Unified Engineering Problem Set  
Week 10  Fall 2006  

**SOLUTIONS**

**m.0.1** \[ U = -\frac{A}{r^n} + \frac{B}{r^m} + U_i \]
with \( m = 9 \)

We are also given:
- **stable separation distance:** \( r_0 = 0.21 \text{nm} = 0.21 \times 10^{-9} \text{m} \)
- **stable energy:** \( U_0 = -26.1 \text{eV} \)
- **bohr energy:** \( U_i = 2.1 \text{eV} \)

(a) **There is a relationship of \( \sqrt{r} \) here, so we can immediately determine this to be ionic bonding.**

To find the constants \( A \) and \( B \), use the value associated with the energy at the stable point \( (r = r_0) \)

\[ U_0 = -\frac{A}{r_0} + \frac{B}{r_0^m} + U_i \]

\[ \Rightarrow -26.1 \text{eV} = -\frac{A}{(0.21 \times 10^{-9} \text{m})} + \frac{B}{(0.21 \times 10^{-9} \text{m})^9} + 2.1 \text{eV} \]
\[(\text{Note}) : 1 \text{ Joule} = 6.24 \times 10^{18} \text{eV} \]
\[\Rightarrow 1 \text{eV} = 1.603 \times 10^{-9} \text{J} \]
\[\Rightarrow -4.52 \times 10^{-18} \text{J} = -\frac{A}{2.1 \times 10^{-10} \text{m}} + \frac{B}{(2.1 \times 10^{-10} \text{m})^9} \quad (1)\]

This gives one equation. To get the second, we know that \(\frac{dU}{dr} = 0\) at the stable point \(r = r_0\) and we apply that:
\[\frac{dU}{dr} = \frac{A}{r^2} - \frac{9B}{r^{10}}\]

and with this \(= 0\) at \(r = r_0\), we get:
\[\Rightarrow 0 = \frac{A}{(2.1 \times 10^{-10} \text{m})^2} - \frac{9B}{(2.1 \times 10^{-10} \text{m})^{10}} \quad (2)\]

Use this second equation \((2)\) in the first \((1)\) to get:
\[-4.52 \times 10^{-18} \text{J} = -\frac{9B}{(2.1 \times 10^{-10} \text{m})^9} + \frac{B}{(2.1 \times 10^{-10} \text{m})^9}\]
\[\Rightarrow B = \frac{1}{8} (4.52 \times 10^{-18} \text{J}) (2.1 \times 10^{-10} \text{m})^9\]

Finding: \(B = 4.49 \times 10^{-10} \text{ J.m}^9\)

Use this in \((2)\) to determine \(A\):
\[A = \frac{9 (4.49 \times 10^{-10} \text{J.m}^9)}{(2.1 \times 10^{-10} \text{m})^8}\]
\[\Rightarrow A = 1.02 \times 10^{-27} \text{ J.m} \]
Summarizing:

\[
\begin{align*}
B &= 4.49 \times 10^{-10} \text{ J.m}^2 \\
A &= 1.07 \times 10^{-27} \text{ J.m}
\end{align*}
\]

(b) We know the stiffness of the bond, \( S_{\text{bond}} \), is related to energy via:

\[
S_{\text{bond}} = \frac{d^2U}{dr^2}
\]

and we need to find this at \( r_0 \) to get \( S_0 \) (the stiffness at the stable point):

\[
\frac{dU}{dr} = \frac{A}{r^2} - \frac{qB}{r^{10}}
\]

\[
S = \frac{d^2U}{dr^2} = -2A + \frac{90B}{r^{11}}
\]

Use the values of \( A, B, \) and \( r_0 \) in this:

\[
S_0 = -\frac{2(1.07 \times 10^{-27} \text{ J.m})}{(2.1 \times 10^{-10} \text{ m})^3} + \frac{90(4.49 \times 10^{-10} \text{ J.m}^2)}{(2.1 \times 10^{-10} \text{ m})^{11}}
\]

\[
= -2.31 \times 10^2 \text{ J.m}^2 + 1.15 \times 10^3 \text{ J/m}^2
\]

Also use \( 1 \text{ J} = 1 \text{ N.m} \) and get:

\[
S_0 = 919 \text{ N/m}
\]

