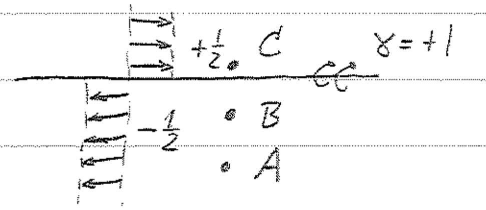
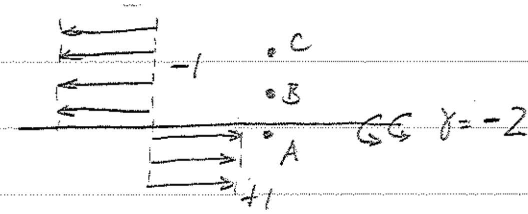
UE Fluids F18 Solution Fall '06

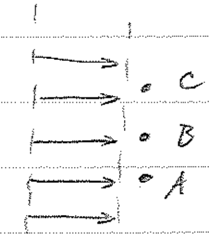
Velocity field of top sheet:



Velocity field of bottom sheet

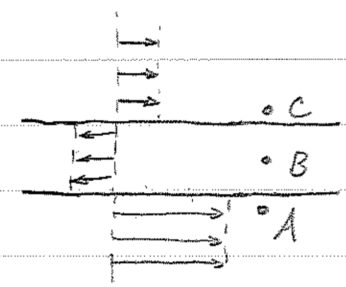


Velocity field of freestream:



Add all three components:

$$\begin{array}{lllll} & & \text{top} & \text{bot} & V_\infty \\ \text{At } A: & u_A = & -\tfrac{1}{2} & +1 & +1 \quad = +\tfrac{3}{2} \\ \text{At } B: & u_B = & -\tfrac{1}{2} & -1 & +1 \quad = -\tfrac{1}{2} \\ \text{At } C: & u_C = & +\tfrac{1}{2} & -1 & +1 \quad = +\tfrac{1}{2} \end{array}$$



UE Fluids F19 Solution Fall '06

a) Minimize $\frac{C_d}{C_\ell} = \frac{C_{d_0}}{C_\ell} + C_{d_4} C_\ell^3$

Set $\frac{d}{dC_\ell}\left(\frac{C_d}{C_\ell}\right) = 0 \Rightarrow -\frac{C_{d_0}}{C_\ell^2} + 3C_{d_4}C_\ell^2 = 0 \Rightarrow \boxed{C_\ell = \left(\frac{C_{d_0}}{3C_{d_4}}\right)^{1/4} = 0.537}$

b) Minimize $\frac{C_d}{C_\ell} = \frac{C_{d_0} + \Delta C_{d_0}}{C_\ell} + C_{d_2} C_\ell + C_{d_4} C_\ell^3$

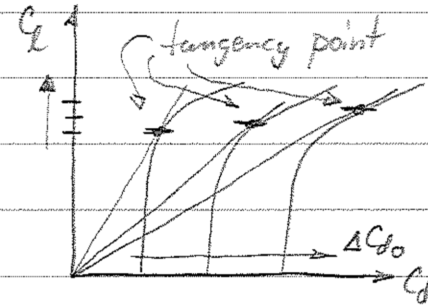
$$\frac{d}{dC_\ell}\left(\frac{C_d}{C_\ell}\right) = 0 \Rightarrow -\frac{C_{d_0} + \Delta C_{d_0}}{C_\ell^2} + C_{d_2} + 3C_{d_4}C_\ell^2 = 0$$

$$\text{or} \quad 3C_{d_4}C_\ell^4 + C_{d_2}C_\ell^2 - (C_{d_0} + \Delta C_{d_0}) = 0$$

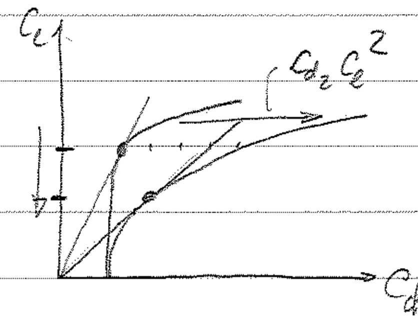
$$C_\ell^2 = \frac{1}{6C_{d_4}}\left\{-C_{d_2} + \sqrt{C_{d_2}^2 + 12C_{d_4}(C_{d_0} + \Delta C_{d_0})}\right\} = 0.2965$$

