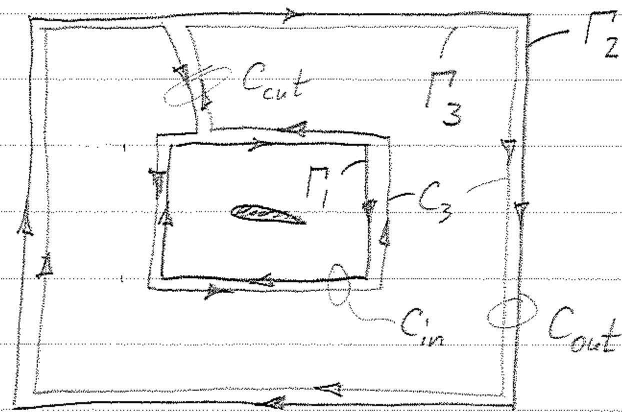
UE Fluids Problem F11 solution Fall '06



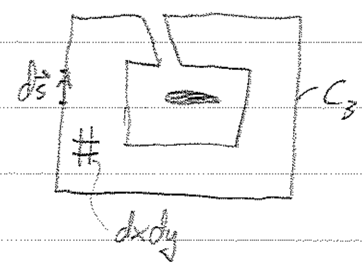
Draw circuit for Γ_3:
Integrating on circuits C_{in} and C_{out}:

$$\Gamma_1 = -\oint_{C_{in}} \vec{V} \cdot d\vec{s}$$

$$\Gamma_2 = -\oint_{C_{out}} \vec{V} \cdot d\vec{s}$$

$$\Gamma_3 = -\oint_{C_{out}} \vec{V} \cdot d\vec{s} + \underbrace{\oint_{C_{in}} \vec{V} \cdot ds}_{} = \Gamma_2 - \Gamma_1$$

this part is + because it's counterclockwise.
Also, there's no C_{cut} contribution to Γ_3 since the two parts cancel.



Now we note that Γ_3's circuit C_3 is "reducible", since the airfoil is topologically outside it.
Hence, Stokes' Theorem can be applied:

$$\Gamma_3 = \oint_{C_3} \vec{V} \cdot d\vec{s} = \iint \nabla \times \vec{V} \, dxdy = \iint 0 \cdot dxdy = 0$$

$$\therefore \quad \Gamma_3 = \Gamma_2 - \Gamma_1 = 0$$

or $\boxed{\Gamma_2 = \Gamma_1}$

UE Fluids F12 Solution Fall '06

a) $u = \frac{\partial \psi}{\partial y} = x^2 \rightarrow \psi = x^2 y + f(x)$

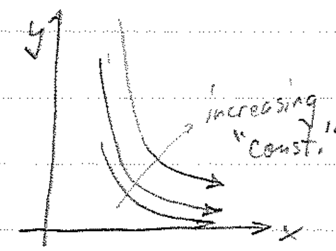
$-v = \frac{\partial \psi}{\partial x} = 2xy \rightarrow \psi = x^2 y + g(y)$

To be consistent, we must have $f(x) = g(y) = ¢$

$$\boxed{\psi(x,y) = x^2 y + \cancel{¢}}$$

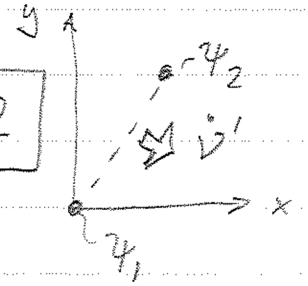
¢ is arbitrary, and can be set ¢ = 0 (no effect on velocities, since it disappears when determining $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$)

b) Set ψ = constant, or $x^2 y$ = Const., or $y = \frac{\text{"const"}}{x^2}$



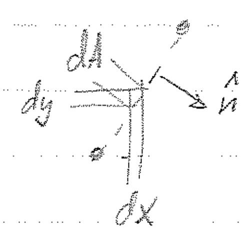
c) Easiest to use streamfunction:

$$\boxed{\dot{V}' = \psi_2 - \psi_1 = (x^2 y)_2 - (x^2 y)_1 = 2^2 \cdot 3 - 0 = 12}$$



OR, ...

Longer way: Integrate $\dot{V}' = \int_1^2 \vec{V} \cdot \hat{n} \, dA$



On line:

$dA = \sqrt{dx^2 + dy^2} = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx \sqrt{1 + \left(\frac{3}{2}\right)^2}$

$dA = dx \cdot \frac{\sqrt{13}}{2}$

$\hat{n} = \frac{1}{\sqrt{3^2 + 2^2}} \left[3\hat{i} - 2\hat{j}\right] = \frac{1}{\sqrt{13}} (3\hat{i} - 2\hat{j})$

$y = (3/2) x$

$u = x^2, \quad v = -2xy = -2x\left(\frac{3}{2}x\right) = -3x^2$

$(\vec{V} \cdot \hat{n}) \, dA = \left(x^2 \frac{3}{\sqrt{13}} + 3x^2 \frac{2}{\sqrt{13}}\right) \cdot dx \frac{\sqrt{13}}{2} = \frac{9}{2} x^2 dx$ whew!

$$\boxed{\int \vec{V} \cdot \hat{n} \, dA = \int_0^2 \frac{9}{2} x^2 dx = \frac{9}{2} \cdot \frac{1}{3} x^3 \Big|_0^2 = 12}$$

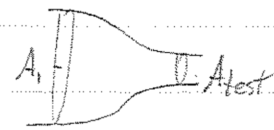
same result.