Finally, get an estimate for \( F \) by recalling:
\[ E = \frac{S_0}{r_0} = \frac{919 \text{ N/m}}{(2.1 \times 10^{-10}) \text{ m}} \]

\[ \Rightarrow E = 4.38 \times 10^2 \text{ N/m}^2 \]

\[ = 4.38 \times 10^9 \text{ N/m}^2 = 4.38 \times 10^9 \text{ Pa} \]

\[ \Rightarrow E = 4.38 \text{ GPa} \]

(c) A modulus for an ionic material is typically in the vicinity of 50 GPa. Discrepancies arise because we have considered only the forces and the influence between the atoms in isolation. We actually need to consider the influence of all the atoms in the lattice, particularly the immediate neighbors.
Mrs. 2. Unidirectional composite material with glass fibers with modulus of 14.0 Msi and epoxy matrix with modulus of 2.1 Msi.

To estimate the composite modulus along and perpendicular to the fiber direction, use the Rule of Mixtures:

\[
\frac{E_L}{E_m} = \frac{E_f}{E_f} \frac{V_f}{V_f} + \frac{E_m}{E_m} (1 - V_f)
\]

where: \(V_f\) = volume fraction of fiber

for \(E_L\): (modulus along fiber direction)

\[
E_L = E_f V_f + E_m (1 - V_f)
\]

for \(E_T\): (modulus perpendicular to fiber direction)

\[
E_T = \frac{E_f E_m}{E_m V_f + E_f (1 - V_f)}
\]

→ Use the given values and make a table and calculate values for increments of 0.1 in \(V_f\) for the fiber system.
<table>
<thead>
<tr>
<th>$v_f$</th>
<th>$E_L$ [ksi]</th>
<th>$E_T$ [ksi]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.1</td>
<td>2.1</td>
</tr>
<tr>
<td>0.1</td>
<td>3.3</td>
<td>2.3</td>
</tr>
<tr>
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<td>2.5</td>
</tr>
<tr>
<td>0.3</td>
<td>5.7</td>
<td>2.8</td>
</tr>
<tr>
<td>0.4</td>
<td>6.9</td>
<td>3.2</td>
</tr>
<tr>
<td>0.5</td>
<td>8.1</td>
<td>3.7</td>
</tr>
<tr>
<td>0.6</td>
<td>9.2</td>
<td>4.3</td>
</tr>
<tr>
<td>0.7</td>
<td>10.7</td>
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</tr>
<tr>
<td>0.8</td>
<td>11.2</td>
<td>6.6</td>
</tr>
<tr>
<td>0.9</td>
<td>12.6</td>
<td>8.9</td>
</tr>
<tr>
<td>1.0</td>
<td>14.0</td>
<td>14.0</td>
</tr>
</tbody>
</table>

*These relations are plotted on the next page*
COMPOSITE PLY MODULUS ESTIMATED FROM CONSTITUENT VALUES

LONGITUDINAL MODULUS

TRANSVERSE MODULUS

COMPOSITE PLY MODULUS [GPa]

FIBER VOLUME FRACTION

Vf
Problem T4 (Unified Thermodynamics): SOLUTIONS

a) Describe the energy exchange processes in the device in terms of heat, work and various forms of energy. (LO’s #1, #2)

External work is done on the upper chamber by the weight. The potential energy of the weight is reduced and the internal energy of the gas in the upper chamber is increased. The process is not quasi-static. Then heat is gradually transferred from the upper chamber to the lower chamber. During this process the internal energy of the upper chamber is decreased and the internal energy of the lower chamber is increased. As the heat is transferred from the upper chamber, work continues to be done on the upper chamber since the piston is free to move (and the potential energy of the weight continues to decrease). When the processes are over, the surroundings have provided energy (from the change in potential energy of the weight). The two chambers have received the energy, and it appears as increased internal energy of each of the chambers.

b) What processes will you use to model this system? Why? (LO’s #2, #4, #5)

The first process is not quasi-static but it is adiabatic because we expect pressure to equilibrate faster than the time it takes for appreciable heat transfer to occur. Therefore the upper chamber should be modeled as adiabatic with the work determined by considering the external pressure times the change in volume. After this first relatively fast process, we can assume that the heat transfer takes place more slowly. It is no longer adiabatic (both chambers have heat transfer with one another). But the upper chamber undergoes constant pressure cooling since the piston is free to move, but the weight remains on it. Since the cooling is relatively slow we can assume that the process in the upper chamber is quasi-static in terms of evaluating the work as pdv. The process in the lower chamber is constant volume heating (no work).

