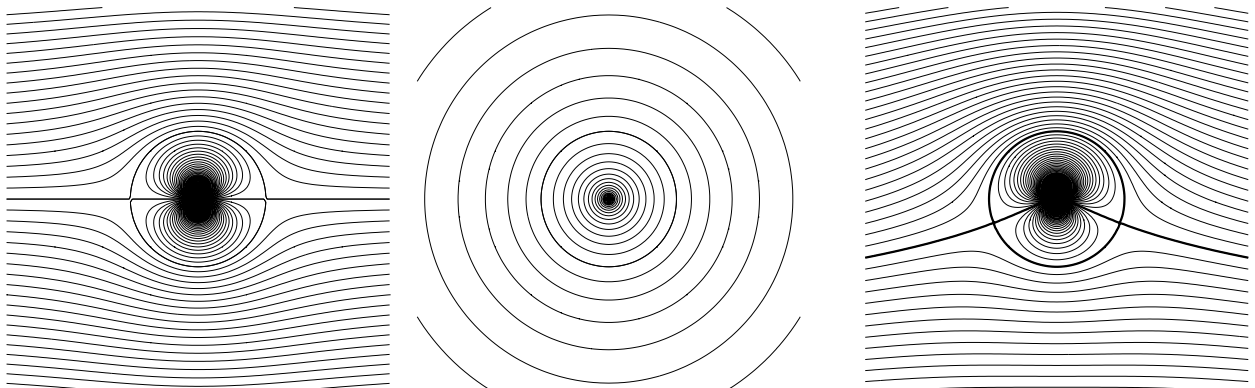Hence, the singular point must be located outside the flow region of interest.

Lifting Flow over Circular Cylinder

Flowfield definition
We now superimpose a uniform flow with a doublet and a vortex.

$$\psi \;=\; V_\infty r \sin\theta\left(1 \;-\; \frac{R^2}{r^2}\right) \;+\; \frac{\Gamma}{2\pi}\ln r$$

This corresponds to the flow about a circular cylinder of radius R as before, but now a top/bottom assymetry is introduced by the vortex.

The radial and tangential velocities can be obtained by differentiating the stream function as follows.

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r}$$

Surface velocities and pressures

On the surface of the cylinder where $r = R$, we have

$$V_r = 0$$

$$V_\theta = -2V_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

The corresponding surface pressure coefficient follows.

$$C_p(\theta) = 1 - \frac{V^2}{V_\infty^2} = 1 - 4\sin^2\theta - \left(\frac{\Gamma}{2\pi V_\infty R}\right)^2 - \left(\frac{2\Gamma}{\pi V_\infty R}\right)\sin\theta \qquad (1)$$
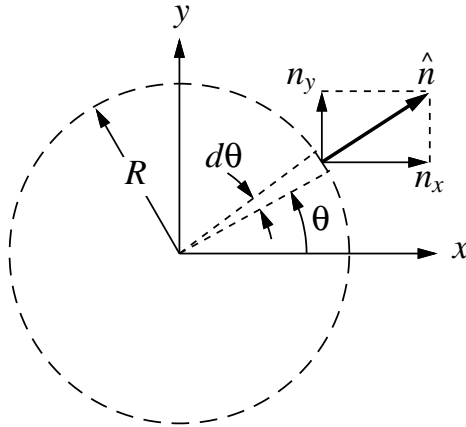
Forces

The resultant force/span is obtained by integrating the pressure forces over the surface of the cylinder.

$$\vec{R}' \equiv D'\,\hat{\imath} + L'\,\hat{\jmath} = \oint -p\,\hat{n}\,dA = \oint -(p - p_\infty)\,\hat{n}\,dA \qquad (2)$$

The constant p_∞ which has been subtracted from p in the integrand does not change the integrated result. This follows from the general identity

$$\oint (\text{constant})\,\hat{n}\,dA = 0$$

which holds for any closed body.