UE Fluids F13 Solution Fall '06

a) Parallel exit flow: Static pressure constant across streamlines: $\boxed{p_{test} - p_{atm} = 0}$

b) Constant mass flow along tunnel: $\rho V_1 A_1 = \rho V_{test} A_{test}$

$$\boxed{V_1 = V_{test}\frac{A_{test}}{A_1} = \frac{1}{36} V_{test}}$$



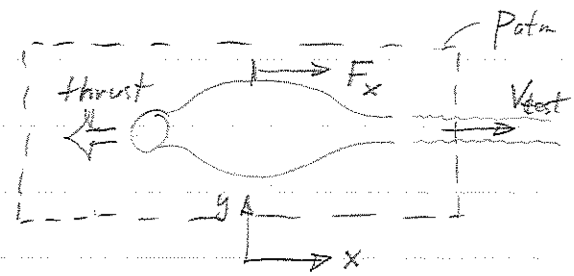
c) Smooth flow: Apply Bernoulli $\quad p_{o1} = p_{o\,test}$

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_{test} + \tfrac{1}{2}\rho V_{test}^2$$

$$p_1 = \overset{p_{atm}}{\cancel{p_{test}}} + \tfrac{1}{2}\rho\left(V_{test}^2 - V_1^2\right) = p_{atm} + \tfrac{1}{2}\rho V_{test}^2\left(1 - \tfrac{1}{36^2}\right)$$

$$\boxed{p_1 = 10^5\,\mathrm{Pa} + \tfrac{1}{2}\cdot 1.2\,\mathrm{kg/m^3}\cdot(50\,\mathrm{m/s})^2\cdot\left(1-\tfrac{1}{36^2}\right) = 101499\,\mathrm{Pa}}$$

d) Using momentum eq'n on C.V. shown:



$$\oint \rho \vec{V}\cdot\hat{n}\, u\, dA + \underbrace{\oint p\, n_x\, dA}_{0\ (p = p_{atm} = \text{const})} + F_x = 0$$

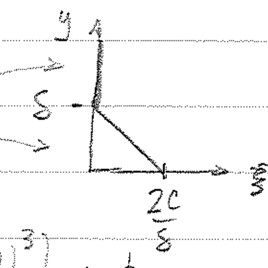
$$\rho V_{test}^2 A_{test} + F_x = 0 \quad \Rightarrow \quad F_x = -\rho V_{test}^2 A_{test}$$

Thrust is simply $-F_x$, $\quad A_{test} = 1\,\mathrm{ft}^2 = \frac{1}{3.28^2}\,\mathrm{m}^2 = 0.09295\,\mathrm{m}^2$

$$\boxed{\text{Thrust} = \rho V_{test}^2 A_{test} = 1.2\cdot(50)^2\cdot 0.09295 = 278.85\ \mathrm{N}}$$

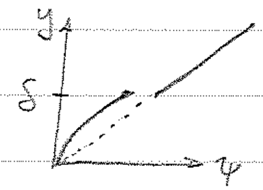
UE Fluids F14 Solution Fall '06

a) $\xi = \cancel{\frac{\partial v}{\partial x}}^{\,0} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = \begin{cases} \frac{2C}{\delta}\left(1-\frac{y}{\delta}\right), & y<\delta \\ 0, & y>0 \end{cases}$



For $y<\delta$: $u = \frac{\partial \psi}{\partial y} = C\left[2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] \rightarrow \psi = C\delta\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right] + c_1$

For $y>\delta$: $u = \frac{\partial \psi}{\partial y} = C \rightarrow \psi = Cy + c_2$



It's OK to leave ψ in the 2-piece definition.

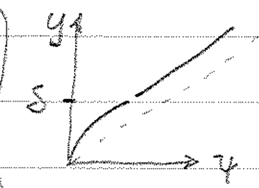
However, we can set c_2 to make the pieces continuous at $y=\delta$

Equate at $y=\delta$: $\psi_1 = \psi_2$

$C\delta\left[1-\frac{1}{3}\right] + c_1 = C\delta + c_2 \rightarrow c_2 = c_1 - \frac{1}{3}C\delta$

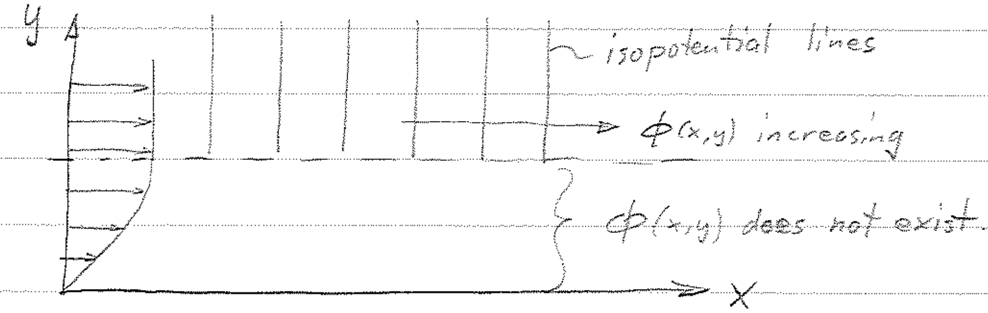
Optional

b) For $y>\delta$, we can write $\phi = Cx$, $\vec{V} = \nabla\phi = C\hat{i} + 0\hat{j}$ ✓



For $y<\delta$, we cannot find any $\phi(x,y)$ which gives $\nabla\phi = u\hat{i} + 0\hat{j}$