c) What is the temperature the gas in the upper chamber comes to shortly after the instantaneous dropping of the weight? (LO #4)

The process is an impulsive compression (not quasi-static). You are given the initial state. You are told it is an adiabatic process, and you know the external pressure that is applied. The first law with q=0 becomes:
\[ \Delta u = -w = -p_{\text{ext}} (v_2 - v_1) \quad \text{or} \quad c_v (T_2 - T_1) = -p_{\text{ext}} (v_2 - v_1) \]

You know \( T_1, v_1 \), but not \( T_2 \) and \( v_2 \). However you know from the ideal gas law that

\[ v_2 = \frac{RT_2}{p_2} = \frac{RT_2}{p_{\text{ext}}} \]

since \( p_2 = p_{\text{ext}} \) when the system comes to pressure equilibrium. Therefore,

\[ c_v (T_2 - T_1) = -p_{\text{ext}} \left( \frac{RT_2}{p_{\text{ext}}} - v_1 \right) = -p_{\text{ext}} \left( \frac{RT_2}{p_{\text{ext}}} - \frac{RT_1}{p_1} \right) = p_{\text{ext}} \frac{RT_1}{p_1} - RT_2 \]

so

\[ c_v T_2 - c_v T_1 = p_{\text{ext}} \frac{RT_1}{p_1} - RT_2 \]

or

\[ (c_v + R)T_2 = p_{\text{ext}} \frac{RT_1}{p_1} + c_v T_1 \]

\[ (c_v + c_p - c_v)T_2 = T_1 \left( \frac{Rp_{\text{ext}}}{p_1} + c_v \right) \]

or

\[ T_2 = \frac{T_1}{c_p} \left( \frac{Rp_{\text{ext}}}{p_1} + c_v \right) \]

\[ T_2 = \frac{300 \text{K}}{1003.5 \text{J/kgK}} \left( \frac{287 \text{J/kgK} (1000 \times 10^3 \text{Pa})}{100 \times 10^3 \text{Pa}} + 716.5 \text{J/kgK} \right) = 1072 \text{K} \]
d) What is the temperature the gas in the lower chamber comes to when the whole system eventually reaches thermodynamic equilibrium? (LO #4)

The upper chamber slowly cools in a constant pressure cooling process. The lower chamber slowly heats in a constant volume heating process. Other than the rigid copper wall separating the two chambers everything else is thermally-insulated, so the heat transferred from the upper chamber is equal to the heat transferred to the lower chamber.

The easiest way to do this is to use two different forms of the first law for the two chambers.

\[ \text{du} = \delta q - pdv \] is convenient for the constant volume process in the lower chamber

\[ \text{dh} = \delta q + vdp \] is convenient for the constant pressure process in the upper chamber

So for the lower chamber

\[ c_v(T_{3\text{lower}} - T_{2\text{lower}}) = \delta q \] since volume is constant

and for the upper chamber

\[ c_p(T_{3\text{upper}} - T_{2\text{upper}}) = \delta q \] since pressure is constant

Here I have assigned state 2 as the state right after the upper chamber comes to pressure equilibrium. For the lower chamber, this is the same as state 1 (for my model, I assumed that no appreciable heat is transferred to the lower chamber during the first process).

The heat that leaves the upper chamber is the same magnitude, but opposite in sign from that which is added to the lower chamber. So with the addition of a negative sign, we can equate the two first law expressions.

\[ c_p(T_{3\text{upper}} - T_{2\text{upper}}) = -c_v(T_{3\text{lower}} - T_{2\text{lower}}) \]

and we know that \( T_{2\text{lower}} = T_1 = 300K \), and \( T_{2\text{upper}} = 1072K \) from above, and \( T_{3\text{upper}} = T_{3\text{lower}} \) since the system comes to thermal equilibrium!

\[ 1003.5J/\text{kgK}(T_{3\text{lower/upper}} - 1072K) = -716.5J/\text{kgK}(T_{3\text{lower/upper}} - 300K) \]

So \( T_{3\text{upper}} = T_{3\text{lower}} = 750.5K \)
SOLUTIONS TO T5 BY WAITEZ

a) All weights removed instantaneously

\[ P_{\text{initial}} = 4 \text{ atm}, \ T_i = 300 \text{ K} \]
\[ V_i = \frac{287.300}{405.3 \text{ kPa}} = 0.21 \text{ m}^3/\text{kg} \]

We know \( P_{\text{initial}} = 4 \text{ atm} \) but we do not know \( T_f \) or \( V_f \).

