Draw circuit for $\Gamma_3$:

Integrating on circuits $C_{in}$ and $C_{out}$:

$$\Gamma_1 = -\oint_{C_{in}} \mathbf{V} \cdot ds$$

$$\Gamma_2 = -\oint_{C_{out}} \mathbf{V} \cdot ds$$

$$\Gamma_3 = -\oint_{C_{out}} \mathbf{V} \cdot ds + \oint_{C_{in}} \mathbf{V} \cdot ds = \Gamma_2 - \Gamma_1$$

This part is + because it’s counterclockwise.

Also, there’s no $C_{cut}$ contribution to $\Gamma_3$ since the two parts cancel.

Now we note that $\Gamma_3$’s circuit $C_3$ is “reducible” since the airfoil is topologically outside it.

Hence, Stokes’ Theorem can be applied:

$$\Gamma_3 = \oint_{C_3} \mathbf{V} \cdot ds = \iint_{C_3} \nabla \times \mathbf{V} \cdot dxdy = \iint_{C_3} 0 \cdot dxdy = 0$$

$$\Rightarrow \Gamma_3 = \Gamma_2 - \Gamma_1 = 0$$

or

$$\Gamma_2 = \Gamma_1$$
a) \[ u = \frac{\partial y}{\partial y} = x^2 \quad \Rightarrow \quad \psi = x^2 y + \phi(x) \]

To be consistent, we must have \[ \phi(x) = g(y) = \phi \]

\[ -\gamma = \frac{\partial y}{\partial x} = 2xy \quad \Rightarrow \quad \gamma = x^2 y + g(y) \]

\[ \psi(x, y) = x^2 y + \phi \]

\( \phi \) is arbitrary, and can be set \( \phi = 0 \) (no effect on velocities, since it disappears when determining \( u = \frac{\partial y}{\partial y}, \gamma = -\frac{\partial y}{\partial x} \)).

b) Set \( \gamma = \text{constant} \), or \( x^2 y = \text{const.} \), or \( y = \frac{\text{"const"}}{x^2} \)

c) Easiest to use stream function:

\[ \sqrt{\psi'} = \psi_2 - \psi_1 = (x^2 y)_2 - (x^2 y)_1 = 2^2 \cdot 3 - 0 = 12 \]

OR...

Longer way: Integrate \( \psi' = \int \nabla \cdot \hat{n} \, dA \)

On line:

\[ dA = \sqrt{dx^2 + dy^2} = dx \cdot \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = dx \sqrt{1 + \left( \frac{3}{2} \right)^2} \]

\[ dA = dx \cdot \frac{\sqrt{13}}{2} \]

\[ \hat{n} = \frac{\left[ 3 \hat{i} - 2 \hat{j} \right]}{\sqrt{13}} = \frac{1}{\sqrt{13}} \left( 3 \hat{i} - 2 \hat{j} \right) \]

\[ y = \left( \frac{3}{2} \right) x \]

\[ u = x^2, \quad \gamma = -2xy = -2x \left( \frac{3}{2} x \right) = -3x^2 \]

\[ \int \nabla \cdot \hat{n} \, dA = \left( x^2 \frac{3}{\sqrt{13}} + 3x^2 \frac{2}{\sqrt{13}} \right) \cdot \frac{\sqrt{13}}{2} = \frac{9}{2} x^2 \quad \text{when!} \]

\[ \int_0^2 \frac{9}{2} x^2 \, dx = \left. \frac{9}{2} \cdot \frac{1}{3} x^3 \right|_0^2 = 12 \quad \text{same result.} \]
a) Parallel exit flow: Static pressure constant across streamlines:
\[ \rho V_1 A_1 = \rho V_{\text{test}} A_{\text{test}} \]
\[ \frac{V_1}{V_{\text{test}}} = \frac{A_{\text{test}}}{A_1} = \frac{1}{36} \frac{V_{\text{test}}}{A_1} \]

