UE Fluids Problem 5 Solution Fall '06

a)

| | $\rho_\infty$ [kg/m³] | $\mu_\infty$ [kg/m-s] | $a_\infty$ [m/s] | $V_\infty$ [m/s] | $c_0$ [m] | $Re \equiv \frac{\rho c V}{\mu}$ — | $M_\infty \equiv \frac{V}{a}$ — |
|---|---|---|---|---|---|---|---|
| Model : | 1.225 | $1.78\times10^{-5}$ | 340 | 18 | 0.957 | $1.185\times10^{6}$ | 0.0529 |
| Full size: | 0.429 | $1.67\times10^{-5}$ | 319.6 | 250 | 45 | $289.0\times10^{6}$ | 0.782 |

b) From dimensional analysis, we know that $\begin{cases} C_L = C_L(\alpha, Re, M_\infty) \\ C_D = C_D(\alpha, Re, M_\infty) \end{cases}$

Since $M_\infty = 0.782$ is not in the low-speed flow regime, $M_\infty$ will clearly have an influence on $C_L$. So $C_{L_{model}} \neq C_{L_{fullsize}}$

Similarly, $Re$ always has a significant effect on $C_D$, so clearly $C_{D_{model}} \neq C_{D_{full\,size}}$

$\boxed{\text{The model's } C_L \text{ and } C_D \text{ values cannot be applied to the full size BWB.}}$

c) To match $M_\infty$, we must have $M_1 = M_2$, or $\frac{V_1}{a_1} = \frac{V_2}{a_2}$, or $V_2 = V_1\left(\frac{a_2}{a_1}\right)$ (*)

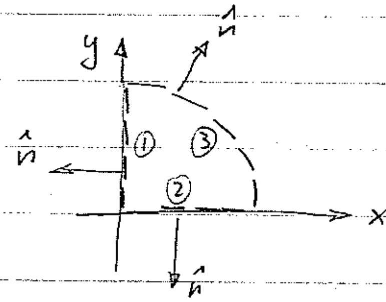
To also match $Re$, we must have $Re_1 = Re_2$ or $\frac{\rho_1 V_1 c_1}{\mu_1} = \frac{\rho_2 V_2 c_2}{\mu_2}$

or $V_2 = V_1\left(\frac{\rho_1 c_1}{\rho_2 c_2}\frac{\mu_2}{\mu_1}\right)$ (**)

Relations (*) and (**) can be compatible only if

$\frac{a_2}{a_1} = \frac{\rho_1 c_1}{\rho_2 c_2}\frac{\mu_2}{\mu_1}$, or $\frac{\rho_1}{\rho_2}\cdot\frac{c_1}{c_2}\cdot\frac{a_1}{a_2}\cdot\frac{\mu_2}{\mu_1} = 1$ which cannot be true

in general, and especially in this case; $0.35\times 47\times 0.94\times\frac{1}{0.94} = 16.45 \neq 1$



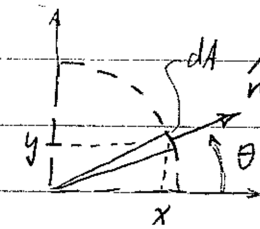

a) $\oint \rho \vec{V}\cdot\hat{n}\, dA = \int_{①} + \int_{②} + \int_{③}$

On ①: $\hat{n} = -\hat{\imath}$, $\vec{V}\cdot\hat{n} = -u = -y$, $dA = dy$

$$\int_{①} \rho \vec{V}\cdot\hat{n}\, dA = \int_0^R \rho(-y)\, dy = -\tfrac{1}{2}\rho R^2$$

On ②: $\hat{n} = -\hat{\jmath}$, $\vec{V}\cdot\hat{n} = -v = -x$, $dA = dx$

$$\int_{②} \rho \vec{V}\cdot\hat{n}\, dA = \int_0^R \rho(-x)\, dx = -\tfrac{1}{2}\rho R^2$$




On ③: $\hat{n} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$, $x = R\cos\theta$, $y = R\sin\theta$, $dA = R\, d\theta$

$$\vec{V} = \hat{\imath}u + \hat{\jmath}v$$

$$\vec{V} = \hat{\imath}y + \hat{\jmath}x$$

$$\vec{V} = \hat{\imath}R\sin\theta + \hat{\jmath}R\cos\theta$$

$$\vec{V}\cdot\hat{n} = R\sin\theta\cos\theta + R\cos\theta\sin\theta = 2R\sin\theta\cos\theta = R\sin 2\theta$$

$$\int_{③} \rho(\vec{V}\cdot\hat{n})\, dA = \int_0^{\pi/2} \rho R\sin 2\theta \cdot R\, d\theta = \rho R^2 \int_0^{\pi/2} \sin 2\theta\, d\theta = \rho R^2\left(-\tfrac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/2} = \rho R^2$$

$$\therefore \boxed{\oint \rho(\vec{V}\cdot\hat{n})\, dA = -\tfrac{1}{2}\rho R^2 - \tfrac{1}{2}\rho R^2 + \rho R^2 = 0}$$

