UE Fluids F16 Solution Fall '06

a) $\Psi(r,\theta) = V_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln r$

In this case, it's easier to work in cartesian coordinates:

$$\Psi(x,y) = V_\infty y\left(1 - \frac{R^2}{x^2+y^2}\right) + \frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2}$$

$$u(x,y) = \frac{\partial\Psi}{\partial y} = V_\infty\left(1 - \frac{R^2}{x^2+y^2}\right) + V_\infty y\frac{R^2\cdot 2y}{(x^2+y^2)^2} + \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$$

On y-axis, $x=0$:

$$u(y) = V_\infty\left(1 - \frac{R^2}{y^2}\right) + V_\infty\frac{R^2\cdot 2y^2}{y^4} + \frac{\overset{4V_\infty R}{\Gamma}}{2\pi}\frac{y}{y^2}$$

or

$$\boxed{u(y) = \underbrace{V_\infty}_{u_{freestream}} + \underbrace{V_\infty\frac{R^2}{y^2}}_{u_{doublet}} + \underbrace{\frac{2}{\pi}V_\infty\frac{R}{y}}_{u_{vortex}}}$$

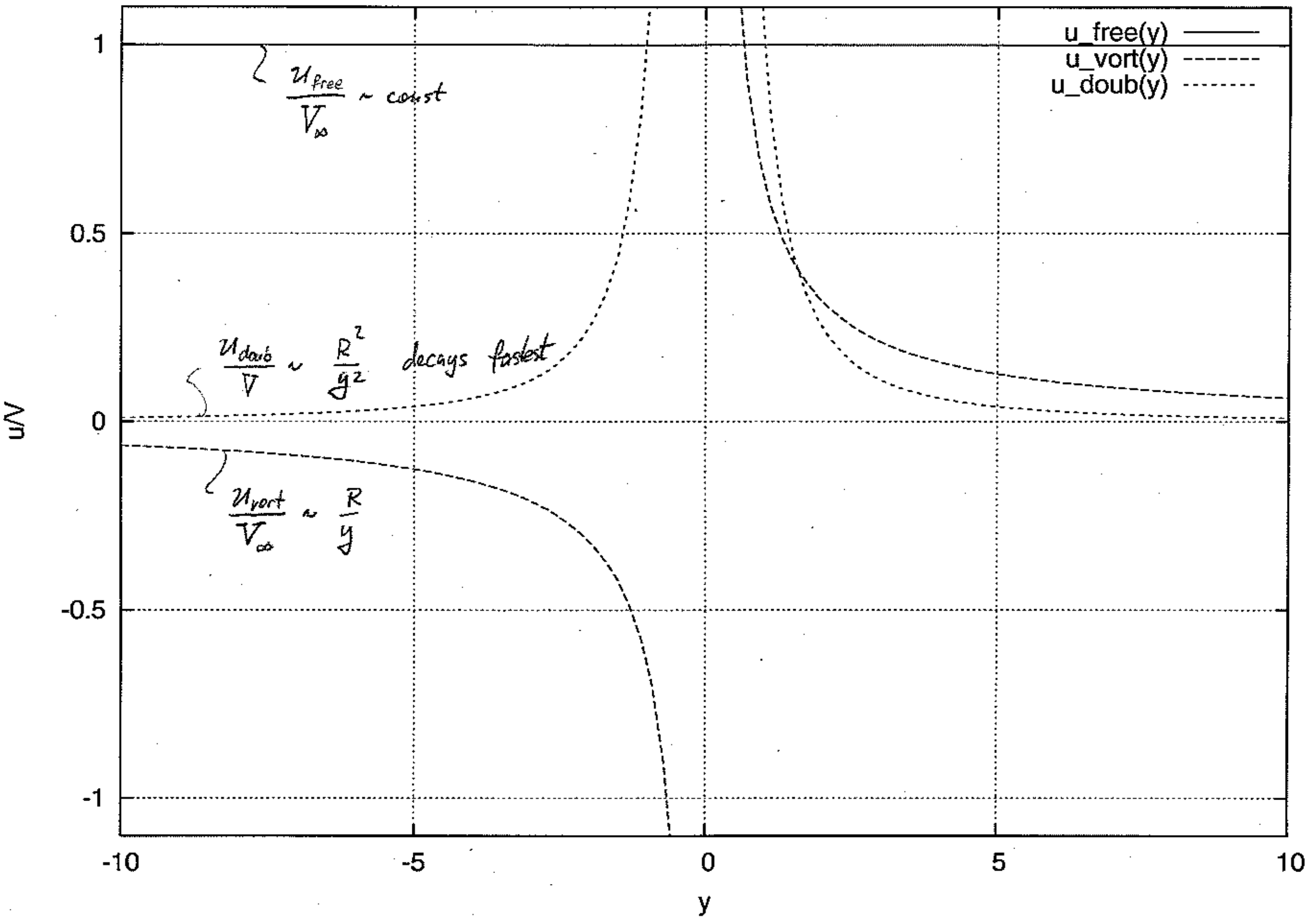
b) plot attached

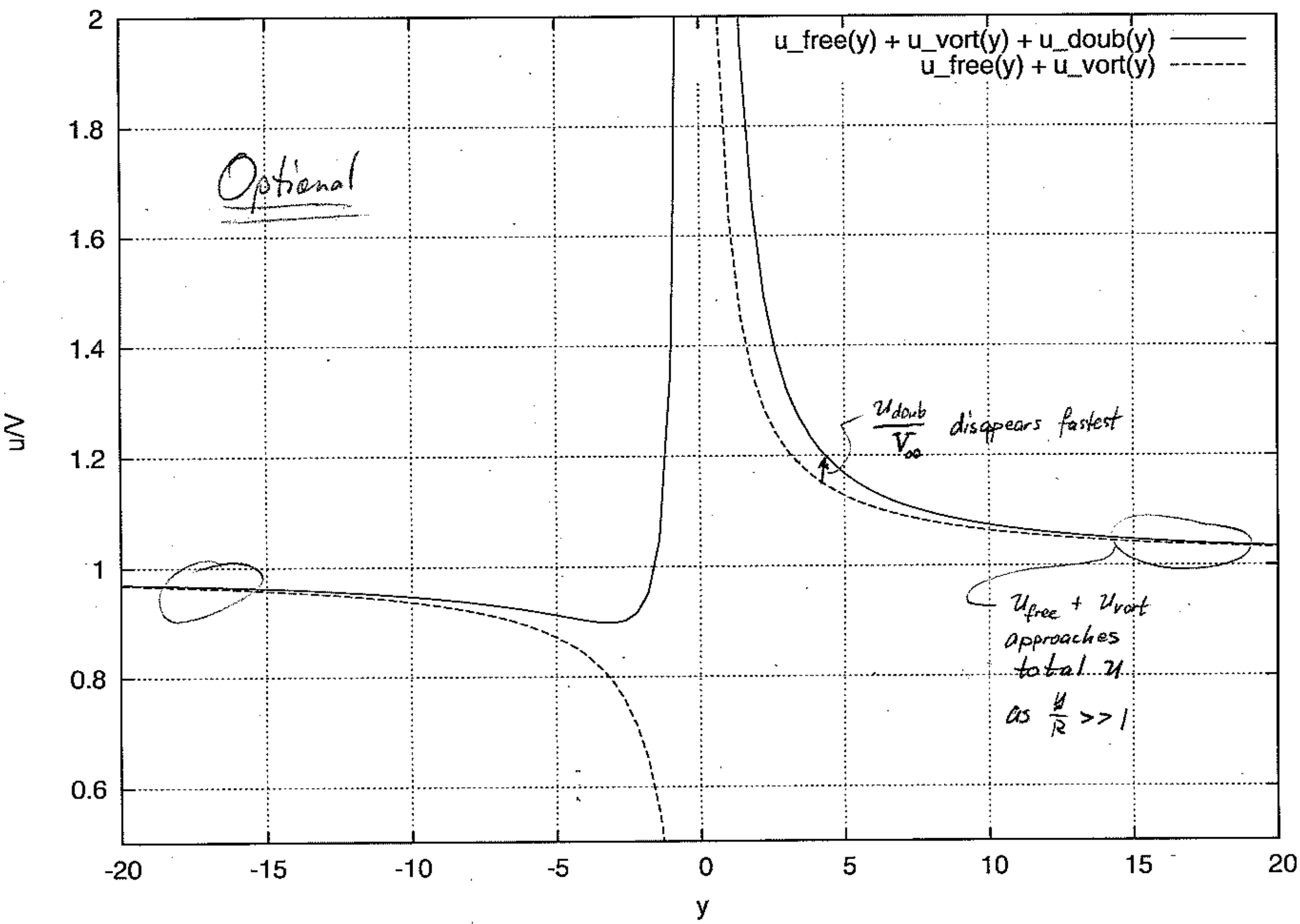
c) i) At small distances, $\boxed{\frac{u_{doublet}}{V_\infty} \sim \frac{R^2}{y^2} \text{ is larger than } \frac{u_{vortex}}{V_\infty} \sim \frac{R}{y}}$