b) Constant mass flow along tunnel:
\[ \frac{V_1}{V_{\text{test}}} = \frac{A_{\text{test}}}{A_1} \]

\[ P_1 - P_{\text{test}} = 0 \]

\[ P_1 = P_{\text{test}} + \frac{1}{2} \rho V_1^2 \]

\[ P_{\text{test}} + \frac{1}{2} \rho V_{\text{test}}^2 = P_{\text{test}} + \frac{1}{2} \rho V_{\text{test}}^2 \]

\[ P_1 = P_{\text{test}} + \frac{1}{2} \rho (V_{\text{test}}^2 - V_1^2) = P_{\text{test}} + \frac{1}{2} \rho V_{\text{test}}^2 \left(1 - \frac{1}{36}\right) \]

\[ P_1 = 10^5 \text{ Pa} + \frac{1}{2} \cdot 1.2 \text{ kg/m}^3 \cdot (50 \text{ m/s})^2 \left(1 - \frac{1}{36}\right) = 101499 \text{ Pa} \]

c) Smooth flow: Apply Bernoulli:
\[ P_1 + \frac{1}{2} \rho V_1^2 = P_{\text{test}} + \frac{1}{2} \rho V_{\text{test}}^2 \]

\[ P_{\text{test}} + \frac{1}{2} \rho (V_{\text{test}}^2 - V_1^2) = P_{\text{test}} + \frac{1}{2} \rho V_{\text{test}}^2 \left(1 - \frac{1}{36}\right) \]

\[ P_1 = 10^5 \text{ Pa} + \frac{1}{2} \cdot 1.2 \text{ kg/m}^3 \cdot (50 \text{ m/s})^2 \left(1 - \frac{1}{36}\right) = 101499 \text{ Pa} \]

\[ P_1 = 10^5 \text{ Pa} + \frac{1}{2} \cdot 1.2 \text{ kg/m}^3 \cdot (50 \text{ m/s})^2 \left(1 - \frac{1}{36}\right) = 101499 \text{ Pa} \]

\[ \rho \frac{V_1^2}{A_1} + F_x = 0 \]

\[ \rho \frac{V_{\text{test}}^2}{A_{\text{test}}} + F_x = 0 \]

\[ F_x = -\rho \frac{V_{\text{test}}^2}{A_{\text{test}}} \]

\[ \text{Thrust is simply } -F_x \]

\[ A_{\text{test}} = 1 \text{ ft}^2 = \frac{1}{3.28^2} \text{ m}^2 = 0.09295 \text{ m}^2 \]

\[ \text{Thrust} = \rho \frac{V_{\text{test}}^2}{A_{\text{test}}} = 1.2 \cdot (50)^2 \cdot 0.09295 = 278.85 \text{ N} \]
a) \( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \begin{cases} \frac{2c^2}{s} (1 - \frac{y}{s}), & y \leq s \\ 0, & y > s \end{cases} \)

For \( y < s \): \( u = \frac{\partial y}{\partial y} = C \left[ 2 \frac{y}{s} - \left( \frac{y}{s} \right)^2 \right] \rightarrow \psi = CS \left[ \frac{y^2}{s} - \frac{1}{3} \frac{y^3}{s^2} \right] + \phi_1 \)

For \( y > s \): \( u = \frac{\partial y}{\partial y} = C \rightarrow \psi = C y + \phi_2 \)

\( \psi \) is OK to leave in the 2-piece definition. However, we can set \( \phi_2 \) to make the pieces continuous at \( y = s \)

Equate at \( y = s \): \( \psi = \psi_1 \)

\( CS \left[ 1 - \frac{1}{3} \right] + \phi_1 = CS + \phi_2 \rightarrow \phi_2 = \phi_1 - \frac{1}{3} CS \)

b) For \( y > s \), we can write \( \phi = C x \), \( \vec{V} = \nabla \phi = C \vec{i} + 0 \vec{j} \)

For \( y < s \), we cannot find any \( \phi(x,y) \) which gives \( \nabla \phi = u \vec{i} + 0 \vec{j} \)

This is because \( \vec{S} \neq 0 \). Flow is rotational, and so \( \phi \) does not exist.
Unified Engineering Problem Set
Week 5 Fall, 2006

SOLUTIONS

(a) Overall there are three pin supports which provide reactions for this system. Draw the free body diagram (labeling the three supports as A, B, C).

