UE Fluids F15 Solution Fall '06

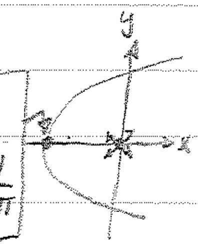
a) $u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = V_\infty + \frac{\Lambda}{2\pi}\frac{x}{x^2+y^2}$

$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2}$

At stagnation point, $u=0$, $v=0$ → find x, y

First use $v=0 \Rightarrow \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} = 0 \longrightarrow \boxed{y=0}$

Then use $u=0 \Rightarrow V + \frac{\Lambda}{2\pi}\frac{x}{\underbrace{x^2+y^2}_{0}} = 0 \longrightarrow \boxed{x = -\frac{\Lambda}{2\pi V_\infty} = \frac{-1}{2\pi}}$

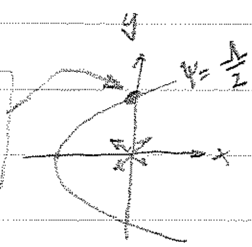


b) At $(x,y) = \left(-\frac{\Lambda}{2\pi V_\infty}, 0\right)$, $\sin\theta = 0$, $r = \frac{\Lambda}{2\pi V_\infty}$, $\theta = \pi$

$\rightarrow \boxed{\psi = V_\infty r \overbrace{\cancel{\sin\theta}}^{0} + \frac{\Lambda}{2\pi}\theta = \frac{\Lambda}{2} = \frac{1}{2}}$

c) Set $\psi = \frac{\Lambda}{2}$, $x = 0$ → find y

$\psi = \frac{\Lambda}{2} = V_\infty y + \frac{\Lambda}{2\pi}\overbrace{\theta}^{\pi/2 \text{ above origin}} = V_\infty y + \frac{\Lambda}{4} \rightarrow \boxed{y = \frac{\Lambda}{4V_\infty} = \frac{1}{4}}$



d) On streamline $\frac{dy}{dx} = \frac{v}{u}$, $u = V_\infty + \frac{\Lambda}{2\pi}\frac{\overbrace{x}^{0}}{x^2+y^2} = 1$

$v = \frac{\Lambda}{2\pi}\frac{y}{x^2+y^2} = \frac{1}{2\pi}\cdot\frac{1/4}{0+(1/4)^2} = \frac{2}{\pi}$

$\therefore \boxed{\frac{dy}{dx} = \frac{2}{\pi}}$

e) a) $y=0$, $x = \frac{-1}{\pi}$ $|x|$ doubles

b) $\psi = 1$

c) $y = \frac{1}{2}$ y doubles

d) $\frac{dy}{dx} = \frac{4}{\pi}$ v doubles, u unchanged

PAL
10/7/06

Unified Engineering Problem Set
Week 6 Fall, 2006

SOLUTIONS

M6.1

$$\begin{bmatrix} S_{111} & 2S_{112} & S_{122} \\ S_{211} & 2S_{212} & S_{222} \\ S_{311} & 2S_{322} & S_{322} \end{bmatrix} \begin{Bmatrix} F_{11} \\ F_{12} \\ F_{22} \end{Bmatrix} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}$$

First write ~~that~~ out in full (as it may help)

$$S_{111} F_{11} + 2 S_{112} F_{12} + S_{122} F_{22} = \delta_1$$
$$S_{211} F_{11} + 2 S_{212} F_{12} + S_{222} F_{22} = \delta_2$$
$$S_{311} F_{11} + 2 S_{312} F_{12} + S_{322} F_{22} = \delta_3$$

Look at/consider this piece by piece:

① The subscript on δ must be a <u>free</u> <u>index</u> because it changes with equation and represents separate equations. It must be <u>latin</u> since it takes on the values 1, 2, 3

$(= \delta_m)$

(2) The subscripts on F take on the values 1 and 2 and therefore must be greek. They change independently and thus must be different. $(F_{\alpha\beta})$

(3) The first subscript on S matches the subscript on δ $(S_{m??} F_{\alpha\beta} = \delta_m)$

(4) The second and third subscripts on S match those on F. By making them the same, they are also summed on (as occurs in the equations)

$$\Rightarrow \boxed{S_{m\alpha\beta} F_{\alpha\beta} = \delta_m}$$

But, one must also make the assumption that $F_{\alpha\beta}$ is symmetric $(F_{\alpha\beta} = F_{\beta\alpha})$ and $S_{m\alpha\beta}$ is symmetric in the last two indices $(S_{m\alpha\beta} = S_{m\beta\alpha})$ to get the factor of 2 in the final equations on the F_{12} terms with S_{m12} as multipliers.