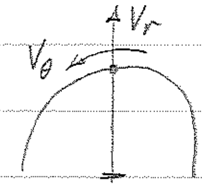
$$\boxed{C_\ell = \sqrt{0.2965} = 0.5446}$$

c) ΔC_{d_0} translates C_d curve to right:
This will clearly increase the C_ℓ value at the tangency point where C_ℓ/C_d is maximum (C_d/C_ℓ is minimum)



C_{d_2} adds a quadratic component which tilts and deforms the C_d curve for large C_ℓ values.
This will decrease the C_ℓ value at the tangency point.





a) Just above point A: $r = R+n$, $\sin\theta = 1$, $\cos\theta = 0$

i) $\boxed{V(n) = \sqrt{\cancelto{0}{V_r}^{2} + V_\theta^2} = |V_\theta| = V_\infty\left(1+\frac{R^2}{(R+n)^2}\right)} = 2V_\infty$ at $n=0$ at A

ii) $\boxed{p(n) = p_{0\infty} - \frac{1}{2}\rho V(n)^2 = p_{0\infty} - \frac{1}{2}\rho V_\infty^2\left(1+\frac{R^2}{(R+n)^2}\right)^2}$

b) $$\frac{\partial p}{\partial n} = -\frac{1}{2}\rho V_\infty^2 \cdot 2\left(1+\frac{R^2}{(R+n)^2}\right)\cdot\frac{-2R^2}{(R+n)^3} = 2\rho V_\infty^2\left(1+\frac{R^2}{(R+n)^2}\right)\frac{R^2}{(R+n)^3}$$

Evaluate $\frac{\partial p}{\partial n}$ at point A, $n=0$: $\boxed{\left.\frac{\partial p}{\partial n}\right|_A = 4\rho V_\infty^2 \frac{1}{R}}$

c) Curvature $\boxed{K = -\frac{1}{R}}$

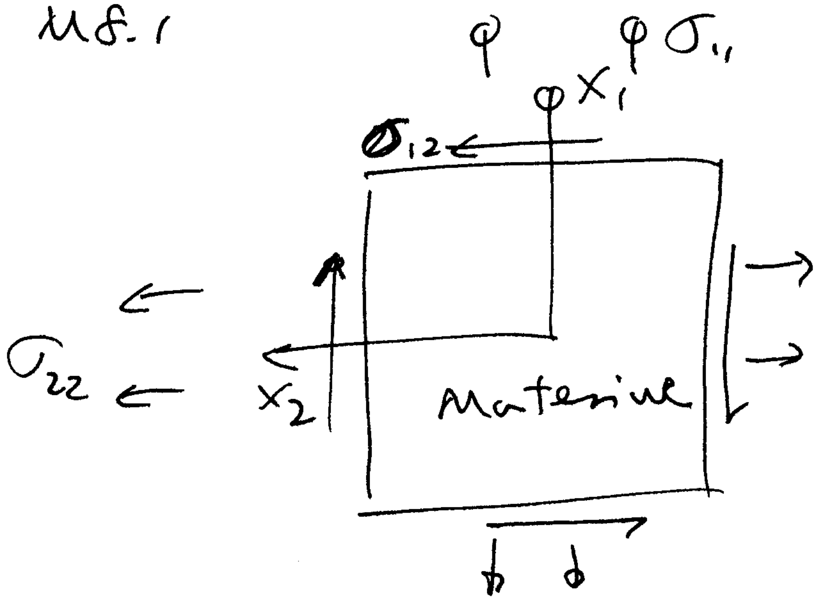
$$\frac{\partial p}{\partial n} = -\rho V^2 K = \rho V^2 \frac{1}{R}, \quad V = 2V_\infty \text{ at point A at } n=0$$

$\Longrightarrow \boxed{\left.\frac{\partial p}{\partial n}\right|_A = \rho(2V_\infty)^2\frac{1}{R} = 4\rho V_\infty^2\frac{1}{R}}$ same as b)