F = M [mass/length]
One can look at this as four different subsystems: the bar and each of the three springs acting on it. [force/length]

\[
F = \frac{Mg}{L}
\]

[mass/length] \( \frac{q}{x_3} \)

The first spring provides a moment. Springs 2 and 3 provide vertical forces. The three or moments on the bar must be equal and opposite to those on the springs. [Sum to zero for the overall system]

Looking at the springs:

Spring 1

\[
M = -M_s_1 \times \frac{q}{x_3}
\]

Spring 2

\[
\phi V_{s_2} \quad H_B \quad V_B
\]

Spring 3

\[
\phi V_{s_3} \quad H_C \quad V_C
\]
(b) This is a 2-D system and has a potential for 3 degrees of freedom of motion:

- lateral in \( x_2 \)
- lateral in \( x_3 \)
- rotational in \( x_2 - x_3 \) plane (about \( x_1 \))

There are 6 reactions:

\[ \# \text{ reactions} \geq \# \text{ d.o.f.} \]

\[ \Rightarrow \text{ statically indeterminate} \]

(c) There are 2 parts that make up the compatibility of displacement here. The primary is that the bar is rigid so the displacement in \( x_3 \) must be a linear function of \( x_2 \):

\[ \delta_3 = mx_2 + b \]

Displacement in \( x_3 \) (set \( x_2 = 0 \) at left end)

Now look at further details. The bar is pinned with the torsional spring at \( x_2 = 0 \) \( \Rightarrow b = 0 \)
With the linear variation and the pin at \( x_2 = 0 \), the displacement at \( x_2 = L \) must be twice that at \( x_2 = \frac{L}{2} \)

\[
\delta_3 = mx_2 \quad \text{(1)}
\]

\[
\Rightarrow \delta_3 (x_2 = L) = mL \quad \text{(1a)}
\]

\[
\delta_3 (x_2 = \frac{L}{2}) = m \frac{L}{2} \quad \text{(1b)}
\]

This does not yet give \( \delta_3 \).

One can also get an expression for the rotation at \( x = 0 \)

\[
\Rightarrow \theta = \delta_3 (x = L) = mL
\]

\[
\Rightarrow \theta = \sin^{-1}(m)
\]

for small displacement

\[
\Rightarrow \theta = m \quad \text{(2)}
\]

Check at \( x_2 = \frac{L}{2} \)

\[
\frac{L}{2} \sin \theta = \delta_3 (x = L) = mL
\]

\[
\Rightarrow \theta = \sin^{-1}(m) \quad \text{check}
\]
(d) Now we call our equation
→ first equilibrium. Doing this for the bar subsystem:
\[ \Sigma F_{x_2} = 0 \quad \Rightarrow \quad H_A = 0 \]
\[ \Sigma F_{x_3} = 0 \quad \Rightarrow \quad V_A - V_{s_2} - V_{s_3} - MLg = 0 \quad (3) \]
\[ \Sigma M_A = 0 \quad \Rightarrow \quad -M_{s_1} - V_{s_2} \frac{y_2}{2} - V_{s_3}L - \int_0^L Mg x_2 dx_2 = 0 \]
\[ \Rightarrow -M_{s_1} - V_{s_2} \frac{y_2}{2} - V_{s_3}L - \frac{MgL^2}{2} = 0 \quad (4) \]
for Spring 1:
\[ \Sigma M_A = 0 \quad \Rightarrow \quad M_{s_1} - M_{s_1} = 0 \]
\[ \Rightarrow \quad M_{s_1} = M_{s_1} \]
for Spring 2:
\[ \Sigma F_{x_2} = 0 \quad \Rightarrow \quad H_B = 0 \]
\[ \Sigma F_{x_3} = 0 \quad \Rightarrow \quad V_B + V_{s_2} = 0 \]
\[ \Rightarrow \quad V_B = -V_{s_2} \quad (5) \]
no moments
for Spring 3:
\[ \Sigma F_{x_2} = 0 \quad \Rightarrow \quad H_c = 0 \]
\[ \Sigma F_{x_3} = 0 \quad \Rightarrow \quad V_C + V_{s_3} = 0 \]
\[ \Rightarrow \quad V_C = -V_{s_3} \quad (6) \]
again, no moment