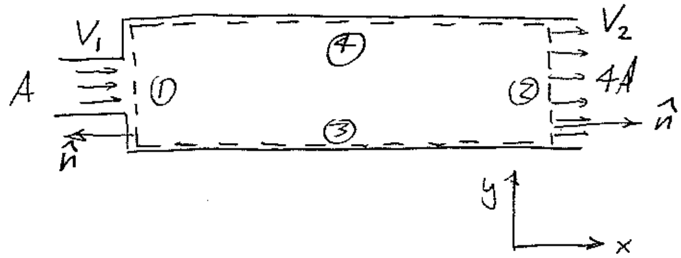
b) For <u>any</u> C.V., we can evaluate $\oint \rho(\vec{V}\cdot\hat{n})\, dA = \iint \rho(\nabla\cdot\vec{V})\, dx\, dy$ via Gauss

For this particular flow, $\boxed{\nabla\cdot\vec{V} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = \underline{0}}$

$$\therefore \oint \rho(\vec{V}\cdot\hat{n})\, dA = \iint \rho\cdot 0\cdot dx\,dy \overset{\checkmark}{=} 0$$

mass is conserved

UE Fluids        Problem 7 Solution        Fall '06




a) Apply $\oint \rho(\vec{V}\cdot\hat{n})\,dA = \int_{①} + \int_{②} + \int_{③} + \int_{④} = 0$

On ①: $-\rho V_1 A$ since $\vec{V}\cdot\hat{n} = -V_1$

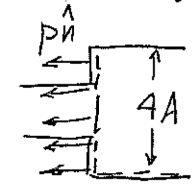
②: $\rho V_2 \cdot 4A$ since $V\cdot\hat{n} = +V_2$

③ $0$ since $\vec{V}\cdot\hat{n} = 0$

④ $+\ 0$ since $\vec{V}\cdot\hat{n} = 0$

$-\rho V_1 A + \rho V_2 4A = 0 \longrightarrow \boxed{V_2 = \frac{1}{4}V_1}$

b) Apply $\oint \rho(\vec{V}\cdot\hat{n})\,u\,dA + \oint p\, n_x\, dA = 0$ (x-component of momentum eq'n)




On ① $(-\rho V_1)V_1 A \quad - p_1 \cdot 4A$ ($p_1$ acts on entire face ①)

② $(\rho V_2)V_2 \cdot 4A \quad + p_2 \cdot 4A$

③ $0 \quad + 0$

④ $0 \quad + 0$

$$-\rho V_1^2 A + \rho V_2^2 4A + (p_2 - p_1)\cdot 4A = 0$$

using $V_2 = \frac{1}{4}V_1$: $\quad -\rho V_1^2 A + \frac{1}{4}\rho V_1^2 A + (p_2 - p_1)4A = 0$

$$-\frac{3}{4}\rho V_1^2 + (p_2 - p_1)\cdot 4 = 0$$

$$\boxed{p_2 - p_1 = \frac{3}{16}\rho V_1^2}$$

PAL
9/14/0

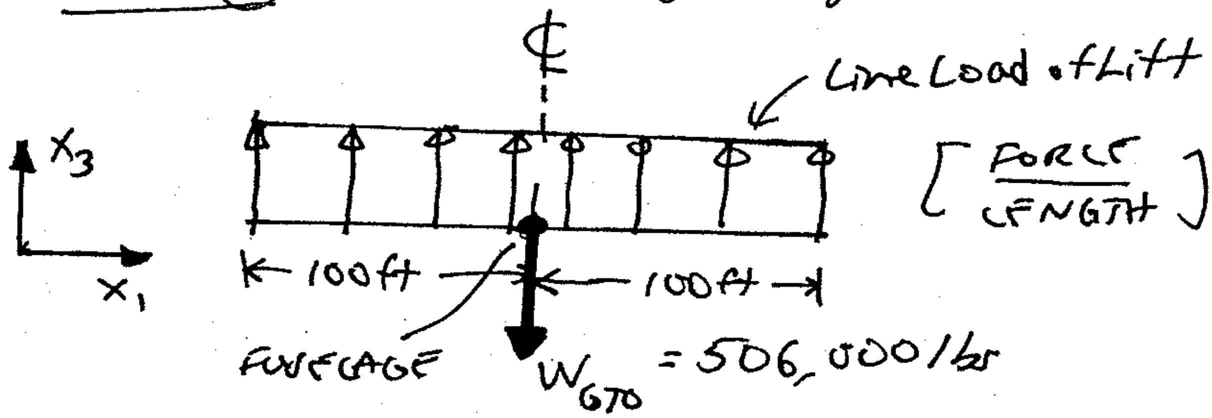
Unified Engineering Problem Set
Week 3 Fall, 2006