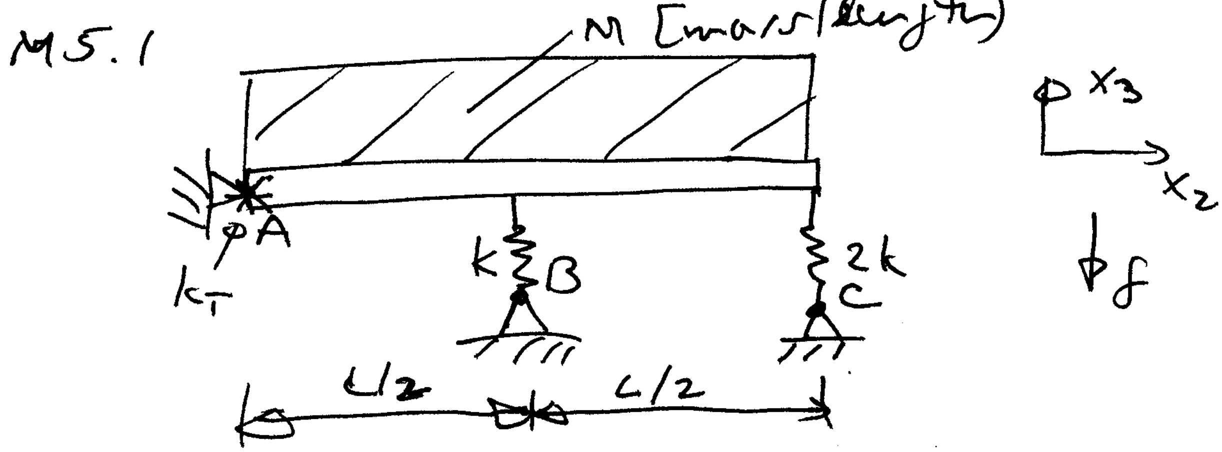
This is because $\xi \neq 0$. Flow is rotational, and so ϕ doesn't exist



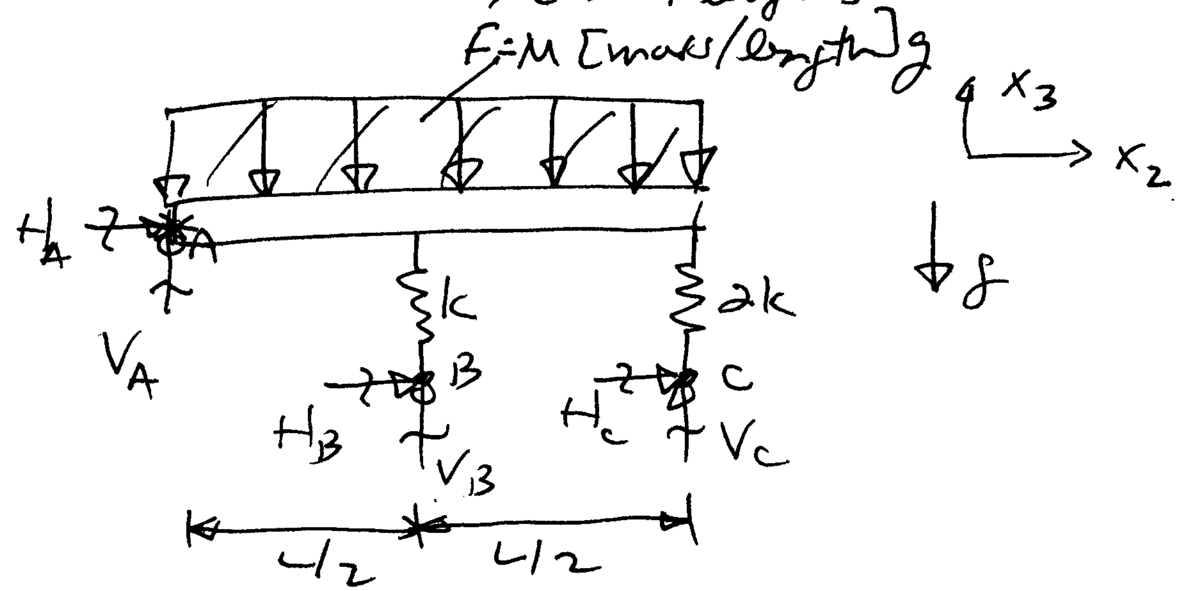
PAL
10/1/06

Unified Engineering Problem Set
Week 5 Fall, 2006
SOLUTIONS

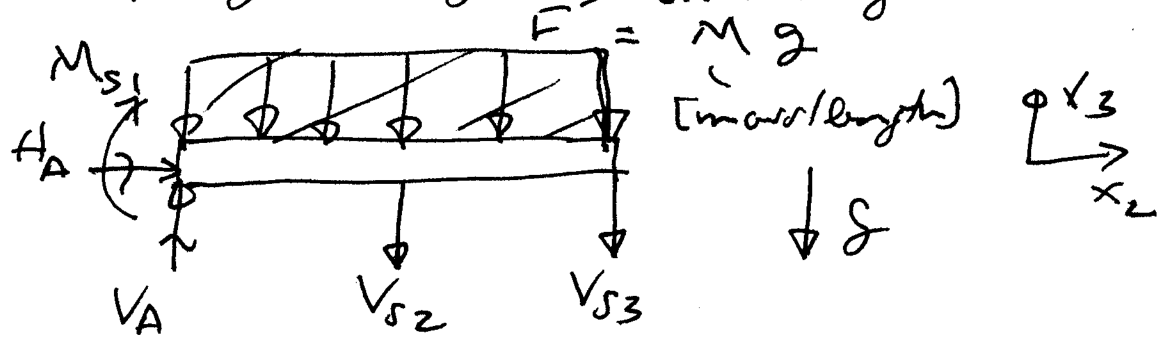
M5.1



(a) Overall there are three pin supports which provide reactions for this system. Draw the free body diagram (labeling the three supports as A, B, C)



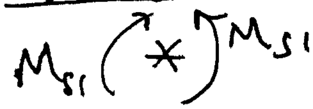
One can look at this as four different subsystems: the bar and each of the three springs acting on it.



