PAL
11/2/06

Unified Engineering Problem Set
Week 10 Fall, 2006
SOLUTIONS

M10.1 $$U = -\frac{A}{r} + \frac{B}{r^m} + U_i$$
with $m = 9$

We are also given:

- stable separation distance:
  $r_0 = 0.21\,nm = 0.21 \times 10^{-9}\,m$
- stable energy $= U_0 = -26.1\,eV$
- base energy $= U_i = 2.1\,eV$

(a) There is a relationship of $1/r$ here, so we can immediately determine this to be $\boxed{\text{ionic bonding}}$

To find the constants A and B, use the values associated with the energy at the stable point $(r = r_0)$

first: $$U_0 = -\frac{A}{r_0} + \frac{B}{r_0^9} + U_i$$

$$\Rightarrow -26.1\,eV = \frac{-A}{(0.21 \times 10^{-9}\,m)} + \frac{B}{(0.21 \times 10^{-9}\,m)^9} + 2.1\,eV$$

(Note): 1 Joule $= 6.24 \times 10^{18}$ eV

$\Rightarrow 1\,eV = 1.603 \times 10^{-19}$ J)

$$\Rightarrow -4.52 \times 10^{-18}\,J = -\frac{A}{2.1 \times 10^{-10}\,m} + \frac{B}{(2.1 \times 10^{-10}\,m)^9} \quad (1)$$

This gives one equation. To get the second, we know that $\frac{dU}{dr} = 0$ at the stable point ($r = r_0$) and we apply that:

$$\frac{dU}{dr} = \frac{A}{r^2} - \frac{9B}{r^{10}}$$

and with this $= 0$ at $r = r_0$ we get:

$$\Rightarrow 0 = \frac{A}{(2.1 \times 10^{-10}\,m)^2} - \frac{9B}{(2.1 \times 10^{-10}\,m)^{10}} \quad (2)$$

Use this second equation (2) in the first (1) to get:

$$-4.52 \times 10^{-18}\,J = -\frac{9B}{(2.1 \times 10^{-10}\,m)^9} + \frac{B}{(2.1 \times 10^{-10}\,m)^9}$$

$$\Rightarrow B = \frac{1}{8}(4.52 \times 10^{-18}\,J)(2.1 \times 10^{-10}\,m)^9$$

giving: $B = 4.49 \times 10^{-106}$ J·m^9

Use this in (2) to determine A:

$$A = \frac{9(4.49 \times 10^{-106}\,J \cdot m^9)}{(2.1 \times 10^{-10}\,m)^8}$$

$$\Rightarrow A = 1.07 \times 10^{-27}\,J \cdot m$$

Summarizing:

$$\boxed{\begin{aligned} B &= 4.49\times 10^{-106}\ \mathrm{J\cdot m^9} \\ A &= 1.07\times 10^{-27}\ \mathrm{J\cdot m} \end{aligned}}$$

(b) We know the stiffness of the bond, S_{bond}, is related to energy via:

$$S_{bond} = \frac{d^2U}{dr^2}$$

and we need to find this at r_0 to get S_0 (the stiffness at the stable point):

$$\frac{dU}{dr} = \frac{A}{r^2} - \frac{9B}{r^{10}}$$

$$S = \frac{d^2U}{dr^2} = \frac{-2A}{r^3} + \frac{90B}{r^{11}}$$

use the values of A, B, and r_0 in this:

$$\Rightarrow S_0 = \frac{-2(1.07\times10^{-27}\ \mathrm{J\cdot m})}{(2.1\times10^{-10}\ \mathrm{m})^3} + \frac{90(4.49\times10^{-106}\ \mathrm{J\cdot m^9})}{(2.1\times10^{-10}\ \mathrm{m})^{11}}$$

$$= -2.31\times10^{2}\ \mathrm{J/m^2} + 1.15\times10^{3}\ \mathrm{J/m^2}$$

also use $1\ \mathrm{J} = 1\ \mathrm{N\cdot m}$ and get:

$$\Rightarrow S_0 = 919\ \mathrm{N/m}$$

Finally, get an estimate for E by recalling:

$$E = \frac{S_0}{r_0} = \frac{919 \text{ N/m}}{(2.1 \times 10^{-10})\text{m}}$$

$$\Rightarrow E = 4.38 \times 10^{12} \text{ N/m}^2$$

$$= 4380 \times 10^9 \text{ N/m}^2 = 4380 \times 10^9 \text{ Pa}$$

$$\Rightarrow \boxed{E = 4380 \text{ GPa}}$$

(c) A modulus for an ionic material is typically in the vicinity of 50 GPa. Discrepancies arise because we have considered only the forces and the influences between the atoms in isolation. We actually need to consider the influence of all the atoms in the lattice, particularly the immediate neighbors.