SOLUTIONS

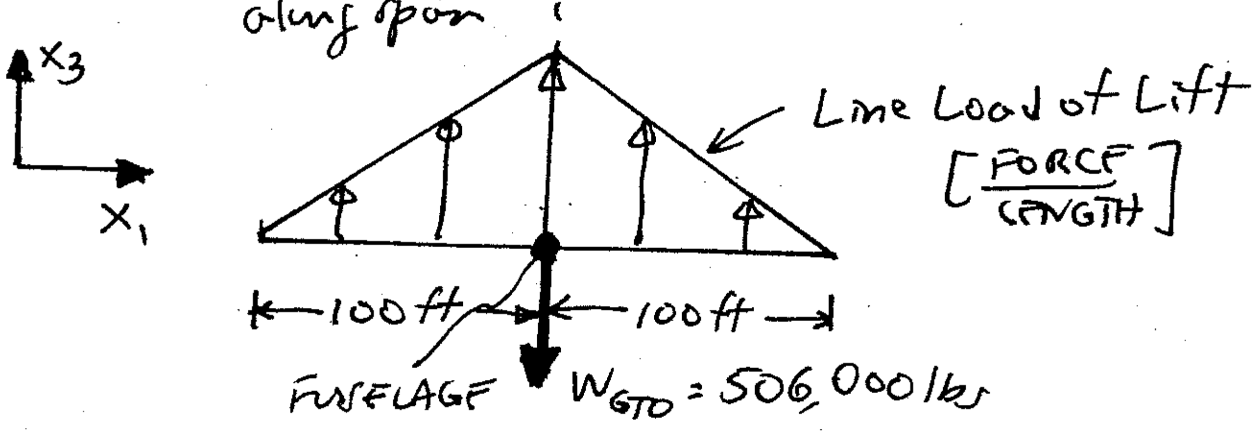
M3-1

(a) The configuration is the same basic one for the three lift models except the loading changes. So:

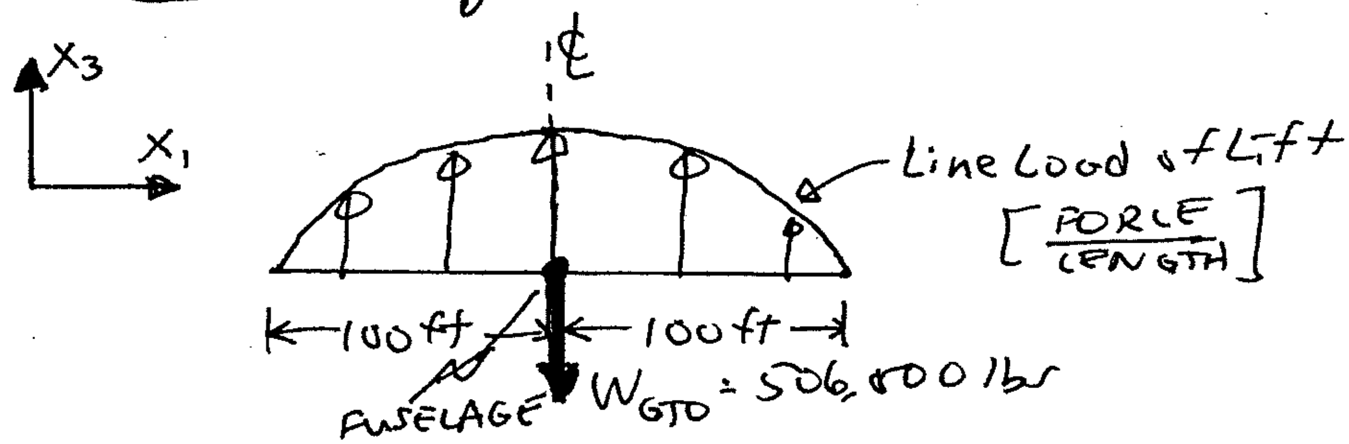
Model (1): constant along the span




Model (2): linear along span



Model (3): quadratic along span




(b) For this we need to consider each lift model and determine values to get an expression for the variation. Place the origin of the $x_1 - x_3$ system at the root ($x_1 = 0$). Here are the steps.

Step 1: Each wing must carry half the weight of the airplane through the counteracting action of the force of lift for the plane to be in level flight. Generally can express lift varying along the wing as:

$$\text{Lift} = f(x_1)$$

for one wing:

$$\frac{W_{GTO}}{2} = 253{,}000 \text{ lbs} = \int_{root}^{tip} f(x_1)\, dx_1$$

for a 100 ft wing: at root, $x_1 = 0$ ft

at tip, $x_1 = 100$ ft

Step 2: Get an expression for the lift as a function of $x_1$. This changes for each case:

(1) it is constant and call it $L_0$

$$\boxed{f_{①}(x_1) = L_0}$$

(2) the lift is maximum at the root (call this $L_R$) and goes to zero at the tip with a linear variation. The general expression is:

$$f_2(x_1) = m x_1 + b$$

First find the slope. This is the ratio of the change in lift to the change in distance:

$$m = \frac{(L_R - 0)}{(0ft - 100ft)} \Rightarrow m = \frac{-L_R}{100ft}$$

To find the constant, b, note that the lift is $L_R$ at the root ($x_1 = 0$). Using this gives

$$b = L_R$$

Finally: $$\boxed{f_{②}(x_1) = -\frac{L_R}{100ft} x_1 + L_R}$$

(NOTE: This is only valid for $x_1 > 0$. For $x_1 < 0$, we need to change the sign of the slope)

It is also worthwhile to check this with the other value we know -- at the tip ($x_1 = 100ft$, $L = 0$)

$$L(100ft) \stackrel{?}{=} -\frac{L_R}{100ft}(100ft) + L_R = 0 \quad \checkmark \text{ checks}$$

(3) We begin with the equation

$$\text{Lift} = f(x_1) = b - ax^2$$

Start by finding the constant, $b$, by noting the maximum occurs at the root $(x_1 = 0)$. Again call this $L_R$ and this gives

$$b = L_R$$

Then use the other point where we know the value of zero at the tip $(x = 100\,ft)$ to work to find $a$. This gives:

$$\cancel{f(100ft)} = 0 = L_R - a(100\,ft)^2$$

$$\Rightarrow a = \frac{L_R}{(100\,ft)^2} = \frac{L_R}{10{,}000\,ft^2}$$

Putting this together:

$$\boxed{f_3(x_1) = L_R\left(1 - \frac{x_1^2}{10{,}000\,ft^2}\right)}$$

<u>Note</u> that the units of $x_1$ are [ft] so the latter part checks for units.

Now proceed to ...

<u>Step 3</u> : Solve using the equations

<u>Model (1)</u> : $253{,}000\,lbs = \int_{0ft}^{100ft} L_0\,dx_1$

$$\Rightarrow 253{,}000\,lbs = L_0\,x_1\Big]_{0ft}^{100ft}$$

$$\Rightarrow \boxed{L_0 = 2{,}530 \ \frac{lbs}{ft}}$$ everywhere along the wing

NOTE: This is in intensity [lbs/ft] since one must multiply by a length to get a force

Model ②: $253{,}000\ lbs = \int_{0ft}^{100ft} \left\{ \left( \frac{-L_R}{100ft} \right) x_1 + L_R \right\} dx_1$

$$\Rightarrow 253{,}000\ lbs = \left( \frac{-L_R}{100ft} \right) \frac{x_1^2}{2} \Bigg]_{0ft}^{100ft} + L_R x_1 \Bigg]_{0ft}^{100ft}$$

So:

$$253{,}000\ lbs = -L_R(50ft) + L_R(100ft)$$
$$= L_R(50ft)$$

$$\Rightarrow \boxed{L_R = 5{,}060\ lbs/ft}$$ at the wing root $(x_1 = 0)$

NOTE: units of intensity as before ✓ check

Model ③: $253{,}000\ lbs = \int_{0ft}^{100ft} L_R \left( 1 - \frac{x_1^2}{10{,}000ft^2} \right) dx_1$

$$\Rightarrow 253{,}000\ lbs = L_R \left( x_1 - \frac{x_1^3}{30{,}000ft^2} \right) \Bigg]_{0ft}^{100ft}$$

So:

$$253{,}000\ lbs = L_R \left( 100\ ft - \frac{1{,}000{,}000\ ft^3}{30{,}000\ ft^2} \right)$$

$$= L_R(100\ ft - 33.3\ ft)$$

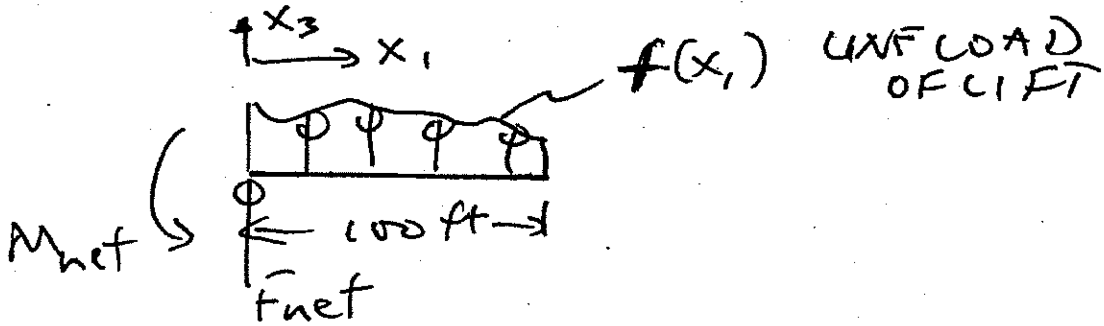
$\Rightarrow$ 253,000 lbs = $L_R$ (66.7 ft)

$\Rightarrow$ $\boxed{L_R = 3{,}793 \text{ lbs/ft}}$ at the wing root ($x_1 = 0$)

NOTE: units of intensity as before ✓check

(c) An equipollent force system is the net sum of forces and moments on the system. Considering this at the root gives us an idea of the loading the wing must take at the attachment to the fuselage based on the different models of lift.