→ Finally, include the **Constitutive Relations**:

\[ \text{Spring 1: } \mathbf{M}_{s_1} = k_T \theta \]  
(7)

\[ \text{Spring 2: } V_{s_2} = k_S (x_2 - y_2) \]  
(8)

\[ \text{Spring 3: } V_{s_3} = 2k_F (x_2 - c) \]  
(9)

→ with the **Compatibility Equation** from part (c) -- equations (10), (16), (22)

There are 10 equations with the following unknowns:

\[ s_3 (x_2 - y_2), s_2 (x_2 - c), \theta, u_1, v_A, v_{s_2}, v_{s_3}, M_{s_0}, V_{s_2}, V_{s_3} \]

\[ \Rightarrow 10 \text{ unknowns} \]

So work the way through the equations already having shown: \[ H_A = H_B = H_C = 0 \]

Use (10) in (9) \[ V_{s_3} = 2kmL \]

and in (6) \[ V_C = -2kmL \]

Use (12) in (8) \[ V_{s_2} = 1kmy_2 \]

and in (5) \[ V_B = -1kmy_2 \]

Use (2) in (7) \[ M_{s_1} = k_T \theta \]  
(14)
Now use these first in (4):

\[-k_Tm - km^2L^2/4 - 2kml^2 - \frac{MgL^2}{2} = 0\]

\[\Rightarrow m(k_T + \frac{9kL^2}{4}) = -\frac{MgL^2}{2}\]

\[\text{For} \quad m = -\frac{MgL^2}{2(k_T + \frac{9kL^2}{2})} \quad (15)\]

**Note:** an is negative \(\Rightarrow\) a negative slope giving a negative displacement, check this with a load of weight \(\vec{m}\) in \(-x_3\) direction.

Use this result for \(m\) with others in (3):

\[V_A - km\frac{v_A}{2} - 2kmL - Mlg = 0\]

\[\Rightarrow V_A + \frac{5kL}{2} \left[ \frac{MgL^2}{2(k_T + \frac{9kL^2}{2})} \right] - Mlg = 0\]

\[\Rightarrow V_A = Mlg - \frac{5MgL^3}{4k_T + 9kL^2} \quad (16)\]

and using (15) in (11) and (13):

\[V_c = \frac{2MgL^3}{2k_T + \frac{9kL^2}{2}} = \frac{4MgL^3}{4k_T + 9kL^2}\]
\[ V_B = \frac{Mg kL^3}{4k_T + 9kL^2} \]

Check: \( V_A + V_B + V_C = MgL \)

**Summarizing:**

\[ V_A = MLg - \frac{5MgL^3}{4k_T + 9kL^2} \]
\[ V_B = \frac{MgL^3}{4k_T + 9kL^2} \]
\[ V_C = \frac{2MgL^3}{4k_T + 9kL^2} \]

Spring 1: \( \theta = -\frac{2MgL^2}{4k_T + 9kL^2} \)

Spring 2: \( S(x=\frac{L}{2}) = -\frac{MgL^2}{4k_T + 9kL^2} \)

Spring 3: \( S(x=L) = -\frac{2MgL^3}{4k_T + 9kL^2} \)

Overall for displacement:

\[ S_3 = -\frac{2MgL^3}{4k_T + 9kL^2} \times 2 \]
MS.2

Place the origin of the coordinate system at point D.