The first spring provides a moment. Springs 2 and 3 provide vertical forces. The forces or moments on the bar must be equal and opposite to those on the springs. [Sum to zero for the overall system]

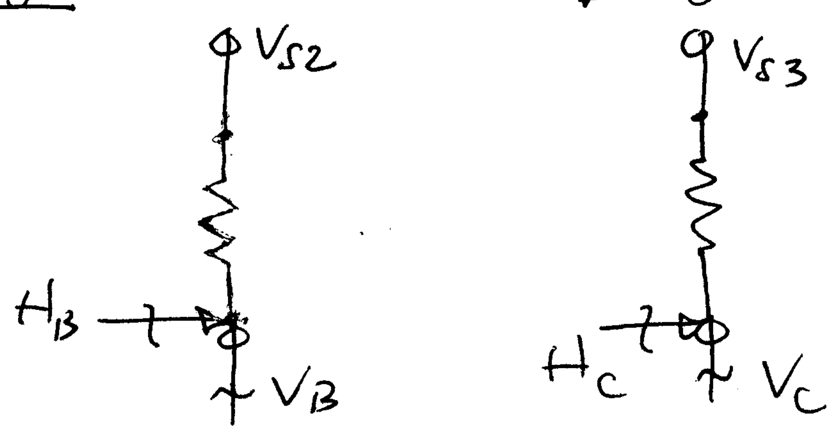
Looking at the springs:

<u>Spring 1</u>



<u>Spring 2</u>

<u>Spring 3</u>



(b) This is a 2-D system and has a potential for 3 degrees of freedom of motion:

- lateral in x_2
- lateral in x_3
- rotational in x_2-x_3 plane (about x_1)

There are 6 reactions,

reactions > # d.o.f.

$\Rightarrow$ $\boxed{\text{statically indeterminate}}$

(c) There are 2 parts that make up the Compatibility of Displacement here. The primary is that the bar is rigid so the displacement in x_3 must be a linear function of x_2:

$$\underbrace{\delta_3}_{\text{displacement in } x_3} = m x_2 + b$$

(set $x_2 = 0$ at left end)

Now look at further details. The bar is pinned with the torsional spring at $x_2 = 0$ $\Rightarrow$ $b = 0$

With the linear variation and the pin at $x_2 = 0$, the displacement at $x_2 = L$ must be twice that at $x_2 = L/2$

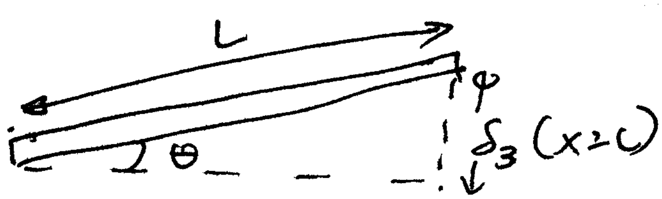
$$\Rightarrow \text{So:} \quad \boxed{\delta_3 = m x_2 \qquad (1)}$$

$$\delta_3 (x_2 = L) = mL \qquad (1a)$$

$$\delta_3 (x_2 = L/2) = m L/2 \qquad (1b)$$

This does not yet give δ_3.

One can also get an expression for the rotation at $x = 0$



$$\Rightarrow L \sin\theta = \delta_3 (x = L) = mL$$

$$\Rightarrow \quad \theta = \sin^{-1}(m)$$

for small displacements

$$\Rightarrow \boxed{\theta = m \qquad (2)}$$

check at $x_2 = L/2$

$$L/2 \sin\theta = \delta_3 (x = L) = m L/2$$

$$\Rightarrow \quad \theta = \sin^{-1}(m) \ \checkmark \text{checks}$$

(d) Now we all our equations

→ First Equilibrium. Doing this for the bar subsystem:

$$\sum F_{x_2} = 0 \xrightarrow{+} \Rightarrow H_A = 0$$

$$\sum F_{x_3} = 0 \quad \uparrow + \Rightarrow V_A - V_{s2} - V_{s3} - MLg = 0 \quad (3)$$

$$\sum M_A = 0 \quad \circlearrowleft + \Rightarrow -M_{s1} - V_{s2} L/2 - V_{s3} L - \int_0^L Mg x_2 dx_2 = 0$$

$$\Rightarrow -M_{s1} - V_{s2} L/2 - V_{s3} L - \frac{MgL^2}{2} = 0 \quad (4)$$

for Spring 1:

$$\sum M_A = 0 \quad \circlearrowleft + \Rightarrow M_{s1} - M_{s1} = 0$$

$$\Rightarrow M_{s1} = M_{s1}$$

for Spring 2:

$$\sum F_{x_2} = 0 \xrightarrow{+} \Rightarrow H_B = 0$$

$$\sum F_{x_3} = 0 \quad \uparrow + \Rightarrow V_B + V_{s2} = 0$$

$$\Rightarrow V_B = -V_{s2} \quad (5)$$

no moments

for Spring 3:

$$\sum F_{x_2} = 0 \xrightarrow{+} \Rightarrow H_c = 0$$

$$\sum F_{x_3} = 0 \quad \uparrow + \Rightarrow V_C + V_{s3} = 0$$

$$\Rightarrow V_C = -V_{s3} \quad (6)$$

again, no moments

→ Finally, include the Constitutive Relations:

Spring 1: $M_{S1} = k_T \theta$ (7)

Spring 2: $V_{S2} = k\delta(x_2 = L/2)$ (8)

Spring 3: $V_{S3} = 2k\delta(x_2 = L)$ (9)

→ with the Compatibility equations from part (c) -- equations (1a), (1b), (2)

There are 10 equations with the following unknowns:

$\delta_3(x_2 = L/2)$, $\delta_3(x_2 = L)$, θ, m, V_A, V_{S2}, V_{S3}, M_{S1}, V_B, V_C

⇒ 10 unknowns

So work the way through the equations already having shown: $H_A = H_B = H_C = 0$

use (1a) in (9) ⇒ $V_{S3} = 2kmL$ (10)

and in (6) ⇒ $V_C = -2kmL$ (11)

use (1b) in (8) ⇒ $V_{S2} = km\,L/2$ (12)

and in (5) ⇒ $V_B = -km\,L/2$ (13)

use (2) in (7) ⇒ $M_{S1} = k_T m$ (14)

Now use these first in (4):

$$-k_T m - km L^2/4 - 2kmL^2 - \frac{MgL^2}{2} = 0$$

$$\Rightarrow \quad m\left(k_T + \frac{9kL^2}{4}\right) = -\frac{MgL^2}{2}$$

gives

$$m = -\frac{MgL^2}{\left(2k_T + \frac{9kL^2}{2}\right)} \qquad (15)$$

Note: m is negative $\Rightarrow$ a negative slope giving negative displacement. Checks with a load of weight in $-x_3$ direction.