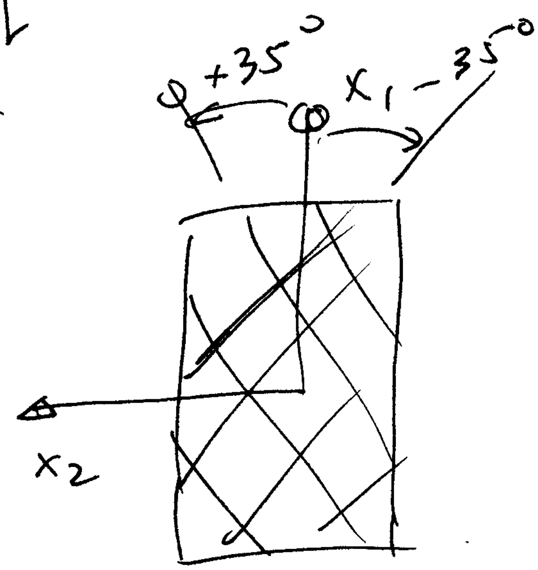
PAL
10/19/06

Unified Engineering Problem Set
Week 8 Fall, 2006
SOLUTIONS

M8.1

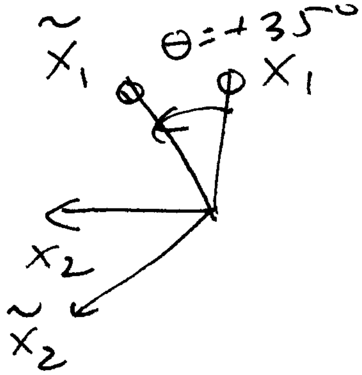


$\sigma_{11} = 40$ MPa
$\sigma_{22} = -15$ MPa
$\sigma_{12} = -20$ MPa

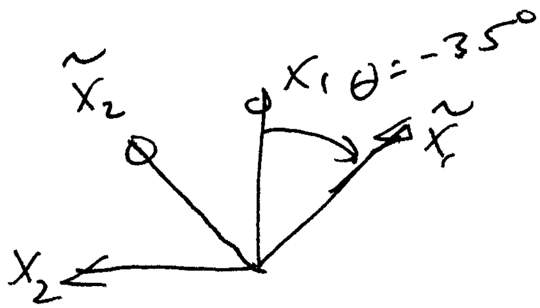


(a) To find the stress state relative to the "fiber axes", align the transformed axis system with its $\tilde{x}_1$ axis along the fiber direction.

So for the $\theta = +35^\circ$ case:



and for the $\theta = -35^\circ$ case:



Then use the transformation equations (in 2-D)

$$\tilde{\sigma}_{11} = \cos^2\theta\,\sigma_{11} + \sin^2\theta\,\sigma_{22} + 2\cos\theta\sin\theta\,\sigma_{12}$$

$$\tilde{\sigma}_{22} = \sin^2\theta\,\sigma_{11} + \cos^2\theta\,\sigma_{22} - 2\cos\theta\sin\theta\,\sigma_{12}$$

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

<u>for the fiber axes for $\theta = 35^\circ$:</u>

$\sin\theta = 0.574$
$\cos\theta = 0.819$

$$\Rightarrow \tilde{\sigma}_{11} = 0.671(40\,\text{MPa}) + 0.329(-15\,\text{MPa}) + 0.940(-20\,\text{MPa})$$

$$\tilde{\sigma}_{22} = 0.329(40\,\text{MPa}) + 0.671(-15\,\text{MPa}) - 0.940(-20\,\text{MPa})$$

$$\tilde{\sigma}_{12} = -0.470(40\,\text{MPa}) + 0.470(-15\,\text{MPa}) + (0.671 - 0.329)(-20\,\text{MPa})$$

working through this gives

for $\theta = +35°$: $\boxed{\begin{aligned} \sigma_{11} &= 3.1 \text{ MPa} \\ \sigma_{22} &= 21.9 \text{ MPa} \\ \sigma_{12} &= -32.7 \text{ MPa} \end{aligned}}$

for the fiber axes for $\theta = -35°$:

$\sin\theta = -0.574$

$\cos\theta = 0.819$

$$\Rightarrow \tilde{\sigma}_{11} = 0.671(40\text{ MPa}) + 0.329(-15\text{ MPa}) - 0.940(-20\text{ MPa})$$

$$\tilde{\sigma}_{22} = 0.329(40\text{ MPa}) + 0.671(-15\text{ MPa}) + 0.940(-20\text{ MPa})$$

$$\tilde{\sigma}_{12} = 0.470(40\text{ MPa}) - 0.470(-15\text{ MPa}) + (0.671 - 0.329)(-20\text{ MPa})$$

working through this gives

for $\theta = -35°$: $\boxed{\begin{aligned} \sigma_{11} &= 40.7 \text{ MPa} \\ \sigma_{22} &= -15.7 \text{ MPa} \\ \sigma_{12} &= -19.0 \text{ MPa} \end{aligned}}$

(b) For plane stress, the principal stresses are the roots of the equation:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

using the stresses in the loading axes as given:

$$\tau^2 - \tau(40\,MPa - 15\,MPa) + [(40\,MPa)(-15\,MPa) - (-20\,MPa)^2] = 0$$

$$\Rightarrow \tau^2 - (25\,MPa)\tau - 1000(MPa)^2 = 0$$

solve via the quadratic formula:

roots: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $ax^2 + bx + c = 0$

$$\Rightarrow \tau = \frac{-(-25\,MPa) \pm \sqrt{(-25\,MPa)^2 - 4(1)(1000\,MPa)^2}}{2(1)}$$

$$= \frac{25 \pm \sqrt{4625}}{2}\,MPa$$

$$= \frac{25 \pm 68}{2}\,MPa$$

$$\Rightarrow \tau = 46.5\,MPa,\ -21.5\,MPa$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_I &= 46.5\,MPa \\ \sigma_{II} &= -21.5\,MPa \end{aligned}}$$

to find the associated directions, use the expression:

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)$$

$$\Rightarrow \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2(-20\,MPa)}{40\,MPa-(-15\,MPa)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(-\frac{40}{55}\right)$$

$$= \frac{1}{2}\tan^{-1}(-0.727)$$

$$\Rightarrow \theta_p = \frac{1}{2}(-36.0^\circ)$$

$$\Rightarrow \theta_p = -18.0^\circ \text{ for } \sigma_{II}$$

with σ_I rotated 90° from that

so:

$$\boxed{\begin{aligned}\theta_{p_{II}} &= -18.0^\circ\\ \theta_{p_I} &= +108.0^\circ = +72.0^\circ\end{aligned}}$$

Check the angles via the transformation equation for shear and that shear stress goes to zero (definition of principal stress):

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\sigma_{12}$$

$$\Rightarrow 0 \stackrel{?}{=} -(-0.309)(0.952)(40\,MPa) + (0.952)(-0.309)(-15\,MPa) + (0.905 - 0.096)(-20\,MPa)$$

$$0 \stackrel{?}{=} +11.77 + 4.41 - 16.18 \checkmark \text{ (yes)}$$

(same as for +72.0°)

As to the effect of the direction of the composite fiber axes....

... this does not change the same stress state and thus the principal stresses $\boxed{\text{do not change}}$

... as to the principal directions, they $\boxed{\text{stay the same}}$ relative to the loading axes x_1-x_2, only the fiber direction (+35° or −35°) must be properly subtracted/added to get the direction relative to the fiber directions.

(c) maximum shear stress(es) occur along planes/directions that are at 45° to the principal axes..

So: direction of maximum shear stress:

$\theta_{p_I} + 45°$ from x_1 $= +117.0° = -63.0°$

$\theta_{p_{II}} + 45°$ from x_1 $= +27.0°$

$\boxed{\text{Directions of maximum shear} = -63.0°, +27.0°}$

the value(s) of the maximum shear stress(es) in the x_1-x_2 plane is:

$$\frac{\sigma_I - \sigma_{II}}{2} = \frac{46.5\,MPa - (-21.5\,MPa)}{2}$$

$\Rightarrow$ $\boxed{\text{value of maximum shear stress} = 34.0\ MPa}$

(this takes on a sign of + and −)

There are two other maximum shear stresses out of the x_1-x_2 plane. For the case of plane stress, the out-of-plane principal stress is zero ($\sigma_{III} = 0$).