We can draw this as "cutting off" the wing and looking at the net force and net moment at the root:




To determine the net force and net moment, $f(x_1)$ needs to be integrated:

Net force: $F_{net} = \int_{0ft}^{100ft} f(x_1)\,dx_1$

Net moment: $M_{net} = \int_{0}^{100ft} f(x_1)\,\underbrace{x_1}_{\text{moment arm}}\,dx_1$

with the moment acting counterclockwise (+ccw)

Do this for each of the three models:

<u>Model ①</u>: $F_{net_{①}} = \int_{0ft}^{100ft} \left(2530 \frac{lbs}{ft}\right) dx_1$

$= \left(2530 \frac{lbs}{ft}\right) x_1 \Big]_{0ft}^{100ft}$

$+\uparrow \Rightarrow \boxed{F_{net_{①}} = 253{,}000 \text{ lbs}}$

NOTE: Balances the gravitational weight (half) of it unit

$M_{net_{①}} = \int_{0}^{100ft} \left(2530 \frac{lbs}{ft}\right) x_1\,dx_1$

$= \left(2530 \frac{lbs}{ft}\right) \frac{x_1^2}{2} \Big]_{0}^{100ft}$

$= \left(2530 \frac{lbs}{ft}\right) \frac{10{,}000 ft^2}{2}$

$\circlearrowleft^{+} \Rightarrow \boxed{\begin{aligned} M_{net_{①}} &= 12{,}650{,}000 \text{ ft·lbs} \\ &= 12.65 \times 10^6 \text{ ft·lbs} \end{aligned}}$

NOTE: Proper unit for a moment

Model ②: $f_{net_②} = \int_0^{100ft} \left\{ \left(-50.6 \frac{lbs}{ft^2}\right) x_1 + 5060 \frac{lbs}{ft} \right\} dx_1$

$$= -\left(50.6 \frac{lbs}{ft^2}\right) \frac{x_1^2}{2} + \left(5060 \frac{lbs}{ft}\right) x_1 \Big]_{0ft}^{100ft}$$

$$= -253{,}000 \, lbs + 506{,}000 \, lbs$$

$$+\uparrow \Rightarrow \boxed{F_{net_②} = 253\,000 \; lbs}$$

NOTE: as before or it must be to balance $W_{GTO}$

$$M_{net_②} = \int_0^{100ft} \left\{ \left(-50.6 \frac{lbs}{ft^2}\right) x_1^2 + \left(5060 \frac{lbs}{ft}\right) x_1 \right\} dx_1$$

$$= \left(-50.6 \frac{lbs}{ft^2}\right) \frac{x_1^3}{3} + \left(5060 \frac{lbs}{ft}\right) \frac{x_1^2}{2} \Big]_{0ft}^{100ft}$$

$$= -16.87 \times 10^6 \, lbs \cdot ft + 25.3 \times 10^6 \, lbs \cdot ft$$

$$\Rightarrow \boxed{M_{net_②} = 8.43 \times 10^6 \, lbs \cdot ft}$$

Model ③: $F_{net_③} = \int_0^{100ft} \left\{ 3793 \frac{lbs}{ft} \left(1 - \frac{x_1^2}{10{,}000ft^2}\right) \right\} dx_1$

$$= \left(3793 \frac{lbs}{ft}\right) \left(x_1 - \frac{x_1^3}{30{,}000ft^2}\right) \Big]_0^{100ft}$$

$$= \left(3793 \frac{lbs}{ft}\right) (100ft - 33.3ft)$$

giving: $F_{net_{3}} = \left(3793 \frac{lbs}{ft}\right)(66.7\ ft)$

$$\Rightarrow \boxed{F_{net_{3}} = 252{,}993\ lbs} \quad +\uparrow$$

NOTE: slight round-off error such that this to significant digits equates to $W_{GTO}$ of 253,000 lbs

$$M_{net_{3}} = \int_0^{100ft} \left\{3793 \frac{lbs}{ft}\left(x_1 - \frac{x_1^3}{10{,}000 ft^2}\right)\right\} dx_1$$

$$= \left(3793 \frac{lbs}{ft}\right)\left(\frac{x_1^2}{2} - \frac{x_1^4}{40{,}000 ft^2}\right)\Big]_0^{100ft}$$

$$= \left(3793 \frac{lbs}{ft}\right)\left(5000\ ft^2 - 2500\ ft^2\right)$$

$$= \left(3793 \frac{lbs}{ft}\right)\left(2500\ ft^2\right)$$

$$\Rightarrow \boxed{M_{net_{3}} = 9.48 \times 10^6\ ft\cdot lbs}$$

EXTRA COMMENTS

Summarizing all these and taking a look:

$F_{net_{1}} = 253{,}000\ lbs$ $\quad M_{net_{1}} = 12.65 \times 10^6\ ft\cdot lbs$

$F_{net_{2}} = 253{,}000\ lbs$ $\quad M_{net_{2}} = 8.43 \times 10^6\ ft\cdot lbs$

$F_{net_{3}} = 253{,}000\ lbs$ $\quad M_{net_{3}} = 9.48 \times 10^6\ ft\cdot lbs$

NOTE THAT

- They all have the same units for forces [lbs] and moments [ft. lbs] as they should
- All 3 models have equipollent forces equal to half the gross takeoff weight as they must to support the plane
- The moments are on the same order of magnitude (approx $10^6$ to $10^7$) but with variation depending upon the distribution of the lift

This latter fact becomes important as we look at a structure called a "beam" in the spring term -- a simple model for a wing

M3.2

(a) To draw the free body diagrams replace the person and the mass with the loads created and the supports with the reaction forces.

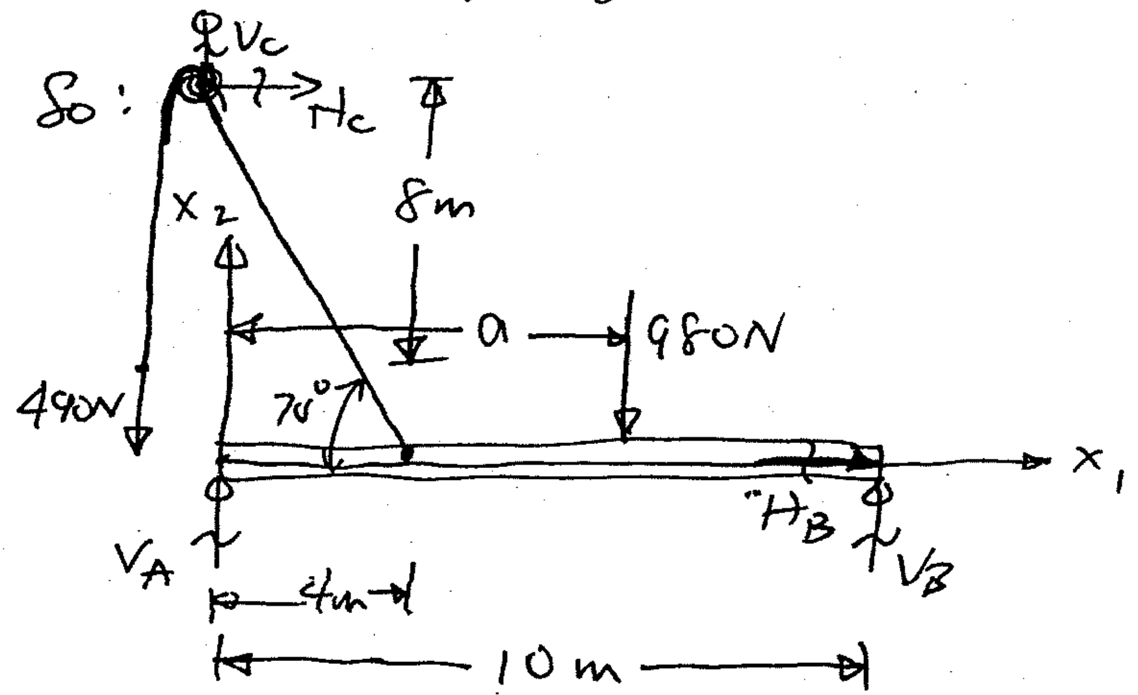
The masses are converted into loads due to gravity:

$$\underset{[N]}{\text{Force}} = \underset{[kg]}{\text{mass}} \times 9.8\ m/s^2$$

$\Rightarrow$ for person: weight = W = 980 N

for mass of pulley = 490 N

So:



where:

A is the roller support at $x_1 = 0$ with vertical reaction

B is the pin support at $x_1 = 10$ m with horizontal and vertical reactions

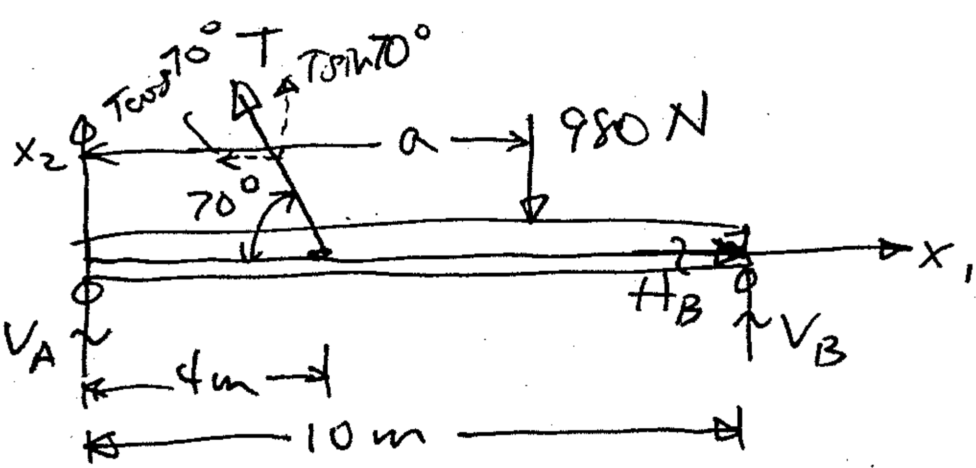
C is the pin support for the pulley with vertical and horizontal reactions

a is the $x_1$-location of the person

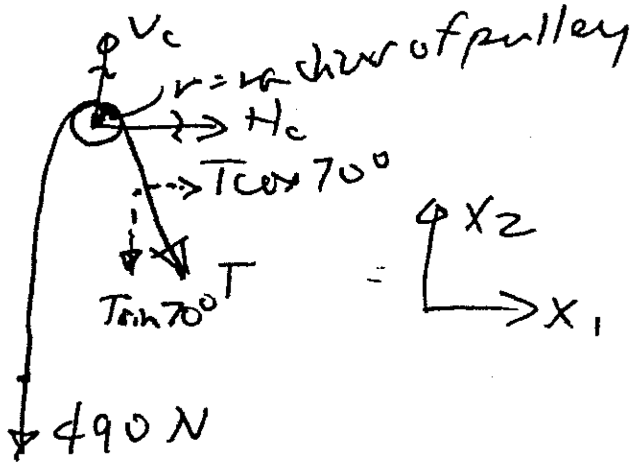
the mass acts at the far side of the pulley

It is useful in this problem to split this into two "subsystems" each with their own Free Body Diagram. Why? We have two structures: the beam and the cable. We "split" this by replacing the beam/cable connection by a force representing the tension in the cable T. Thus:

Beam system



## Pulley system




(b) For it to be possible to determine the reactions, the system must be statically determinate (or we need constitutive relations). In this case, that means each of these subsystems must be statically determinate.

→ Beam system:

\# of dof = 3

- lateral in $x_1$
- lateral in $x_2$
- rotation in $x_1$-$x_2$ plane

\# of reactions = 3 ($V_A$, $H_A$, $H_B$)

(# of dof) = (# of reactions) ⇒ Statically Determinate

→ Pulley system:

# of dof = 2
- lateral in $x_1$
- lateral in $x_2$

(NOTE: A cable is not rigid, so it cannot rotate as a rigid body)

# of reactions = 2 ($H_c$, $V_c$)

(# of dof) = (# of reactions) ⇒ Statically Determinate

What about T?

This is a transmission across two systems, so there will be "an equation" as well as "across the cable....

So apply equilibrium within each of the subsystems:

→ Beam system

$\Sigma F_1 = 0 \quad \overset{+}{\rightarrow} \Rightarrow H_B - 0.34T = 0 \qquad (1)$

$\Sigma F_2 = 0 \quad \uparrow + \Rightarrow V_A + V_B + 0.94T - 980N = 0 \ (2)$

$\Sigma M = 0 \quad \circlearrowleft + \Rightarrow (0.94T)(4m) - (980N)a + V_B(10m) = 0 \ (3$

(take about $x_1 = 0$)

giving $3.76T\,[m] - (980N)a + V_B(10m) = 0$ (

→ Pulley system

$$\Sigma F_1 = 0 \quad \overset{+}{\rightarrow} \quad \Rightarrow \quad H_C + 0.34T = 0 \qquad (4)$$

$$\Sigma F_2 = 0 \quad \uparrow + \quad \Rightarrow \quad V_C - 0.94T - 490N = 0 \qquad (5)$$

$$\Sigma M = 0 \quad \circlearrowleft + \quad \Rightarrow \quad Tr - (490N)r = 0 \qquad (6)$$

(about pulley pin) where: r = pulley radius

Rearranging and summarizing the equations:

$$H_B = 0.34\,T \qquad (1)$$

$$V_A + V_B = 980N - 0.94T \qquad (2)$$

$$10V_B = (980\tfrac{N}{m})a - 3.76T \qquad (3)^*$$

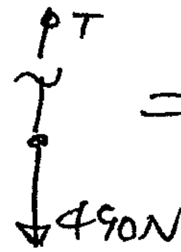
$$H_C = -0.34T \qquad (4)$$

$$V_C = 490N + 0.94T \qquad (5)$$

$$T = 490N \qquad (6)^{**}$$

(*NOTE 1: the unit [m] has been canceled on both sides of the equation, so the distance a to the person must be in [m]