Breaking up the resultant force/span (2) into separate x- and y-components, and dividing by $\frac{1}{2}\rho V_\infty^2\, 2R$, we obtain expressions for the drag and lift coefficients.

$$c_d = \frac{1}{2R}\oint -C_p\, n_x\, dA$$

$$c_\ell = \frac{1}{2R}\oint -C_p\, n_y\, dA$$

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Using the cylinder geometry relations

$$n_x = \cos\theta \quad , \quad n_y = \sin\theta \quad , \quad dA = R\, d\theta$$

and substituting the $C_p(\theta)$ result (1) gives

$$c_d = \frac{1}{2}\int_0^{2\pi} -C_p\,\cos\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\left[-1 + 4\sin^2\theta + \left(\frac{\Gamma}{2\pi V_\infty R}\right)^2 + \left(\frac{2\Gamma}{\pi V_\infty R}\right)\sin\theta\right]\cos\theta\, d\theta$$

$$c_\ell = \frac{1}{2}\int_0^{2\pi} -C_p\,\sin\theta\, d\theta = \frac{1}{2}\int_0^{2\pi}\left[-1 + 4\sin^2\theta + \left(\frac{\Gamma}{2\pi V_\infty R}\right)^2 + \left(\frac{2\Gamma}{\pi V_\infty R}\right)\sin\theta\right]\sin\theta\, d\theta$$

After evaluating the integrals we obtain the final results.

$$c_d = 0$$

$$c_\ell = \frac{\Gamma}{V_\infty R}$$

The equivalent dimensional forms are

$$D' = 0 \tag{3}$$

$$L' = \rho V_\infty \Gamma \tag{4}$$

The result of zero drag (3) is known as *d'Alembert's Paradox*, since it's in direct conflict with the observation that $D' > 0$ for all real bodies in a uniform flow. The explanation is of course that viscosity has been neglected. The lift result (4) is known as the *Kutta-Joukowsky Theorem*, which will turn out to be valid for a 2-D body of any shape, not just for a circular cylinder.

Real Cylinder Flows

Real viscous flow about a circular cylinder at large Reynolds numbers exhibits large amounts of flow separation and drag. Normally the flow is symmetric between top and bottom, and hence the lift is zero. However, if the cylinder has a rotational velocity, the separation is pushed aft on the aft-going side and pushed forward on the forward-going side, resulting in a flow assymetry. This assymetry has an associated nonzero circulation and a corresponding lift. This phenomenon is known as the *Magnus effect*. Although the lift generated by a rotating cylinder can match or exceed the lift achievable by a wing of similar size, the cylinder is not a satisfactory lifting device because of its unavoidably large drag.

boundary layer separation
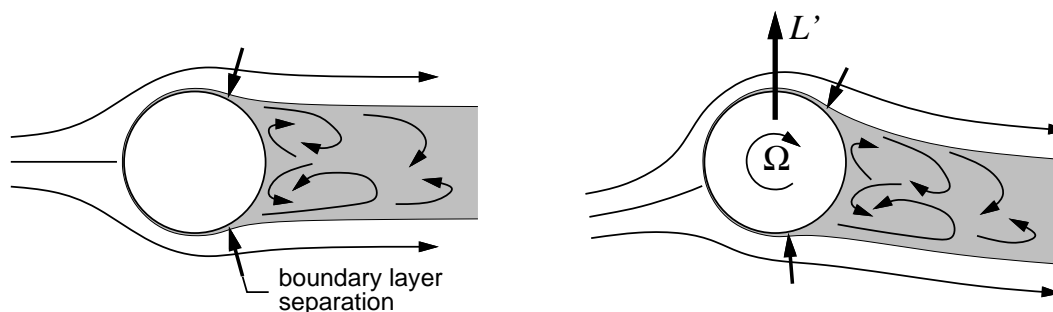
A rotating sphere also exhibits the Magnus effect, and here it has a strong influence on many ball sports. The curveball pitch in baseball, the diving topspin volley in tennis, and the sideways curving flight of a sliced golf ball are all due to the Magnus effect.