(a) Draw the Free Body Diagram for the overall system
One can also consider each joint as a separate subsystem (Note: This is fundamentally what is done via the Method of Joints):

**Joint A:**

![Joint A Diagram]

**Joint B:**

![Joint B Diagram]

**Joint C:**

![Joint C Diagram]
Joint D:

\[ F_4 \leftarrow F_5 \]

(5) Consider the overall Free Body Diagram of part (a). There are 3 degrees of freedom (2 in translation, 1 in rotation) and 5 reaction forces. So there are 3 equations of equilibrium in 5 unknowns.

3 d.o.f. < 5 reactions \[ \Rightarrow \text{Statically Indeterminate} \]

(c) Since the entire system is statically indeterminate, must apply the 3 "great principles" to solve for the unknowns -- i.e. use equilibrium, constitutive relations, compatibility.

First do a bit of geometry to resolve box forces.
Bar #2 is at 45° and is 1.41 L in length.

So \( F_{3x3} = 0.707 F_3 \)

\( F_{3x1} = 0.707 F_3 \)

Bar #3 is at:

\[ \theta = \tan^{-1} \frac{L}{2L} = 26.6° \]

So:

\( F_{5x3} = F_5 \sin \theta = 0.45 F_5 \)

\( F_{5x1} = F_5 \cos \theta = 0.89 F_5 \)

and the bar = 2.23L in length

→ Now, apply equilibrium

at Joint A:

\[ \Sigma F_{x_1} = 0 \quad \Rightarrow \quad H_A + 0.707 F_3 + 0.89 F_5 = 0 \quad (1) \]

\[ \Sigma F_{x_3} = 0 \quad \Rightarrow \quad -V_A - F_5 - 0.707 F_3 - 0.45 F_5 = 0 \quad (2) \]
\[ \Rightarrow \text{at Joint B:} \]
\[ \sum F_{x_1} = 0 \Rightarrow F_2 = 0 \]
\[ \sum F_{x_3} = 0 \quad \Rightarrow \quad V_B + F_1 = 0 \quad \Rightarrow \quad V_B = -F_1 \quad (3) \]

\[ \Rightarrow \text{at Joint C:} \]
\[ \sum F_{x_1} = 0 \Rightarrow -F_2 + F_4 - 0.707F_3 = 0 \quad \Rightarrow \quad F_4 = 0.707F_3 \quad (4) \]
\[ \sum F_{x_3} = 0 \Rightarrow V_C + 0.707F_3 = 0 \quad \Rightarrow \quad V_C = -0.707F_3 \quad (5) \]

\[ \Rightarrow \text{at Joint D:} \]
\[ \sum F_{x_1} = 0 \Rightarrow -F_4 - 0.89F_5 + P = 0 \quad (6) \]
\[ \sum F_{x_3} = 0 \Rightarrow V_D + 0.45F_5 = 0 \quad \Rightarrow \quad V_D = -0.45F_5 \quad (7) \]

So there are now 7 equations in 9 unknowns \((H_A, V_A, V_B, V_C, V_D, F_1, F_3, F_4, F_5)\)

\[ \Rightarrow \text{Second, apply constitutive relations for} \]
\[ \text{the overall deformation of each} \]
\[ \text{bar} \]
Need the expression of the basic relation:

\[ F = k \delta \]

For an overall deflection of bar

for a bar: \[ k = \frac{EA}{L} \]

all bars have the same \( E \) and \( A \) (set as "known" constants) with only length varying:

\[ k_1 = \frac{EA}{L} = K - \text{we call bar parameter} \]
\[ k_2 = \frac{EA}{L} = K \]
\[ k_3 = \frac{EA}{1.44L} = 0.707 \frac{EA}{L} = 0.707 K \]
\[ k_4 = \frac{EA}{L} = K \]
\[ k_5 = \frac{EA}{2.23L} = 0.45 \frac{EA}{L} = 0.45 K \]