ii) At large distances, $\boxed{\frac{u_{doublet}}{V} \sim \frac{R^2}{y^2} \text{ is smaller than } \frac{u_{vortex}}{V_\infty} \sim \frac{R}{y}}$

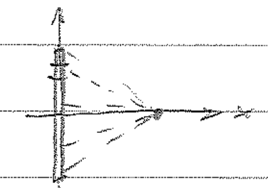
d) Because $u_{doublet} \sim \frac{R^2}{y^2}$, it decays faster with distance than $u_{freestream} \sim$ const or $u_{vortex} \sim \frac{R}{y}$.
So far away, $u_{doublet}$ will disappear first, leaving only $u_{freestream} + u_{vortex}$. If an object has $L' \neq 0$ then also it must have $V_\infty \neq 0$ and also $\Gamma \neq 0$ ($L' = \rho V_\infty \Gamma$).
So if there's lift, both $u_{freestream}$ and u_{vortex} are present, and these are dominant far away.

P6 Solution


u_free(y)
u_vort(y)
u_doub(y)
$\frac{u_{free}}{V_\infty} \sim$ const
$\frac{u_{doub}}{V} \sim \frac{R^2}{y^2}$ decays fastest
$\frac{u_{vort}}{V_\infty} \sim \frac{R}{y}$
u/V
y


u_free(y) + u_vort(y) + u_doub(y)
u_free(y) + u_vort(y)
Optional
$\frac{u_{doub}}{V_\infty}$ disapears fastest
$u_{free} + u_{vort}$ approaches total u as $\frac{y}{R} >> 1$
u/V
y

UE Fluids F17 Solution Fall '06



a) Adding up contributions of all dy pieces:

$$\phi(x) = \int d\phi = \int_{-l/2}^{l/2} \frac{\lambda}{2\pi} \ln\sqrt{x^2+y^2}\, dy$$

Since $\lambda = \text{const}$, and $\ln\sqrt{(\,)} = \frac{1}{2}\ln(\,)$: $\phi(x) = \frac{\lambda}{2\pi}\int_{-l/2}^{l/2} \frac{1}{2}\ln\left(x^2+y^2\right) dy$