*NOTE THAT WE EXPECT \( T_f < T_i \)  
SINCE IT IS A THERMALLY-INNULABE (ADIABATIC) CYLINDER AND WE ARE GETTING WORK OUT SO \( \Delta U \) SHOULD BE NEGATIVE.*

Consider 1st law for adiabatic process:

\[ \Delta U = \frac{P_f}{V_i} - \omega \]
\[ C_v \Delta T = -P_{\text{ext}} \Delta V \]
\[ C_v (T_f - T_i) = -P_{\text{ext}} (V_f - V_i) \]

ideal gas: \( P_f V_f = R T_f \) \( \left\{ \begin{array}{l} \text{2 eqns} \\text{unknowns} \end{array} \right\} \)

Let \( V_f = \frac{R T_f}{P_f} \)

\[ C_v (T_f - T_i) = -P_{\text{ext}} \left( \frac{R T_f}{P_f} - V_i \right) \]

At the final state \( P_f = P_{\text{ext}} \) (it came to thermodynamic equilibrium)

\[ C_v T_f - C_v T_i = -R T_f + P_{\text{ext}} V_i \]

\[ (C_v + R) T_f = P_{\text{ext}} V_i + C_v T_i \]

\[ T_f = \frac{P_{\text{ext}} V_i + C_v T_i}{C_v + R} \]
Plug in some numbers

\[ T_f = \frac{(101325)(0.21) + (7165)(300)}{7165 + 287} = 235.4 \text{ K} \]

\[ P_f = P_{ext} = 101325 \text{ N/m}^2 \]

\[ PV = RT \implies V_f = 0.67 \text{ m}^3/\text{kg} \]

\[ \Delta u = -\Delta w \]

\[ \omega = -C_v (235.4 - 300) = +46.3 \text{ kJ/kg} \]

Work by system

b) Two step process (expect more work out of system!)

\[ W_m = P_{ext0} \frac{V_i + C_v T_i}{C_v + R} = \frac{253312 \cdot 0.21 + 7165 \cdot 300}{7165 + 287} = 267.2 \text{ K} \]

\[ P_m = 253312 \text{ N/m}^2 \]

\[ V_m = 0.30 \text{ m}^3/\text{kg} \]

\[ W_0 = 23.5 \text{ kJ/kg} \]

\[ T_f = \frac{P_{ext0} V_m + C_v T_m}{C_v + R} = \frac{101325 \cdot 0.30 + 7165 \cdot 267.2}{7165 + 287} \]
\[ T_f = 221 \text{ K}, \quad P_f = 101325 \text{ Pa}, \quad V_f = 0.68 \text{ m}^3/\text{kg} \]

\[ \omega_0 = -716.5 \times (221 - 267.2) = 33.1 \text{ kJ/kg} \]

\[ \omega_{\text{total}} = \omega_0 + \omega_0 = 56.6 \text{ kJ/kg} \]

c) **Quasi-static process (expect the most work)**

\[ PV^\gamma = \text{const.} \quad \frac{T_f}{T_i} = \left( \frac{P_f}{P_i} \right)^{\gamma/\delta} = \left( \frac{1}{4} \right)^{0.4/4} \]

\[ \therefore \quad T_f = 202 \text{ K} \]

\[ P_f = 101325 \text{ Pa} \]

\[ V_f = 0.572 \text{ m}^3/\text{kg} \]

\[ \Delta u = q - \omega \quad C_v \Delta T = -\omega \]

\[ \omega = -716.5 \times (202 - 300) = 70.2 \text{ kJ/kg} \]

**MESSAGES:**

1. THE AMOUNT OF WORK EXTRACTED DEPENDS ON PATH
2. IN CALCULATING WORK FOR NON-QUASI-STATIC PROCESSES, NEED TO USE \( \omega_{\text{ext}} \).
3. THE CLOSER WE APPROACH A QUASI-STATIC PROCESS, THE MORE WORK WE GET OUT OF THE SYSTEM. THEREFORE IN DESIGNING AN ENGINE WE WANT TO LIMIT NON-QUASI-STATIC PROCESSES TO EVERY EXTENT POSSIBLE.