Use this result for m with others in (3):

$$V_A - km L/2 - 2kmL - MLg = 0$$

$$\Rightarrow V_A + \frac{5kL}{2}\left[\frac{MgL^2}{2k_T + \frac{9kL^2}{2}}\right] - MLg = 0$$

$$\Rightarrow V_A = MLg - \frac{5MgkL^3}{4k_T + 9kL^2} \qquad (16)$$

and using (15) in (11) and (13):

$$V_c = \frac{2MgkL^3}{2k_T + \frac{9kL^2}{2}} = \frac{4MgkL^3}{4k_T + 9kL^2}$$

$$V_B = \frac{MgkL^3}{4k_T + 9kL^2}$$

Check: $V_A + V_B + V_C = MgL$ ✓

Summarizing:

$$V_A = MLg - \frac{5MgkL^3}{4k_T + 9kL^2}$$

$$V_B = \frac{MgL^3}{4k_T + 9kL^2}$$

$$V_C = \frac{4MgL^3}{4k_T + 9kL^2}$$

spring 1: $\theta = -\dfrac{2MgL^2}{4k_T + 9kL^2}$

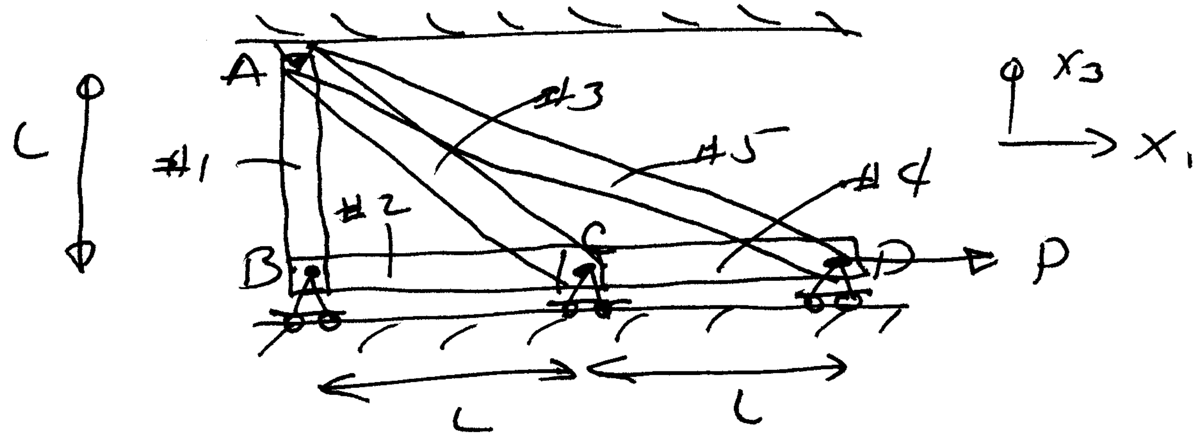
spring 2: $\delta(x = L/2) = -\dfrac{MgL^3}{4k_T + 9kL^2}$

spring 3: $\delta(x = L) = -\dfrac{2MgL^3}{4k_T + 9kL^2}$

Overall bar displacement:

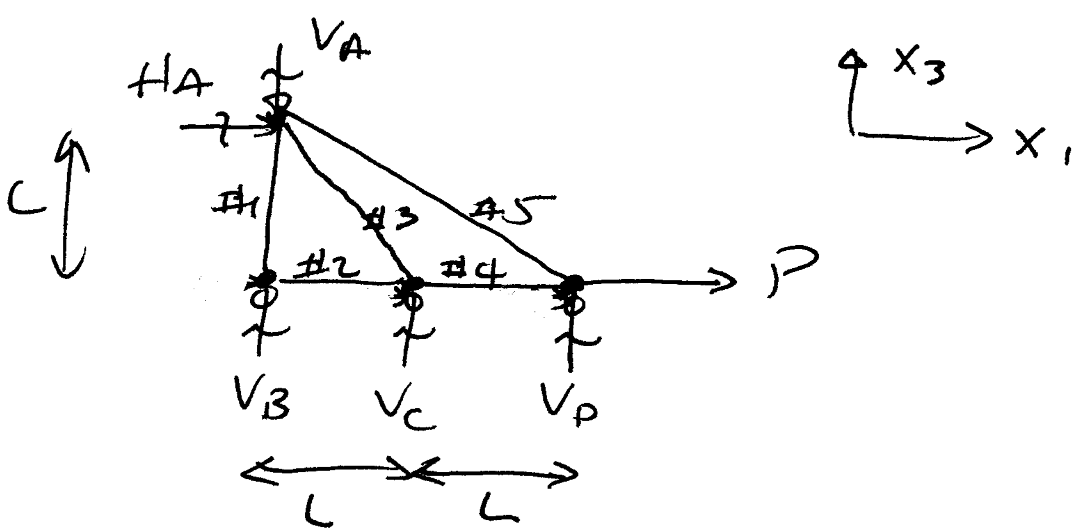
$$\delta_3 = -\frac{2MgL^3}{4k_T + 9kL^2}\, x_2$$

M5.2



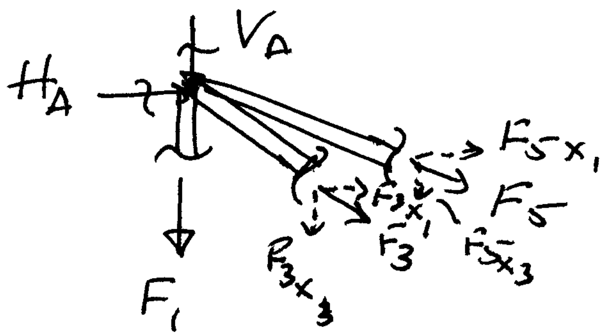
Place the origin of the coordinate system at point D.