Using given integral: $x \to y$, $a \to x$

$$\int_{-l/2}^{l/2} \frac{1}{2}\ln\left(y^2+x^2\right) dy = \left[\frac{1}{2} y \ln\left(y^2+x^2\right) - y + x \arctan\left(\frac{y}{x}\right)\right]_{-l/2}^{l/2}$$

$$= \frac{1}{2}\left\{\frac{l}{2}\ln\left[\left(\frac{l}{2}\right)^2+x^2\right] + \frac{l}{2}\ln\left[\left(\frac{l}{2}\right)^2+x^2\right]\right\} - l + x\left\{\arctan\left(\frac{l/2}{x}\right) - \arctan\left(\frac{-l/2}{x}\right)\right\}$$

$$\phi(x) = \frac{\lambda}{2\pi}\left\{\frac{1}{2} l \cdot \ln\left[\left(\frac{l}{2}\right)^2 + x^2\right] - l + 2x \arctan\left(\frac{l/2}{x}\right)\right\}$$

b) $$u(x) = \frac{\partial\phi}{\partial x} = \frac{\lambda}{2\pi}\left\{\cancel{\frac{xl}{(l/2)^2+x^2}} + 2\arctan\left(\frac{l/2}{x}\right) - \cancel{\frac{2x \cdot l/2}{(l/2)^2+x^2}}\right\}$$

$$\boxed{\begin{aligned} u(x) &= \frac{\lambda}{2\pi} \cdot 2\arctan\left(\frac{l/2}{x}\right) \\ u_{point}(x) &= \frac{\Lambda}{2\pi x} \end{aligned}}$$

$\boxed{\Lambda = \lambda l}$ for same total strength

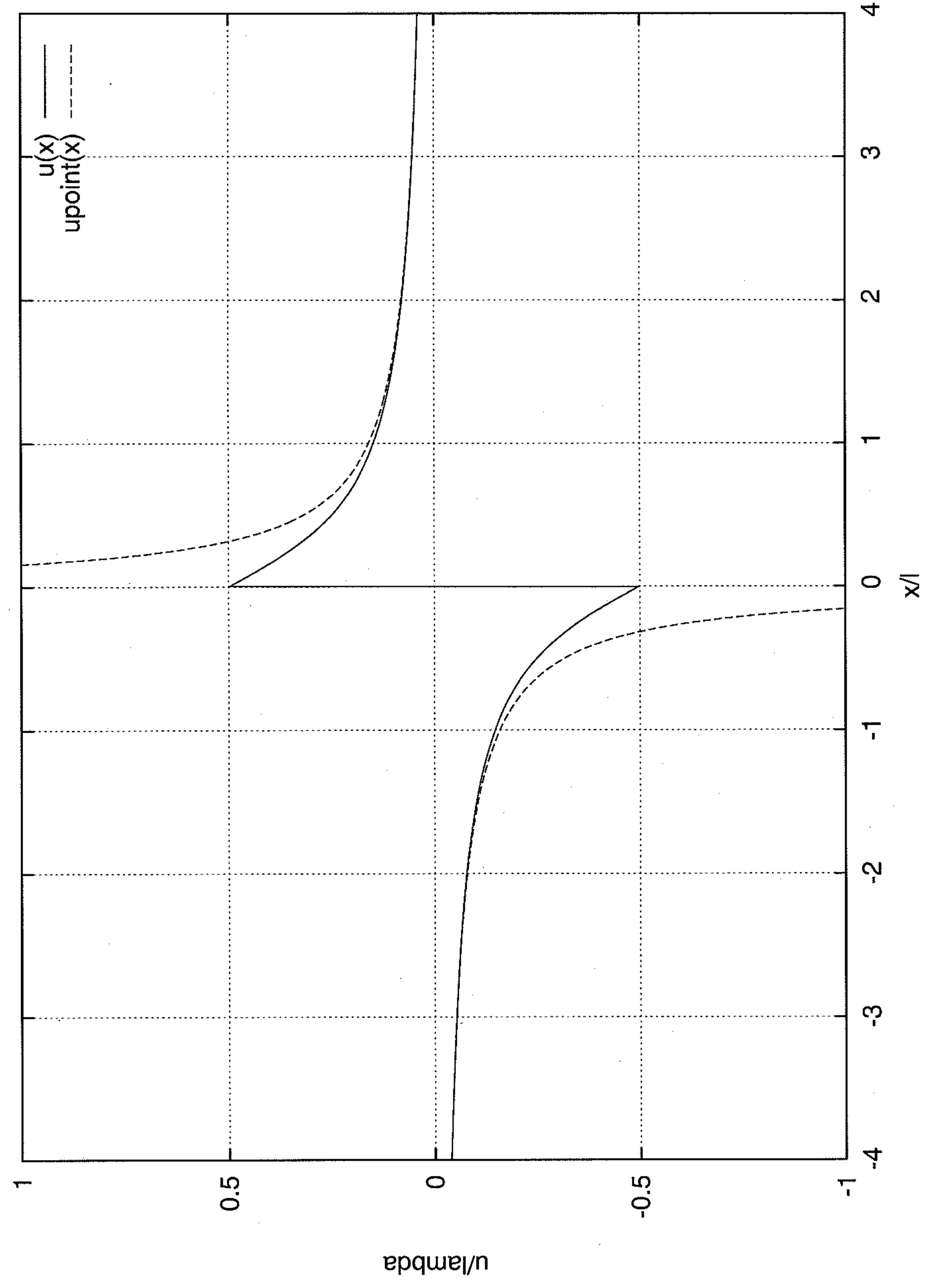
c) See plot. For $x >> l/2$, or $\frac{l/2}{x} << 1$, $\rightarrow \arctan\left(\frac{l/2}{x}\right) \approx \frac{l/2}{x}$