We get the most area under the curve
### T6 Solutions by Waitz

#### Table

<table>
<thead>
<tr>
<th>LEG</th>
<th>Q</th>
<th>W</th>
<th>PROCESS</th>
</tr>
</thead>
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<tr>
<td>1-2</td>
<td>0</td>
<td>-</td>
<td>ADIABATIC COMPRESSION</td>
</tr>
<tr>
<td>2-3</td>
<td>+</td>
<td>+</td>
<td>CONST. PRESSURE EXPANSION</td>
</tr>
<tr>
<td>3-4</td>
<td>+</td>
<td>+</td>
<td>ISOThERMAL EXPANSION</td>
</tr>
<tr>
<td>4-1</td>
<td>-</td>
<td>0</td>
<td>CONST. VOLUME COOLING</td>
</tr>
</tbody>
</table>

#### b) LEG 1-2

**ADIABATIC COMPRESSION**, \( PV^γ = \text{constant} \Rightarrow \frac{P_2}{P_1} = \left( \frac{T_2}{T_1} \right)^{\frac{γ}{γ-1}}\)

\[
\frac{P_2}{P_1} = 20 = \left( \frac{T_2}{300} \right)^{\frac{1.4}{0.4}} \Rightarrow T_2 = 706 K
\]

\[
\Delta u = 0 - w = C_v (T_2 - T_1) \Rightarrow w = -291 \text{kJ/kg}
\]

\[
\Delta h = C_p (T_2 - T_1) = 1003.5 (706 - 300) \Rightarrow w = 907 \text{kJ/kg}
\]

\[
W = -291 \text{kJ/kg}, \quad 0 = 0, \quad \Delta u = 291 \text{kJ/kg}, \quad \Delta h = 907 \text{kJ/kg}
\]

#### LEG 2-3

**CONSTANT PRESSURE EXPANSION**, \( P = \text{const.} \)

\[
\Delta h = \delta q + \nu d\phi
\]

\[
\delta q = C_p (T_3 - T_2) = q = 1003.5 (1800 - 706) = 1098 \text{kJ/kg}
\]

\[
\Delta u = C_v (T_3 - T_2) = 716.5 (1800 - 706) = 784 \text{kJ/kg}
\]

\[
\Delta u = q - \Delta w \Rightarrow \Delta w = q - \Delta u = 1098 - 784 = 314 \text{kJ/kg}
\]

\[
W = 314 \text{kJ/kg}, \quad q = 1098 \text{kJ/kg}, \quad \Delta u = 784 \text{kJ/kg}, \quad \Delta h = 1098 \text{kJ/kg}
\]

#### LEG 3-4

**ISOThERMAL EXPANSION**, \( T = \text{const.} \)

\[
\Delta u = 0
\]

\[
\delta q = \Delta u \Rightarrow q = \Delta u \quad ; \quad W = RT \ln \frac{V_2}{V_3}
\]

\[
P_3 = 20 \times 10^6 \text{Pa} \quad ; \quad \rho_3 = 2 \times 10^6 \text{Pa} \]

\[
T_3 = 1800 K \quad ; \quad V_3 = \frac{RT_3}{P_3} = 0.258 \text{m}^3 \text{kg}^{-1}
\]

\[
V_4 = V_1 = \frac{RT_1}{P_1} = \frac{287 \times 300}{100 \times 10^3} = 0.861 \text{m}^3 \text{kg}^{-1}
\]

\[
\left\{ \begin{array}{l}
W = 287 (1800) \ln \left( \frac{0.861}{0.258} \right) \\
W = 623 \text{kJ/kg}, \quad q = 623 \text{kJ/kg}, \quad \Delta u = 0, \quad \Delta h = 0
\end{array} \right.
\]
LEG 4.1

**CONSTANT VOLUME COOLING, V = CONST.**

\[ d\mu = dq - \frac{dW}{V} \quad \text{const volume} \quad \therefore \quad \Delta \mu = CV(T_f - T_i) = 0 \]

\[ \Delta \mu = 716.5 \text{ (300-1800)} = -1075 \text{kJ/kg} = q \]

\[ \Delta h = 1003.5 \text{ (300-1800)} = -1505 \text{kJ/kg} \]

\[ W = 0, \quad q = -\frac{1075}{kg}, \quad \Delta \mu = -1075 \text{kJ/kg}, \quad \Delta h = -1505 \text{kJ/kg} \]