So we have:

$$|\tau_{max}| = \left|\frac{\sigma_I - \sigma_{III}}{2}\right| = \frac{46.5\,MPa}{2} = \boxed{23.3\,MPa}$$

This is in a plane at 45° to the x_1-x_2 plane rotated about the x_2-axis

$$|\tau_{max}| = \left|\frac{\sigma_{II} - \sigma_{III}}{2}\right| = \frac{21.5\,MPa}{2} = \boxed{10.8\,MPa}$$

This is in a plane at 45° to the x_1-x_2 plane rotated about the x_1-axis

(d) Draw the Mohr's circle as specified in the instructions (see associated figure)

① Plot σ_{11}, σ_{12} (40 MPa, +20 MPa) as Point A

② Plot σ_{22}, σ_{21} (−15 MPa, −20 MPa) as Point B

③ Connect A and B

④ Draw circle of diameter of the line AB about the point where the line AB crosses the horizontal axis (denote this point as C).

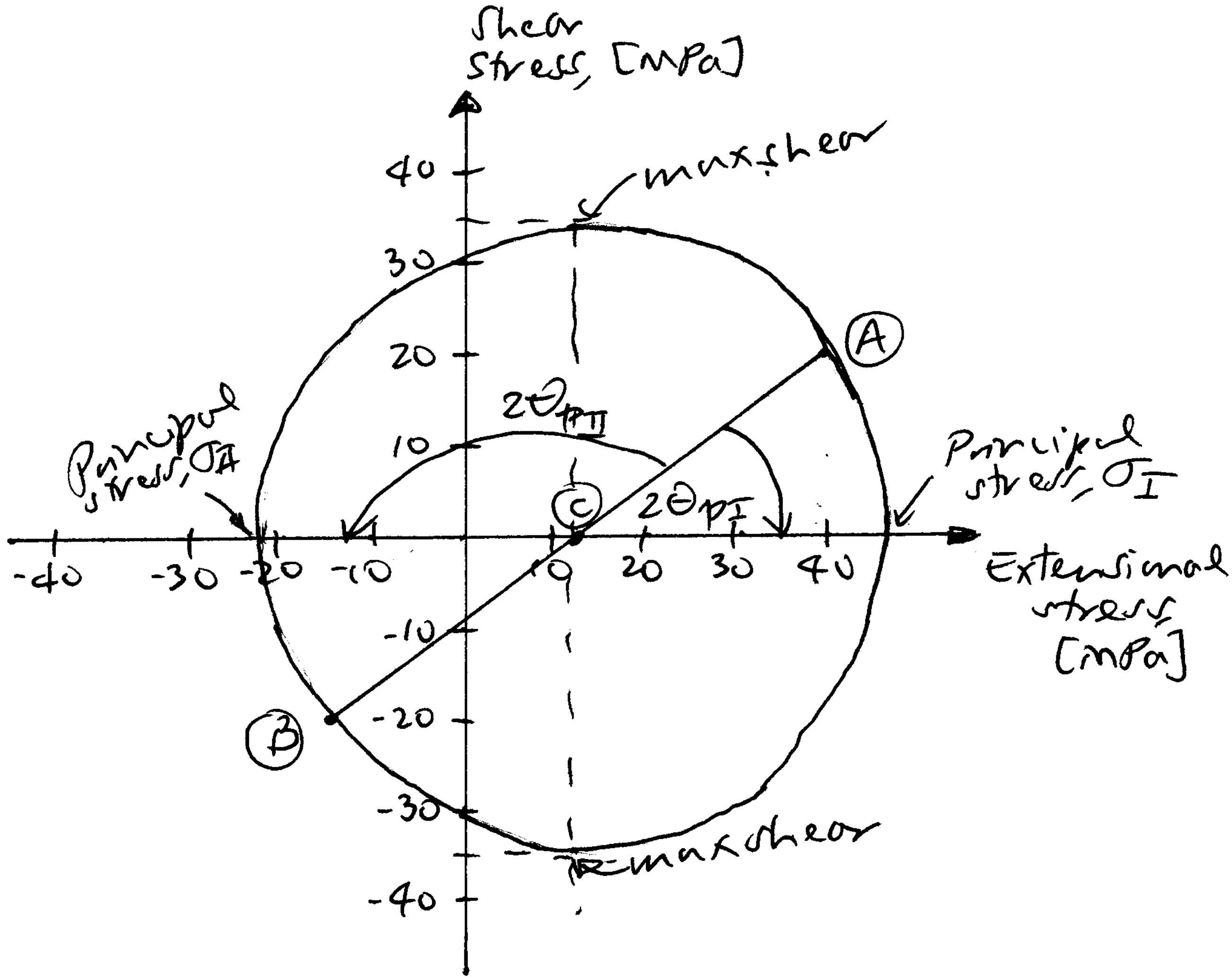
$$\text{point } C = \text{midpoint} = \frac{\sigma_{11} + \sigma_{22}}{2} = \frac{40\text{ MPa} - 15\text{ MPa}}{2}$$