$$\therefore \boxed{u(x) \approx \frac{\lambda}{2\pi} \cdot 2\left(\frac{l/2}{x}\right) \approx \frac{\lambda l}{2\pi x} \approx \frac{\Lambda}{2\pi x} \approx u_{point}(x)}$$

$u(x)$ approaches $u_{point}(x)$ for $x >> \frac{l}{2}$ (far away)

Sheet looks like a point source far away.

F17 Solution



PAL
10/9/06

Unified Engineering Problem Set
Week 7 Fall, 2007
SOLUTIONS

M7.1 Begin by writing out the stress equilibrium equations as we have them in tensorial notation:

$$\frac{\partial \sigma_{mn}}{\partial x_n} + f_n = 0$$

expand this:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

Recall the symmetry of the stress tensor: $\sigma_{mn} = \sigma_{nm}$

(a) To go from tensorial to engineering notation, recall:

$$x_1 \rightarrow x$$
$$x_2 \rightarrow y$$
$$x_3 \rightarrow z$$

and a similar conversion on subscripts on the stresses. So:

$$\boxed{\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0 \\
&\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0 \\
&\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0
\end{aligned}}$$

Notes: • expression of stress with subscripts still remains symmetric

• τ can be used in place of σ for shear stresses (2 different subscripts: τ_{xy}, τ_{xz}, τ_{yz})

(b) A state of plane stress has:

• no out-of-plane components

$$\Rightarrow \sigma_z = \sigma_{yz} = \sigma_{xz} = 0$$

• no out of plane gradient

$$\Rightarrow \frac{\partial}{\partial z} = 0$$

And, since there are no forces in the out-of-plane (z) direction, the body force in that direction (f_z) must be zero

We thus end up with two equations:

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0 \end{aligned}}$$

This could also be done in tensorial notation:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

M 7.2 Given:

$$\sigma_{11} = A \; ; \; \sigma_{22} = B \; ; \; \sigma_{33} = C$$

$$\sigma_{23} = D x_2 - E x_3 + C_{23} + f_{23}(x_1)$$

(since σ_{23} has a + linear variation of D in x_2 and a − linear variation of E in x_3. Can add a constant to that [use C_{23}] as well as a possible variation in x_1 [use $f_{23}(x_1)$])

<u>NOTE</u>: In most general case, all stresses are functions of all three dimensions

$$\sigma_{mn}(x_1, x_2, x_3)$$

Use this information with the stress equations of equilibrium to determine information about the other stresses. First write the 3-D stress equilibrium equations -- there are no body forces.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0 \qquad (2)$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \qquad (3)$$

Take partial derivatives of what is known about stresses to give information:

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0$$

$$\frac{\partial \sigma_{22}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{33}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{23}}{\partial x_2} = D$$

$$\frac{\partial \sigma_{23}}{\partial x_3} = -E$$

use in (2):

$$\frac{\partial \sigma_{12}}{\partial x_1} + 0 - E = 0$$

$$\Rightarrow \frac{\partial \sigma_{12}}{\partial x_1} = E$$

using multi-variable calculus and getting functions of integration:

$$\sigma_{12} = E x_1 + f_{12}(x_2, x_3)$$

use differential results in (3):

$$\frac{\partial \sigma_{13}}{\partial x_1} + D + 0 = 0$$

$$\Rightarrow \frac{\partial \sigma_{13}}{\partial x_1} = -D$$

So: $\sigma_{13} = -D x_1 + f_{13}(x_2, x_3)$

Taking appropriate partials of these two stresses for use in equation (1)

$$\frac{\partial \sigma_{12}}{\partial x_2} = \frac{\partial f_{12}(x_2, x_3)}{\partial x_2}$$

and
$$\frac{\partial \sigma_{13}}{\partial x_3} = \frac{\partial f_{13}(x_2, x_3)}{\partial x_3}$$

This yields:
$$0 + \frac{\partial f_{12}(x_2, x_3)}{\partial x_2} + \frac{\partial f_{13}(x_2, x_3)}{\partial x_3} = 0$$

This is all the information we can get. So this is what can be said about the stress field:

- The extensional stresses are constant with:
$$\boxed{\sigma_{11} = A, \ \sigma_{22} = B, \ \sigma_{33} = C}$$

- The shear stress in the x_2-x_3 plane has linear variations in x_2 and x_3 with a value of $+D$ in x_2 and of $-E$ in x_3, plus some function related to x_1 including a constant:
$$\boxed{\sigma_{23} = D x_2 - E x_3 + f_{23}(x_1) + C_{23}}$$

- The shear stresses involving the x_1-direction have a linear variation in x_1 (σ_{12} of E, σ_{13} of $-D$) and have functional variation in x_2 and x_3 linked by partial derivatives. In addition, constants must be added:

$$\boxed{\begin{aligned} \sigma_{12} &= Ex_1 + f_{12}(x_2, x_3) + C_{12} \\ \sigma_{13} &= -Dx_1 + f_{13}(x_2, x_3) + C_{13} \\ &\text{where: } \frac{\partial f_{12}(x_2, x_3)}{\partial x_2} = -\frac{\partial f_{13}(x_2, x_3)}{\partial x_3} \end{aligned}}$$

M7.3 We have a two-dimensional field of displacement, thus all out-of-plane strains are zero:

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$$

Define the in-plane strain-displacement relations:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

<u>and</u> $\underline{u} = u_1 \underline{i}_1 + u_2 \underline{i}_2$

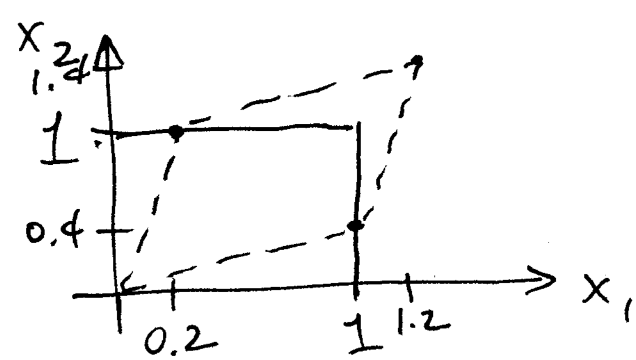
so:

(a) $\underline{u} = (0.020 x_2)\underline{i}_1 + (0.040 x_1)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0.020 + 0.040) = 0.030$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0.030 \end{aligned}}$$

This is pure shear

(b) $\underline{u} = -(0.030\,x_2)\underline{i}_1 + (0.030\,x_1)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(-0.030 + 0.030) = 0$$