M6.2

(a) $a_{11} q_1 q_1 + a_{21} q_2 q_1 + a_{12} q_1 q_2 + a_{22} q_2 q_2 = f_1 g_1 + f_2 g_2$

→ each subscript on a is summed on from 1 to 2 and thus are free. Being summed on requires them to be repeated and the first is repeated in the first q. the second in the second q:

$$a_{\alpha\beta} q_\alpha q_\beta = f_1 g_1 + f_2 g_2$$

→ the subscripts on f and g are the same and are summed on from 1 to 2 and thus are free. Summed on is provided by being in both f and g:

$$\Rightarrow \boxed{a_{\alpha\beta} q_\alpha q_\beta = f_\gamma g_\gamma}$$

(b) $R_1 = b_{11} \sin \pi x$

$R_2 = b_{22} \sin 2\pi x$

$R_3 = b_{33} \sin 3\pi x$

→ the subscript on R must be a free index since there are separate equations and it must be latin since there are 3 equations. ($= R_m$)

→ the subscripts on b are the same as on R but they cannot be the same or they would be repeated. So use

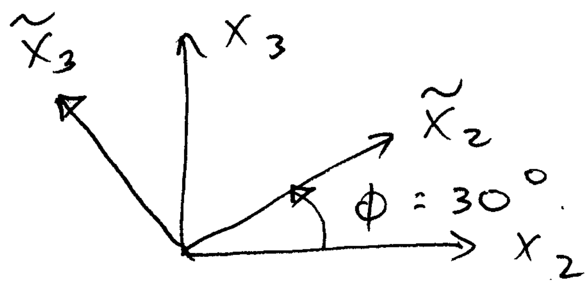
b_{mn} $(m=n)$

→ Also note that the multiplier in $\sin(a\pi x)$ is the same as the subscript.

Thus: $\boxed{R_m = b_{mn} \sin m\pi x \quad (m=n)}$

M6.3 $\underline{F} = 8\underline{i}_1 - 5\underline{i}_2 + 3\underline{i}_3$

(a) Draw the rotation:



with x_1 out of the paper

~ represents the rotated system

$\tilde{x}_1 = x_1$

Know that the rotation can be represented via direction cosines

$$\tilde{F}_{\tilde{i}} = l_{\tilde{i}j} F_j$$

Determine the direction cosines:

$$l_{\tilde{1}1} = \cos(0°) = 1$$
$$l_{\tilde{1}2} = \cos(90°) = 0$$
$$l_{\tilde{1}3} = \cos(90°) = 0$$
$$l_{\tilde{2}2} = \cos\phi = \sqrt{3}/2 = 0.866$$
$$l_{\tilde{2}1} = \cos(-90°) = 0$$
$$l_{\tilde{2}3} = \cos(90 - \phi) = \sin\phi = 0.5$$

$$\ell_{\tilde{3}1} = \cos(-90°) = 0$$
$$\ell_{\tilde{3}2} = \cos(-90-\phi) = \cos(90+\phi) = -\sin\phi = -0.5$$
$$\ell_{\tilde{3}3} = \cos(-\phi) = \cos\phi = 0.866$$

$$\tilde{F}_1 = \ell_{\tilde{1}1} F_1 + \ell_{\tilde{1}2} F_2 + \ell_{\tilde{1}3} F_3 = F_1$$
$$\tilde{F}_2 = \ell_{\tilde{2}1} F_1 + \ell_{\tilde{2}2} F_2 + \ell_{\tilde{2}3} F_3 = (0.866)(-5) + (0.5)(3)$$
$$= -2.83$$
$$\tilde{F}_3 = \ell_{\tilde{3}1} F_1 + \ell_{\tilde{3}2} F_2 + \ell_{\tilde{3}3} F_3 = (-0.5)(-5) + (0.866)(3)$$
$$= 5.10$$

$$\boxed{\underline{\tilde{F}} = 8\underline{\tilde{i}}_1 - 2.83\underline{\tilde{i}}_2 + 5.10\underline{\tilde{i}}_3}$$

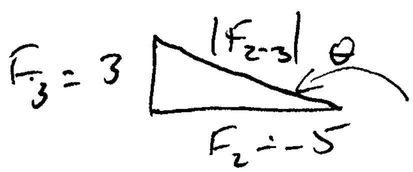
(5) Prove

1. First note that the magnitude in the 1-direction is the same since there is no rotation there (that axis is rotated about)

2. The 2-D vector creates a right triangle with the x_2-magnitude as one side and the x_3 magnitude as the other.

With this, determine the overall magnitude of the vector via:

$$|F_{2-3}| = \sqrt{(F_2)^2 + (F_3)^2}$$



$$= \sqrt{(-5)^2 + (3)^2}$$

$$= 5.83$$

and the angle relative to $x_2 - x_3$:

$$\tan^{-1}\left(\frac{F_3}{F_2}\right) = 149^\circ$$

③ Now do the same for the rotated $\tilde{x}_2 - \tilde{x}_3$ system:

$$|\tilde{F}_{2-3}| = \sqrt{(\tilde{F}_2)^2 + (\tilde{F}_3)^2}$$

$$= \sqrt{(-2.83)^2 + (5.10)^2}$$

$$= 5.83$$

<u>the same</u> ✓

and the angle relative to $\tilde{x}_2 - \tilde{x}_3$:

$$\tan^{-1}\left(\frac{\tilde{F}_3}{\tilde{F}_2}\right) = 119^\circ$$

Note rotation of $+30^\circ$ from $\tan^{-1}\left(\frac{F_3}{F_2}\right)$

→ <u>same</u> as $\phi = 30^\circ$ ✓

<u>Proven</u>