**c) WORK OF CYCLE**

\[ -291 + 314 + 623 = 646 \text{ kJ/kg} \]

**d)**

\[ \eta = \frac{\text{Work cycled}}{\text{Q added}} = \frac{646}{1098 + 623} = 0.375 \]

**e)**

In reverse, work cycle = -646 kJ/kg and all signs on heat are reversed. So the heat flows into the system (from food), heat in would be \( \frac{1075}{kg}. \)

So for each joule of work in, could remove \( \frac{1075}{646} = 1.66 \text{ J} \) of heat from food.
Problem S1 (Signals and Systems) SOLUTION

1. Consider the system of equations

\[
\begin{align*}
1x + 2y + 2z &= 3 \\
3x + 2y + 1z &= 2 \\
1x + 3y + 3z &= 4
\end{align*}
\]

Solve for \(x\), \(y\), and \(z\), in three separate ways.

(a) Determine \(x\), \(y\), and \(z\) using (symbolic) elimination of variables.

To solve by eliminating variables, solve for \(x\) in terms of \(y\) and \(z\), using the first equation, so that

\[x = 3 - 2y - 2z\]  
(1)

Plug this into the second and third equations, and simplify, to obtain

\[-4y - 5z = -7 \\
y + z = 1\]

Solve the first of these for \(y\), so that

\[y = \frac{7}{4} - \frac{5}{4}z\]  
(2)

Plug this into the second to obtain

\[-\frac{1}{4}z = \frac{3}{4}\]

so that \(z = 3\). Plug this back into Equation (2) to obtain \(y = -2\). Finally, plug both back into Equation (1) to obtain \(x = 1\). This result is correct, and even organized, but a little bit prone to error, especially as the variable names are written over and over.

(b) Determine \(x\), \(y\), and \(z\) by Gaussian reduction.

Form the array that describes the problem:

\[
\begin{bmatrix}
1 & 2 & 2 & 3 \\
3 & 2 & 1 & 2 \\
1 & 3 & 3 & 4
\end{bmatrix}
\]

Subtract 3 times the first row from the second; subtract the first row from the third to obtain

\[
\begin{bmatrix}
1 & 2 & 2 & 3 \\
0 & -4 & -5 & -7 \\
0 & 1 & 1 & 4
\end{bmatrix}
\]

Divide the second row by -4:

\[
\begin{bmatrix}
1 & 2 & 2 & 3 \\
0 & 1 & 5/4 & 7/4 \\
0 & 1 & 1 & 4
\end{bmatrix}
\]
Subtract the second row from the third:

\[
\begin{bmatrix}
1 & 2 & 2 \\
0 & 1 & \frac{3}{4} \\
0 & 0 & -\frac{3}{4}
\end{bmatrix}
\begin{bmatrix}
3 \\
\frac{7}{4} \\
\frac{7}{4}
\end{bmatrix}
\]

Divide the last row by \(-1/4\):

\[
\begin{bmatrix}
1 & 2 & 2 \\
0 & 1 & \frac{3}{4} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
\frac{7}{4} \\
3
\end{bmatrix}
\]

Back substitute to obtain \(z = 3, y = -2, x = 1\).

In practice, we don’t write the array over and over — we instead modify the rows in place, and cross out the old rows.

(c) Determine \(x, y, \) and \(z\) using Cramer’s rule.

Cramer’s rule says that the solution for \(x\) is given by

\[
x = \frac{\begin{vmatrix}
3 & 2 & 2 \\
2 & 2 & 1 \\
4 & 3 & 3
\end{vmatrix}}{\begin{vmatrix}
1 & 2 & 2 \\
3 & 2 & 1 \\
1 & 3 & 3
\end{vmatrix}} = \frac{3 \cdot 2 \cdot 3 + 2 \cdot 1 \cdot 4 + 2 \cdot 2 \cdot 3 - 4 \cdot 2 \cdot 2 - 3 \cdot 1 \cdot 3 - 3 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3 + 2 \cdot 1 \cdot 1 + 2 \cdot 3 \cdot 3 - 1 \cdot 2 \cdot 2 - 3 \cdot 1 \cdot 1 - 3 \cdot 3 \cdot 2} = 1
\]

Similarly,

\[
y = \begin{vmatrix}
1 & 3 & 2 \\
3 & 2 & 1 \\
1 & 4 & 3
\end{vmatrix} = -2
\]

\[
z = \begin{vmatrix}
1 & 2 & 3 \\
3 & 2 & 2 \\
1 & 3 & 4
\end{vmatrix} = 3
\]

Of course, this solution hides the pain of finding the last two determinants.