(a) Draw the Free Body Diagram for the overall system

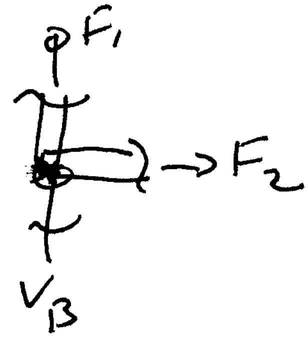


One can also consider each joint as a separate subsystem (Note: This is fundamentally what is done via the Method of Joints):

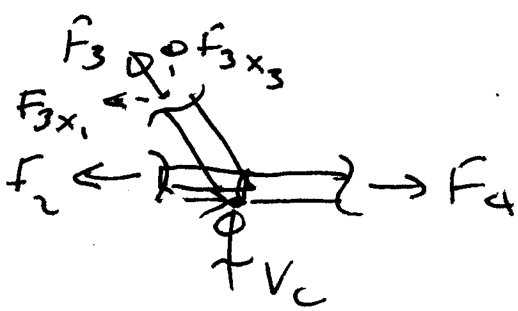
Joint A:



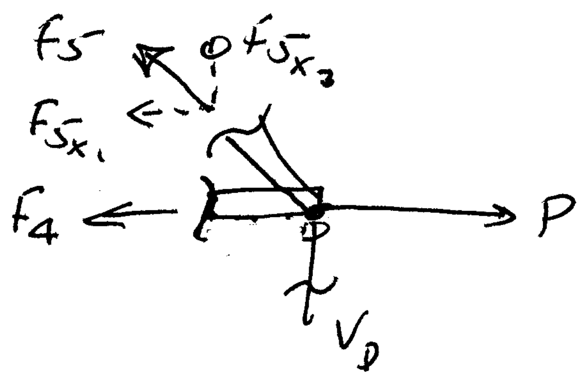
Joint B:



Joint C:



Joint D:



(b) Consider the overall Free Body Diagram of part (a). There are 3 degrees of freedom (in 2-D ... 2 in translation, 1 in rotation) and 5 reaction forces. So there are 3 equations of equilibrium in 5 unknowns.

3 d.o.f. < 5 reactions $\Rightarrow$ $\boxed{\text{Statically Indeterminate}}$

(c) Since the entire system is statically indeterminate, must apply the 3 "great principles" to solve for the unknowns -- i.e. use equilibrium, constitutive relations, compatibility.

First do a bit of geometry to involve bar forces.

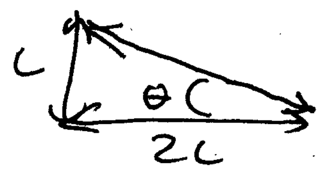
Bar # 3 is at 45°

and is 1.41 L in length.

So $F_{3x_3} = 0.707 F_3$

$F_{3x_1} = 0.707 F_3$

Bar # 5 is at:



$\theta = \tan^{-1} \frac{L}{2L} = 26.6^{\circ}$

So: $F_{5x_3} = F_5 \sin\theta = 0.45 F_5$

$F_{5x_1} = F_5 \cos\theta = 0.89 F_5$

and the bar = 2.23L in length

→ Now, apply equilibrium

→ at Joint A:

$\sum F_{x_1} = 0 \xrightarrow{+} \Rightarrow H_A + 0.707 F_3 + 0.89 F_5 = 0$ (1)

$\sum F_{x_3} = 0 \quad \uparrow + \Rightarrow -V_A - F_1 - 0.707 F_3 - 0.45 F_5 = 0$ (2)

→ at Joint B:

$\sum F_{x_1} = 0 \overset{+}{\rightarrow} \Rightarrow \boxed{F_2 = 0}$

$\sum F_{x_3} = 0 \uparrow + \Rightarrow V_B + F_1 = 0$

$\Rightarrow V_B = -F_1 \quad (3)$

→ at Joint C:

$\sum F_{x_1} = 0 \overset{+}{\rightarrow} \Rightarrow -F_2 + F_4 - 0.707 F_3 = 0$

$\Rightarrow F_4 = 0.707 F_3 \quad (4)$

$\sum F_{x_3} = 0 \uparrow + \Rightarrow V_C + 0.707 F_3 = 0$

$\Rightarrow V_C = -0.707 F_3 \quad (5)$

→ at Joint D:

$\sum F_{x_1} = 0 \overset{+}{\rightarrow} \Rightarrow -F_4 - 0.89 F_5 + P = 0 \quad (6)$

$\sum F_{x_3} = 0 \uparrow + \Rightarrow V_D + 0.45 F_5 = 0$

$\Rightarrow V_D = -0.45 F_5 \quad (7)$

So there are now $\underline{7}$ equations in $\underline{9}$ unknowns ($H_A, V_A, V_B, V_C, V_D, F_1, F_3, F_4, F_5$)

→ Second, apply <u>constitutive relation</u> for the overall deformation of each bar

Need the expression of the basic relation:

$$F = k\,\delta$$

F ← force on bar; δ ← overall deflection of bar

for a bar: $k = \dfrac{EA}{L}$

all bars have the same E and A (set as "known" constants) with only length varying:

$k_1 = EA/L = K$ ← use as bar parameter

$k_2 = EA/L = K$

$k_3 = EA/1.414L = 0.707\dfrac{EA}{L} = 0.707K$

$k_4 = EA/L = K$

$k_5 = EA/2.23L = 0.45\dfrac{EA}{L} = 0.45K$

So the overall deflection of each bar is:

$\delta_1 = F_1/K$ (8)

$\boxed{\delta_2 = 0}$ since $F_2 = 0$

$\delta_3 = F_3/0.707K = 1.414\dfrac{F_3}{K}$ (9)

$\delta_4 = F_4/K$ (10)

$\delta_5 = F_5/0.45K = 2.23\dfrac{F_5}{K}$ (11)

Now there are <u>11</u> equations but <u>13</u> unknowns as 4 more have been introduced ($\delta_1, \delta_3, \delta_4, \delta_5$)

→ Finally, enforce compatibility at each joint. Define the displacement of each joint:

A is pinned so $\delta_A = 0$

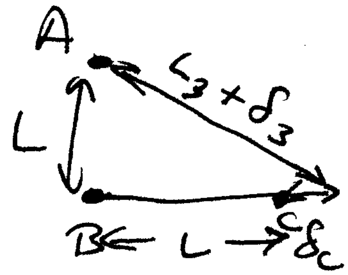
B, C, and D are rollers, so they only have displacement in x: $\delta_B, \delta_C, \delta_D$.