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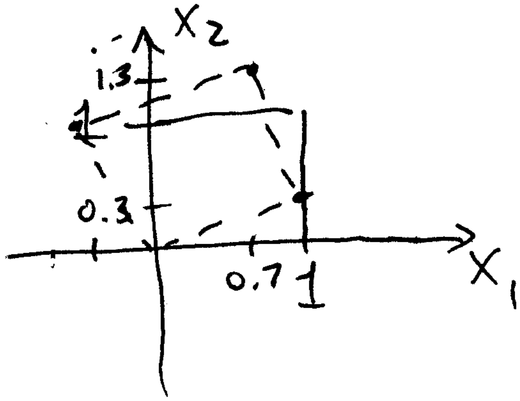

$$\boxed{\begin{aligned}\epsilon_{11} &= 0\\ \epsilon_{22} &= 0\\ \epsilon_{12} &= 0\end{aligned}}$$

This is <u>pure</u> <u>rotation</u>

(c) $\underline{u} = (-0.050\,x_1 + 0.030\,x_2)\underline{i}_1 + (0.030\,x_1 + 0.010\,x_2)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = -0.050$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0.010$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(+0.030 + 0.030) = 0.030$$

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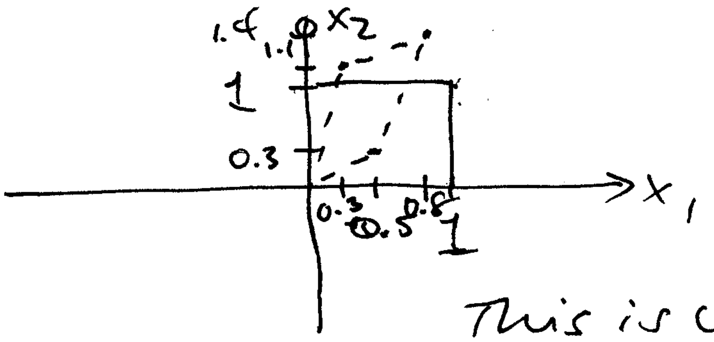


$$\boxed{\begin{aligned}\epsilon_{11} &= -0.050\\ \epsilon_{22} &= 0.010\\ \epsilon_{12} &= 0.030\end{aligned}}$$

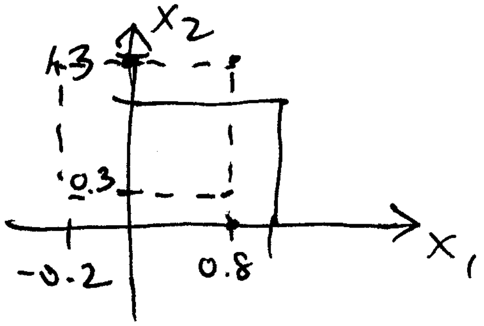
This is <u>combined</u> <u>elongation</u> and <u>shear</u>

(d) $\underline{u} = -(0.020)\,\underline{i}_1 + (0.030)\,\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is pure translation in x_1 and x_2

(e) $\underline{u} = -(0.030\,x_1)\,\underline{i}_1 + (0.020\,x_2)\,\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = -0.030$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0.020$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

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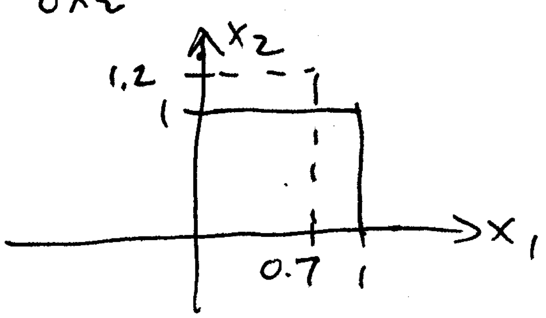

$$\boxed{\begin{aligned} \epsilon_{11} &= -0.030 \\ \epsilon_{22} &= 0.020 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is pure elongation in two directions (one being positive, one being negative)

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