So the overall deflection of each bar is:

\[ \delta_1 = F_1 / K \tag{8} \]
\[ \delta_2 = 0 \text{ since } F_2 = 0 \]
\[ \delta_3 = F_3 / 0.707 K = 1.414 \frac{F_3}{K} \tag{9} \]
\[ \delta_4 = F_4 / K \]
\[ \delta_5 = F_5 / 0.45 K = 2.23 \frac{F_5}{K} \tag{10} \]

Now there are 10 equations but 13 unknowns as 4 more have been introduced (\( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_10 \))
Finally, enforce compatibility at each joint. Define the displacement of each joint:

- A is pinned, so \( \delta_A = 0 \)
- B, C, and D are rollers, so they only have displacement in \( x \): \( \delta_3, \delta_c, \delta_d \).

Define the change in length (overall deflection) of each bar by the difference in displacement of its joints:

Bar #2: \( \delta_2 = \delta_c - \delta_B = 0 \)  \( \Rightarrow \delta_c = \delta_B \)  \( \text{(12)} \)

Bar #4: \( \delta_4 = \delta_D - \delta_c \)  \( \text{(13)} \)

Bar #3: Consider the following height graph:

\[ (L_3 + \delta_3)^2 = L^2 + (L + \delta_c)^2 \]

Earlier found \( L_3 = 1.41L \)

\[ 2L^2 + 2.82L_3 + \delta_3^2 = L^2 + L^2 + 2L\delta_c + \delta_c^2 \]

Simplifies: \( 2.82L_3 + \delta_3^2 = 2L\delta_c + \delta_c^2 \)
Assuming deflections are small, one can ignore higher order term terms giving:

\[ 2 \times 2 \delta_3 = 2 \delta_c \]

\[ \Rightarrow 1.41 \delta_3 = \delta_c \quad (14) \]

→ in a similar way for Bar 5:

\[ (L_5 + \delta_5 -) \]
\[ \Rightarrow (L_5 + \delta_5 -) = L^2 + (2L + \delta_D)^2 \]

Earlier found \( L_5 = 2.23L \)

\[ \Rightarrow 5L^2 + 4.46 \delta_5 - L + \delta_5^2 = L^2 + 4\delta_c^2 + 4L_\delta + \delta_D^2 \]

Again ignoring higher order terms with the assumption of small displacements:

\[ 4.46 \delta_5 = 4\delta_D \]

\[ \Rightarrow \delta_D = 1.12 \delta_5 \quad (15) \]

Finally for Bar 1, any change in length will be equal to a displacement of \( L \):

\[ (L + \delta_L)^2 = L^2 + \delta_L^2 \]

\[ \Rightarrow L^2 + 2L \delta_L + \delta_L^2 = L^2 + \delta_B^2 \]
To let \( s_0 = s_c \) \hspace{1cm} (16)

There are now 15 equations in a total of 16 unknowns as 3 have been added \((s_3, s_c, s_0)\).

Now work through the case it is a matter of simply manipulating the equations.

→ Put \((6) + (12): \ s_c = s_c \) and use in \((8): \ F_1 = Ks_c \) \hspace{1cm} (17)

→ Put the results of \((14): \ s_3 = 0.707s_c \) in \((9): \ F_3 = 0.5Ks_c \) \hspace{1cm} (18)

and use this in \((4): \ F_4 = 0.35Ks_c \) \hspace{1cm} (19)

→ Put \((15) + (13): \ s_4 = 1.12s_5 - s_c \) and use \((10)\) and \((11)\) to get:

\[
\frac{F_4}{K} = 1.12 \left(2.23 \frac{F_5}{K}\right) - s_c
\]

and with \((19): \)

\[
0.35s_c = 2.23 \frac{F_5}{K} - s_c
\]

\[\Rightarrow F_5 = 0.54Ks_c \hspace{1cm} (20)\]