Define the change in length (overall deflection) of each bar by the difference in displacement of its joints:

Bar #2: $\delta_2 = \delta_C - \delta_B = 0 \Rightarrow \delta_C = \delta_B$ (12)

Bar #4: $\delta_4 = \delta_D - \delta_C$ (13)

Bar #3: Consider the following right triangle:



$$\Rightarrow (L_3 + \delta_3)^2 = L^2 + (L + \delta_C)^2$$

earlier found $L_3 = 1.41\,L$

$$\Rightarrow 2L^2 + 2.82\,\delta_3 L + \delta_3^2 = L^2 + L^2 + 2L\delta_C + \delta_C^2$$

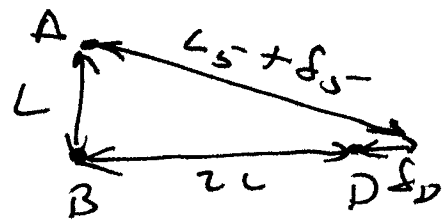
gives: $2.82\,\delta_3 L + \delta_3^2 = 2L\delta_C + \delta_C^2$

Assuming deflections are small, one can ignore higher order terms giving:

$$2.82\delta_3 = 2\delta_C$$

$$\Rightarrow 1.41\delta_3 = \delta_C \qquad (14)$$

→ in a similar way for Bar 5:



$$\Rightarrow (L_5+\delta_5)^2 = L^2 + (2L+\delta_D)^2$$

earlier found $L_5 = 2.23L$

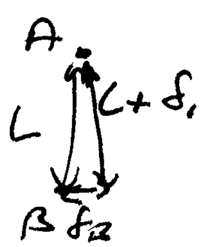
$$\Rightarrow 5L^2 + 4.46\delta_5 L + \delta_5^2 = L^2 + 4L^2 + 4L\delta_D + \delta_D^2$$

again ignoring higher order terms with the assumption of small displacements:

$$4.46\delta_5 = 4\delta_D$$

$$\Rightarrow \delta_D = 1.12\delta_5 \qquad (15)$$

→ Finally for Bar 1, any change in length will be equal to a displacement of B



$$\Rightarrow (L+\delta_1)^2 = L^2 + \delta_B^2$$

$$\Rightarrow L^2 + 2L\delta_1 + \delta_1^2 = L^2 + \delta_B^2$$

so let $\delta_B = \delta_1$ (16)

There are now <u>16</u> equations in a total of <u>16</u> unknowns as 3 have been added (δ_B, δ_C, δ_D)

Now work through the eqns it is a matter of simply manipulating the equations.

→ Put (16) in (12): $\delta_1 = \delta_C$

and use in (8): $F_1 = K\delta_C$ (17)

→ Put the results of (14): $\delta_3 = 0.707\delta_C$

in (9): $F_3 = 0.5K\delta_C$ (18)

and use this in (4): $F_4 = 0.35K\delta_C$ (19)

→ Put (15) in (13): $\delta_4 = 1.12\delta_5 - \delta_C$

and use (10) and (11) to get:

$$\frac{F_4}{K} = 1.12\left(2.23\frac{F_5}{K}\right) - \delta_C$$

and with (19):

$$0.35\delta_C = 2.\cancel{50}\frac{F_5}{K} - \delta_C$$

$$\Rightarrow F_5 = 0.54K\delta_C \quad (20)$$

→ Put (19) and (20) in (6):

$$-0.35K\delta_C - 0.89(0.54K\delta_C) + P = 0$$

$$\Rightarrow 0.83K\delta_C = P$$

$$\Rightarrow \delta_C = 1.2\frac{P}{K}$$

Now use the result for δ_c in:

$$
\boxed{
\begin{array}{ll}
(20): & F_5 = 0.65P \\
(19): & F_4 = 0.42P \\
(18): & F_3 = 0.60P \\
(17): & F_1 = 1.20P \\
\text{and} & \text{from before } F_2 = 0
\end{array}
}
$$

Bar Loads

Bar deflections are determined from (8) through (11):

$$
\boxed{
\begin{array}{l}
\delta_1 = 1.20\dfrac{P}{K} \\
\delta_2 = 0 \\
\delta_3 = 0.85\dfrac{P}{K} \\
\delta_4 = 0.42\dfrac{P}{K} \\
\delta_5 = 1.45\dfrac{P}{K}
\end{array}
}
$$

Bar deflections

and the reactions are determined via:

(1): $H_A + 0.707(0.60P) + 0.89(0.65P) = 0$

$$\Rightarrow H_A = -P$$

(2): $-V_A - 1.20P - 0.707(0.60P) - 0.45(0.65P) = 0$

$$\Rightarrow V_A = -1.91P$$

$$(3): V_B = -1.20P$$

$$(5): V_C = -0.707(0.60P) = -0.42P$$

$$(7): V_D = -0.45(0.65P) = -0.29P$$

Summarizing:

$$\boxed{\begin{aligned} H_A &= -P \\ V_A &= -1.91P \\ V_B &= -1.20P \\ V_C &= -0.42P \\ V_D &= -0.29P \end{aligned}}$$

Do a final check via the overall system:

$$\Sigma F_{x_1} = 0 \quad \overset{+}{\rightarrow} \quad \Rightarrow H_A + P = 0 \quad \checkmark$$

$$\Sigma F_{x_3} = 0 \quad \uparrow + \quad \Rightarrow -V_A + V_B + V_C + V_D = 0 \quad \checkmark$$

(d) In the solution, we assumed "small" deflections. If they were to become "large", we could not ignore the higher order terms that resulted in equations (14), (15), and (16). Furthermore the angular orientation of the bars would also change and this would need to be included in resolving the bar forces. These geometric relations would not add additional variables but the equations would become quite nonlinear and would need to be solved using numerical approximation or other techniques.