(d) Which method is fastest?

For this problem, with \(n = 3\), row reduction and Cramer’s rule are about the same speed. However, for larger \(n\), row reduction is much better.

2. Consider the system of equations

\[
\begin{align*}
2x + 3y + 9z &= 1 \\
5x - 3y - 3z &= 6 \\
2x + 1y + 4z &= 2
\end{align*}
\]
This time, solve for $x$, $y$, and $z$, using only Gaussian elimination.

Write the equation in array format:

\[
\begin{bmatrix}
2 & 3 & 9 & 1 \\
5 & -3 & -3 & 6 \\
2 & 1 & 4 & 2
\end{bmatrix}
\]

Divide first row by 2:

\[
\begin{bmatrix}
1 & 3/2 & 9/2 & 1/2 \\
5 & -3 & -3 & 6 \\
2 & 1 & 4 & 2
\end{bmatrix}
\]

Subtract 5 times first row from second row and 2 times first row from third:

\[
\begin{bmatrix}
1 & 3/2 & 9/2 & 1/2 \\
0 & -21/2 & -51/2 & 7/2 \\
0 & -2 & -5 & 1
\end{bmatrix}
\]

Divide second row by $-21/2$:

\[
\begin{bmatrix}
1 & 3/2 & 9/2 & 1/2 \\
0 & 1 & 17/7 & -1/3 \\
0 & -2 & -5 & 1
\end{bmatrix}
\]

Add twice the second row to the third:

\[
\begin{bmatrix}
1 & 3/2 & 9/2 & 1/2 \\
0 & 1 & 17/7 & -1/3 \\
0 & 0 & -1/7 & 1/3
\end{bmatrix}
\]

Divide third row by $-1/7$:

\[
\begin{bmatrix}
1 & 3/2 & 9/2 & 1/2 \\
0 & 1 & 17/7 & -1/3 \\
0 & 0 & 1 & -7/3
\end{bmatrix}
\]

Back substitute to obtain $z = -7/3$, $y = -1/3 - (17/7)(-7/3) = 16/3$, $x = 1/2 - (3/2)(16/3) - (9/2)(-7/3) = 3$. 

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1. Label the circuit as below. Note: For each resistor, labeling of the +/- terminals is arbitrary, but the current must be into the + terminal. If you used a different labeling, your final answer may differ by a sign.

2. KVL gives

\[-v_1 + v_2 + v_4 = 0 \quad \text{(left loop)}\]
\[-v_3 - v_4 + v_5 = 0 \quad \text{(right loop)}\]

3. KCL gives

\[i_1 + i_2 = 0 \quad \text{(upper left node)}\]
\[i_4 - i_2 - i_3 = 0 \quad \text{(upper middle node)}\]
\[i_3 + i_5 = 0 \quad \text{(upper right node)}\]
\[-i_1 - i_4 - i_5 = 0 \quad \text{(lower node)}\]

Note that one of the above equations is redundant — any three form a linearly independent set of equations.

4. The constitutive laws are

\[v_1 = V_1 = 7 \text{ V}\]
\[v_2 = R_2I_2 = (1 \Omega)I_2\]
\[v_3 = R_3I_3 = (2 \Omega)I_3\]
\[v_4 = R_4I_4 = (2 \Omega)I_4\]
\[v_5 = V_5 = 2 \text{ V}\]
5. There are 10 equations (2 from KVL, 3 from KCL, 5 constitutive relations) and 10 unknowns; hence, we should be able to solve uniquely for each variable. The unknowns can be solved for using, for example, row reduction. The result is

\[ v_1 = 7 \, \text{V}, \quad i_1 = -3 \, \text{A} \]
\[ v_2 = 3 \, \text{V}, \quad i_2 = 3 \, \text{A} \]
\[ v_3 = -2 \, \text{V}, \quad i_3 = -1 \, \text{A} \]
\[ v_4 = 4 \, \text{V}, \quad i_4 = 2 \, \text{A} \]
\[ v_5 = 2 \, \text{V}, \quad i_5 = 1 \, \text{A} \]