→ Put \((19)\) and \((20)\) in \((6): \)

\[-0.35Ks_c - 0.89 \left(0.54Ks_c\right) + P = 0\]

\[\Rightarrow 0.83Ks_c = P\]

\[\Rightarrow s_c = 1.2 \frac{P}{K}\]
Now use the result for $F_c$ in:

(20): \[ F_5 = 0.65 P \]
(19): \[ F_4 = 0.42 P \]
(18): \[ F_3 = 0.60 P \]
(17): \[ F_1 = 1.20 P \]

and therefore $F_2 = 0$

Bar deflections are determined from (8) through (11):

\[
\begin{align*}
\delta_1 &= 1.20 \frac{P}{K} \\
\delta_2 &= 0 \\
\delta_3 &= 0.85 \frac{P}{K} \\
\delta_4 &= 0.42 \frac{P}{K} \\
\delta_5 &= 1.45 \frac{P}{K}
\end{align*}
\]

and the reactions are determined via:

(1): \[ H_A + 0.707(0.60 P) + 0.89(0.65 P) = 0 \]
\[ \Rightarrow H_A = -P \]

(2): \[ -V_A - 1.59 P - 0.707(0.60 P) - 0.45(0.65 P) = 0 \]
\[ \Rightarrow V_A = -1.91 P \]
(3) $V_B = -1.20P$

(5) $V_C = -0.707(0.60P) = -0.42P$

(7) $V_D = -0.45(0.65P) = -0.29P$

Summary:

\[
\begin{array}{c}
H_A = -P \\
V_A = -1.91P \\
V_B = -1.20P \\
V_C = -0.42P \\
V_D = -0.29P \\
\end{array}
\]

Do a final check via the overall system:

$\Sigma F_x = 0 \implies H_A + P = 0 \checkmark$

$\Sigma F_{x_3} = 0 \implies -V_A + V_B + V_C + V_D = 0 \checkmark$

(d) In the solution, we assumed "small" deflections. If they were to become large, we could not ignore the higher-order terms that resulted in equations (14), (15), and (16). Furthermore,
the angular orientation of the bars would also change and this would need to be included in resolving the bar forces. These geometric relations would not add additional variables but the equations would become quite non-linear and would need to be solved using numerical approximation or other techniques.

A Special Note re/ (c) and Compatibility:

Different assumptions can be made about deflection (example: bar angles do not change, bar length + bar f times cosine of angle equals length (if floor from A to bar junction plus deflection of that junction). Answers change because of this. This shows that different assumptions yield different results. Only accounting for full effects yields actual results.......

PAL
M.3

For all cases recall rules for tensorial/indicial notation:

- **Latin subscripts** take on values 1, 2, 3
- **Greek subscripts** take on values 1, 2
- When subscript is repeated in one term, it is a "dummy index" and is summed on
- When subscript appears only once on left side of equation in one term, it is a "free index" and represents separate equations

So...

\[(\sigma) \sigma_{ij} = l_{ij} k_{ik} \]

- \(i'\) and \(k'\) are dummy indices and are summed on from 1 to 3 (they are Latin subscripts)

Therefore:

\[
\sigma_{ij} = l_{ij} - (l_{31} \sigma_{i1} + l_{32} \sigma_{i2} + l_{33} \sigma_{i3})
+ l_{12} (l_{31} \sigma_{12} + l_{32} \sigma_{12} + l_{33} \sigma_{12})
+ l_{13} (l_{31} \sigma_{13} + l_{32} \sigma_{13} + l_{33} \sigma_{13})
\]
(5) \( L_{\alpha \beta} = \sum_{i,j} C_{\alpha i} M_{\beta j} \)