A Special Note re/ (c) and Compatibility:

Different assumptions can be made about deflection (example: bar angle does not change, so bar length + bar δ times cosine of angle equals length of floor from A to bar junction plus deflection of that junction). Answers change because of this. This shows that different assumptions yield different results. Only accounting for all effects yields actual results.....

M5.3

for all cases recall rules for tensorial/indicial notation:

- Latin subscripts take on values 1, 2, 3
- Greek subscripts take on values 1, 2
- when subscript is repeated in one term, it is a "dummy index" and is summed on
- when subscript appears only once on left side of equation in one term, it is a "free index" and represents separate equations

So....

(a) $\sigma_{13} = l_{1i'} l_{3k'} \sigma'_{i'k'}$

- i' and k' are dummy indices and are summed on from 1 to 3 (They are Latin subscripts)

$\Rightarrow$ same as: $\sigma_{13} = \sum_{i'=1}^{3} \sum_{k'=1}^{3} l_{1i'} l_{3k'} \sigma'_{i'k'}$

Thus:

$$\boxed{\begin{aligned}\sigma_{13} = {} & l_{11'}(l_{31'}\sigma'_{1'1'} + l_{32'}\sigma'_{1'2'} + l_{33'}\sigma'_{1'3'}) \\ & + l_{12'}(l_{31'}\sigma'_{2'1'} + l_{32'}\sigma'_{2'2'} + l_{33'}\sigma'_{2'3'}) \\ & + l_{13'}(l_{31'}\sigma'_{3'1'} + l_{32'}\sigma'_{3'2'} + l_{33'}\sigma'_{3'3'})\end{aligned}}$$

(b) $L_{\alpha\beta} = \delta_{ij} C_{\alpha i} M_{\beta j}$

- i and j are <u>dummy indices</u> and are summed on from 1 to 3 (they are latin subscripts)
- α and β are <u>free indices</u> and indicate separate equations (2 per free subscript gives 4 total)

δ_{ij} = kronecker delta $= 0$ if $i \neq j$
$= 1$ if $i = j$

$\Rightarrow$ same as:

$$L_{\alpha\beta} = \sum_{i=1}^{3} \sum_{j=1}^{3} C_{\alpha i} M_{\beta j} \delta_{ij}$$

with terms nonzero only if $i = j$

Thus:

$$\boxed{\begin{aligned}
(\alpha=1, \beta=1) \quad L_{11} &= C_{11} M_{11} + C_{12} M_{12} + C_{13} M_{13} \\
(\alpha=2, \beta=1) \quad L_{21} &= C_{21} M_{11} + C_{22} M_{12} + C_{23} M_{13} \\
(\alpha=1, \beta=2) \quad L_{12} &= C_{11} M_{21} + C_{12} M_{22} + C_{13} M_{23} \\
(\alpha=2, \beta=2) \quad L_{22} &= C_{21} M_{21} + C_{22} M_{22} + C_{23} M_{23}
\end{aligned}}$$

(c) $D_{pq} = S_{pqrs} z_r \beta_s$ (for $p = 3, q = 2$)

$\Rightarrow D_{32} = S_{32rs} z_r \beta_s$

- r and s are dummy indices and are summed on from 1 to 3 (they are latin subscripts)

$\Rightarrow$ same as $D_{32} = \sum_{r=1}^{3} \sum_{s=1}^{3} S_{32rs} z_r \beta_s$

Thus:

$$\boxed{\begin{aligned} D_{32} = {} & S_{3211} z_1 \beta_1 + S_{3212} z_1 \beta_2 + S_{3213} z_1 \beta_3 \\ & + S_{3221} z_2 \beta_1 + S_{3222} z_2 \beta_2 + S_{3223} z_2 \beta_3 \\ & + S_{3231} z_3 \beta_1 + S_{3232} z_3 \beta_2 + S_{3233} z_3 \beta_3 \end{aligned}}$$

(d) $E = \frac{1}{2} \sigma_{\alpha\beta} \epsilon_{\alpha\beta}$

- α and β are dummy indices and are summed on from 1 to 2 (they are greek subscripts)

$\Rightarrow$ same as $E = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \epsilon_{\alpha\beta}$

Thus:

$$\boxed{E = \tfrac{1}{2}\{\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{21}\epsilon_{21} + \sigma_{22}\epsilon_{22}\}}$$

(e) $B_{ij}\,(\partial b/\partial x_j) + f_i = 0$

- j is a dummy index and is summed on from 1 to 3 (it is a latin subscript)
- i is a free index and indicates there are 3 equations (latin subscript ⇒ 1, 2, 3)

⇒ same as: $\sum_{j=1}^{3} B_{ij}\,(\partial b/\partial x_j) + f_i = 0$

Thus:

$$\boxed{\begin{aligned} p=1:&\quad B_{11}\frac{\partial b}{\partial x_1} + B_{12}\frac{\partial b}{\partial x_2} + B_{13}\frac{\partial b}{\partial x_3} + f_1 = 0\\ p=2:&\quad B_{21}\frac{\partial b}{\partial x_1} + B_{22}\frac{\partial b}{\partial x_2} + B_{23}\frac{\partial b}{\partial x_3} + f_2 = 0\\ p=3:&\quad B_{31}\frac{\partial b}{\partial x_1} + B_{32}\frac{\partial b}{\partial x_2} + B_{33}\frac{\partial b}{\partial x_3} + f_3 = 0 \end{aligned}}$$