$$= 12.5\text{ MPa}$$

→ Principal stresses and directions (2-D)

The intersection of the circle with the horizontal axis gives the two values of the principal stresses. By sight, these are approximately +46 MPa and −22 MPa which correspond to the results in (b). One can be more formal by finding the circle diameter $\left(= 2\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}\right)$ and then adding and subtracting half of this from the midpoint value (point C = 12.5 MPa)

Directions (angles) can be measured via a protractor or ~~that~~ half the angle from line AB to the horizontal axis at the two points (2 directions)



Shear stress, [MPa]
max shear
40
30
20
10
-10
-20
-30
-40
A
B
C
$2\theta_{pII}$
$2\theta_{pI}$
Principal stress, σ_{II}
Principal stress, σ_{I}
-40
-30
-20
-10
10
20
30
40
Extensional stress, [MPa]
max shear

→ Maximum shear stress

This is the upper and lower "reaches" of the circle along the vertical direction. It can be read off to be just about 35 MPa (and −35 MPa). This is exactly the radius of the circle and is in correspondence to the results in (c).

The direction(s) of the maximum shear stress(es) are 90° on Mohr's circle from that of the principal stresses. Divide by 2, since this is twice the rotation angle, and that is 45° added on to $\theta_{p_{I,II}}$ as in (c).

Note: Only the maximum shear stress in the $x_1 - x_2$ plane can be determined since Mohr's circle only allows rotation in the $x_1 - x_2$ plane.

M8.2 Begin by writing out the transformation equations for in-plane strain (as for the case of plane stress):

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{22} = \sin^2\theta\, \epsilon_{11} + \cos^2\theta\, \epsilon_{22} - 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{12} = -\sin\theta\cos\theta\, \epsilon_{11} + \cos\theta \sin\theta\, \epsilon_{22} + (\cos^2\theta - \sin^2\theta)\, \epsilon_{12}$$

Considering this, one can see that no matter what direction an elongation strain is measured, one cannot get a result for the shear strain ϵ_{12} -- ϵ_{11} and ϵ_{22} are involved in these expressions. So:

measuring <u>one</u> <u>elongation strain</u> is $\boxed{\text{not sufficient}}$

Further consideration of these transformation equations shows that knowing three strains (ϵ_{11}, ϵ_{22}, ϵ_{12}) and an angle of rotation allows one to characterize the state of strain in any direction. Working with this one can say:

① Any $\underline{\underline{3}}$ strains fully characterize the state of in-plane strain of a body along with the knowledge of the strain directions.

② ~~M~~easuring $\underline{\underline{3}}$ elongation strains and using the knowledge of their directions will yield the shear strain in one set of axes.