**NOTE 2: can also get this by noting that a cable transmits axial force only, so the tension in the cable must be equal to the weight hanging at its end. So by cutting the cable near the end:




$\Rightarrow T = 490N$

Using (6) to get $T$ in the other equation results in the following for the reactions.

$$\boxed{\begin{aligned} H_c &= -167\ N \\ V_c &= 951\ N \\ H_B &= 167\ N \\ V_B &= \left(98\ \tfrac{N}{m}\right)a - 184\ N \\ V_A &= 703\ N - \left(98\ \tfrac{N}{m}\right)a \end{aligned}}$$

One can check this by performing an equilibrium assessment on the overall free body diagram first drawn:

$$\Sigma F_1 = 0 \ \overset{+}{\rightarrow} \ \Rightarrow H_c + H_B \overset{?}{=} 0 \qquad \checkmark \ \underline{\text{yes}}$$

$$\Sigma F_2 = 0 \ \uparrow+ \ \Rightarrow V_c - 490\ N + V_A - 980\ N + V_B \overset{?}{=} 0$$

$$951\ N - 490\ N + 703\ N - \left(98\ \tfrac{N}{m}\right)a - 980\ N - 184\ N + 98\left(\tfrac{N}{m}\right)a \overset{?}{=} 0$$

$\checkmark$ <u>yes</u>

A similar thing can be done with moments

(e) If one wants $V_A$ to be zero, all the equilibrium equations stay the same except (2) becomes

$$V_B = 980\,N - 0.94\,T$$

use this with (3):

$$V_B = (98\,\tfrac{N}{m})a - 0.376\,T$$

Putting the two together gives:

$$980\,N - 0.94\,T = (98\,\tfrac{N}{m})a - 0.376\,T$$

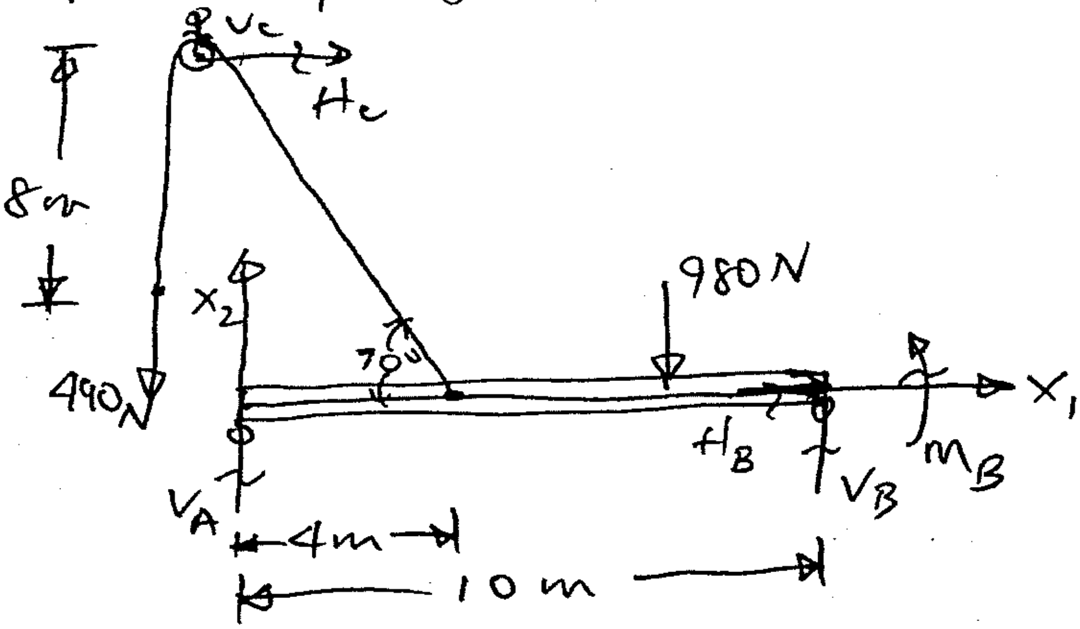
$$\Rightarrow 980\,N - (98\,\tfrac{N}{m})a = 0.564\,T$$

Recalling that $T = \text{mass} \times 9.8\,m/s^2$ then a mass can be determined for any location, $a$ of the person.

$\Rightarrow \boxed{\text{Yes}}$ it can be done

<u>NOTE</u>: The reaction is still there, it is just zero. Forces balance at this point. One can also think of this as the cable providing the other vertical support

(d) If a clamped support replaces a pin, a moment reaction is added. Thus, the free Body Diagram becomes:




The extra moment reaction, $M_B$, is thus added to the Beam (sub)system. This adds this term to equilibrium equation (3) ($\Sigma M = 0$ for Beam system) giving:

$$(0.94T)(4m) - (980\ N)a + V_B(10\ m) + M_B = 0 \quad (3'$$

There is now an additional reaction giving 4 reactions with 3 degrees of freedom

$\Rightarrow$ $\boxed{\text{Statically indeterminate}}$