- \( i \) and \( j \) are dummy indices and are summed over from 1 to 3 (they are Latin subscripts)
- \( \alpha \) and \( \beta \) are free indices and indicate separate equations (2 per free transcript gives 4 total)

\( \delta_{ij} = \text{Kronecker Delta} = 0 \) if \( i \neq j \)
\( = 1 \) if \( i = j \)

\( \Rightarrow \) same as:

\( L_{\alpha \beta} = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{\alpha i} M_{\beta j} \delta_{ij} \)

with terms nonzero only if \( i = j \)

Thus:

\( \alpha = 1, \beta = 1 \) \( L_{11} = C_{11} M_{11} + C_{12} M_{12} + C_{13} M_{13} \)

\( \alpha = 2, \beta = 1 \) \( L_{21} = C_{21} M_{11} + C_{22} M_{12} + C_{23} M_{13} \)

\( \alpha = 1, \beta = 2 \) \( L_{12} = C_{11} M_{21} + C_{12} M_{22} + C_{13} M_{23} \)

\( \alpha = 2, \beta = 2 \) \( L_{22} = C_{21} M_{21} + C_{22} M_{22} + C_{23} M_{23} \)
(c) \( D_{pq} = S_{pqrs} \varepsilon_r \beta_s \) (\( \varepsilon_p = 3, q = 2 \))

\[ \Rightarrow D_{32} = S_{32rs} \varepsilon_r \beta_s \]

- \( r \) and \( s \) are dummy indices and are summed on from 1 to 3 (they are both subscripts)

\[ \Rightarrow \text{same as } D_{32} = \sum_{r=1}^{3} \sum_{s=1}^{3} S_{32rs} \varepsilon_r \beta_s \]

Thus:

\[ D_{32} = S_{321r} \varepsilon_r \beta_1 + S_{3212} \varepsilon_r \beta_2 + S_{3213} \varepsilon_r \beta_3 \]

\[ + S_{322r} \varepsilon_r \beta_2 + S_{3222} \varepsilon_r \beta_2 + S_{3223} \varepsilon_r \beta_3 \]

\[ + S_{323r} \varepsilon_r \beta_3 + S_{3232} \varepsilon_r \beta_2 + S_{3233} \varepsilon_r \beta_3 \]

(d) \( E = \frac{1}{2} \sigma_{\alpha\beta} \varepsilon_{\alpha\beta} \)

- \( \alpha \) and \( \beta \) are dummy indices and are summed on from 1 to 2 (they are both subscripts)

\[ \Rightarrow \text{same as } E = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sigma_{\alpha\beta} \]

Thus:

\[ E = \frac{1}{2} \left( \sigma_{11} \varepsilon_{11} + \sigma_{12} \varepsilon_{12} + \sigma_{21} \varepsilon_{21} + \sigma_{22} \varepsilon_{22} \right) \]
(e) $B_{ij} \left( \frac{\partial b}{\partial x_j} \right) + f_i = 0$

- $j$ is a dummy index and is summed on from 1 to 3 (it is a Latin subscript)
- $i$ is a free index and indicates there are 3 equations (Latin subscript $= 1, 2, 3$)

$\Rightarrow$ same as: $\sum_{j=1}^{3} B_{ij} \left( \frac{\partial b}{\partial x_j} \right) + f_i = 0$

Thus:

- $p=1$: $B_{11} \frac{\partial b}{\partial x_1} + B_{12} \frac{\partial b}{\partial x_2} + B_{13} \frac{\partial b}{\partial x_3} + f_1 = 0$
- $p=2$: $B_{21} \frac{\partial b}{\partial x_1} + B_{22} \frac{\partial b}{\partial x_2} + B_{23} \frac{\partial b}{\partial x_3} + f_2 = 0$
- $p=3$: $B_{31} \frac{\partial b}{\partial x_1} + B_{32} \frac{\partial b}{\partial x_2} + B_{33} \frac{\partial b}{\partial x_3} + f_3 = 0$