Demonstrate this via the equations:

1. Measure $\tilde{\epsilon}_{11}$ and have θ
2. Measure ϵ_{11}
3. Measure ϵ_{22}
4. Use all in the first transformation equation:

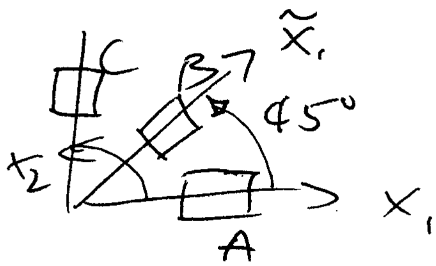
$$\tilde{\epsilon}_{11} = \cos^2\theta\,\epsilon_{11} + \sin^2\theta\,\epsilon_{22} + 2\cos\theta\sin\theta\,\epsilon_{12}$$

and solve for ϵ_{12}:

$$\boxed{\epsilon_{12} = \frac{\tilde{\epsilon}_{11} - \cos^2\theta\,\epsilon_{11} - \sin^2\theta\,\epsilon_{22}}{2\cos\theta\sin\theta}}$$

So: $\boxed{\underline{\underline{3}}\text{ measurements}}$ are needed

<u>Note 1</u>: Such a "shear strain gage" normally measures elongational strain along axes of 0°, 45°, and 90°:



So in the ϵ_{12} equation, $\theta = 45°$. If the reading for ϵ_{11} is A; for $\tilde{\epsilon}_{11}$ is B; for ϵ_{22} is C; then:

$$\epsilon_{12} = B - \frac{1}{2}(A + C)$$

<u>CAUTION</u>!! This is <u>tensorial</u> shear strain.

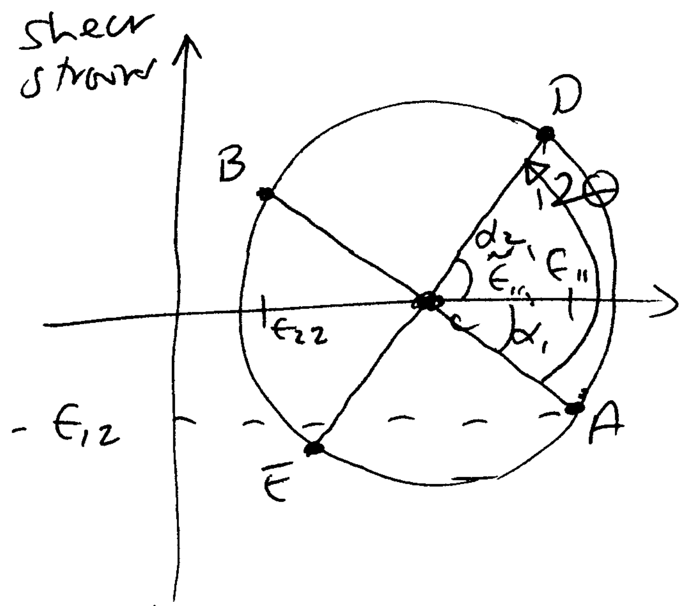
For engineering shear strain, there is a factor of 2: $\gamma_{12} = 2B - (A+C)$

<u>Note 2</u>: The same reasoning can be followed through using the Mohr's circle as a basis. Again, three readings (and associated directions) of longitudinal strain are needed to characterize the circle.

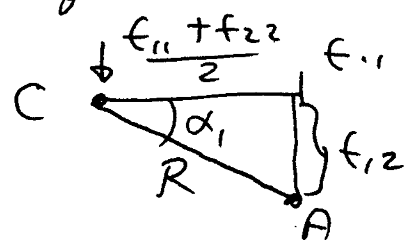
How? Consider the invariants first. The pertinent one is that which gives the midpoint: $\frac{\epsilon_{11} + \epsilon_{22}}{2}$

Measuring a second $\tilde{\epsilon}_{11}$ gives an $\tilde{\epsilon}_{22}$

Now draw the circle:



Considering geometry one can show the angle to principal strains is:



$\Rightarrow R\cos\alpha_1 = \epsilon_{11} - \frac{\epsilon_{11}+\epsilon_{22}}{2} = \frac{\epsilon_{11}-\epsilon_{22}}{2}$

and $R\sin\alpha_1 = \epsilon_{12}$

In a similar way:

$$R\cos\alpha_2 = \frac{\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}}{2}$$

factor of 2 in with the Mohr's circle

Finally: $\alpha_2 = 2\theta - \alpha_1$

$\Rightarrow \cos\alpha_2 = \cos(\theta - \alpha_1) = \cos 2\theta \cos\alpha_1 + \sin 2\theta \sin\alpha_1$

(using trigonometric expansions)

working this last item with what is known:

$$\frac{\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}}{2R} = \cos 2\theta \left(\frac{\epsilon_{11} - \epsilon_{22}}{2R}\right) + \sin 2\theta \, \frac{\epsilon_{12}}{R}$$

→ R cancels out

→ θ, ϵ_{11}, ϵ_{22}, $\tilde{\epsilon}_{11}$ are measured

→ $\tilde{\epsilon}_{22}$ results since $\epsilon_{11} + \epsilon_{22} = \hat{\epsilon}_{11} + \tilde{\epsilon}_{22}$

$$\Rightarrow \tilde{\epsilon}_{22} = \epsilon_{11} + \epsilon_{22} - \tilde{\epsilon}_{11}$$

So: $2\tilde{\epsilon}_{11} - \epsilon_{11} - \epsilon_{22} = (\epsilon_{11} - \epsilon_{22}) \cos 2\theta + 2 \sin 2\theta \, \epsilon_{12}$

and solve for ϵ_{12}

for the case of $\theta = 45°$

and $\tilde{\epsilon}_{11} = B$

$\epsilon_{11} = A$

$\epsilon_{22} = C$

$\Rightarrow \cos 2\theta = 0$, $\sin 2\theta = 1$

giving: $2B - A - C = 2\epsilon_{12}$

$$\Rightarrow \epsilon_{12} = B - \tfrac{1}{2}(A + C)$$

as <u>before</u>!

M8.3 The following answers, as asked for in the problem statement, include a brief sentence on the functional requirement that needs to be met for each of the given cases. This includes the loads (e.g. tension, compression, shear, impact, cyclic, thermal, electrical) and five material properties that are most relevant to meeting this requirement. Note that the items listed are just ***some*** of the possible requirements, loads, and properties. (**NOTE:** Problem set answers will vary according to what the individual student indicates are the relevant loads and properties.)

(a) <u>Components of a truss used in a building</u>: Must provide load-carrying capacity for loads that a building truss undergoes.

Types of loads:
1. & 2. Tension and Compression (depending on design)
3. Assembly
4. Environmental (Thermal, Fire Resistance)

Material properties:
1. Modulus - High
2. Strength - Medium
3. Fabrication & Joining - High
4. Price - Low
5. Longevity - Medium

(b) <u>Components of a space truss</u>: Must provide load-carrying capacity for loads that a space truss undergoes.

Types of loads:
1. Impact (docking)
2. Thermal (solar)
3. & 4. Tension and Compression (depending on design)
5. Cyclic

Material properties:
1. Thermal - High
2. Density - Low
3. Modulus - High
4. Joining - Medium
5. Longevity - High

(c) <u>Reentry shield on a person-carrying space capsule</u>: Must insulate the capsule structure and its passengers from the extreme heat of reentry (Note: no need for reuse).

Types of loads:
1. Thermal
2. Cyclic
3. Impact

Material properties:
1. Thermal - High
2. Density - Low

3. Oxidation Resistance - High
4. Hardness - Medium
5. Strength - Medium

(d) Tiles for a house floor: Must provide an "aesthetic" and durable surface for a house floor.

Types of loads:
1. Impact
2. Compression
3. Thermal
4. Environmental

Material properties:
1. Price - Low
2. Availability - High
3. Hardness - Medium
4. Appearance - High
5. Finishing - High

(e) Cable used for towing cars: Must provide load-carrying capacity and resistance to environment for loads and items encountered in towing

Types of loads:
1. Tension (pulling, from bumps)
2. Impact
3. Thermal (due to baseline temperature from environment)
4. Wear

Material properties:
1. Strength - High
2. Abrasion and wear - High
3. Modulus - High
4. Corrosion - High
5. Price - Low

(f) Compressor blades of a jet engine: Must maintain desired shape during operation of engine.

Types of loads:
1. Impact
2. Tension and Compression
3. Aerodynamic/Pressure
4. Wear

Material properties:
1. Strength - High
2. Modulus - High
3. Impact - High
4. Corrosion - High
5. Cyclic - High