Prob. 2 Unidirectional composite material with glass fibers with modulus of 14.0 Msi and epoxy matrix with modulus of 2.1 Msi

To estimate the composite modulus along and perpendicular to the fiber direction, use the

Rule of Mixtures

for E_L: (modulus along fiber direction)

$$E_L = E_f V_f + E_m (1 - V_f)$$

where: V_f = volume fraction of fiber

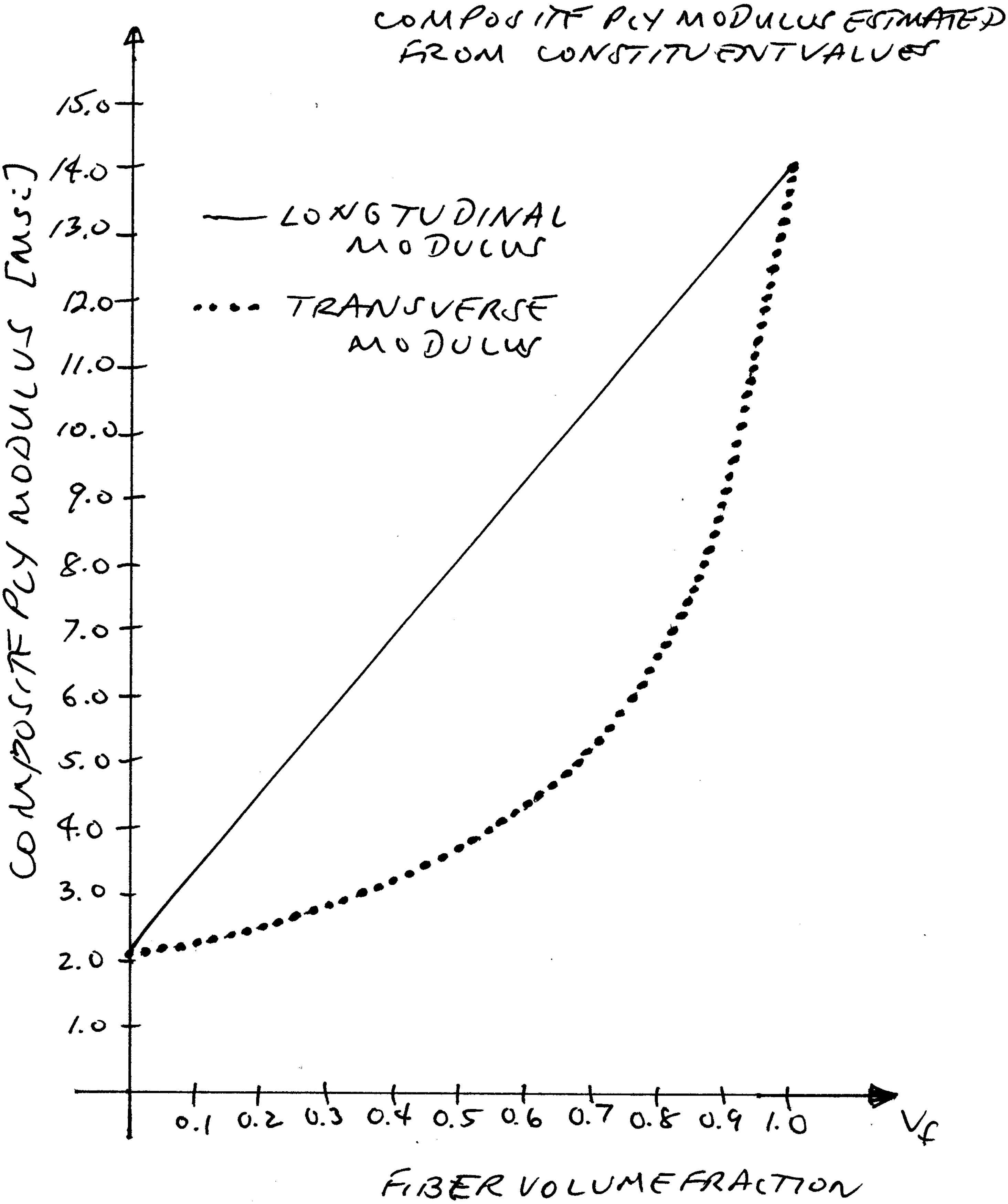
for E_T: (modulus perpendicular to fiber direction)

$$E_T = \frac{E_f E_m}{E_m V_f + E_f (1 - V_f)}$$

→ Use the given values and make a table and calculate values for increments of 0.1 in V_f for the fiber system.

V_f	E_L [Msi]	E_T [Msi]
0	2.1	2.1
0.1	3.3	2.3
0.2	4.5	2.5
0.3	5.7	2.8
0.4	6.9	3.2
0.5	8.1	3.7
0.6	9.2	4.3
0.7	10.4	5.2
0.8	11.2	6.6
0.9	12.6	8.9
1.0	14.0	14.0

These relations are plotted on the next page


COMPOSITE PLY MODULUS ESTIMATED FROM CONSTITUENT VALUES
— LONGTUDINAL MODULUS
•••• TRANSVERSE MODULUS
COMPOSITE PLY MODULUS [Msi]
15.0
14.0
13.0
12.0
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
V_f
FIBER VOLUME FRACTION

Problem T4 (Unified Thermodynamics): SOLUTIONS

a) Describe the energy exchange processes in the device in terms of heat, work and various forms of energy. (LO's #1, #2)

External work is done on the upper chamber by the weight. The potential energy of the weight is reduced and the internal energy of the gas in the upper chamber is increased. The process is not quasi-static. Then heat is gradually transferred from the upper chamber to the lower chamber. During this process the internal energy of the upper chamber is decreased and the internal energy of the lower chamber is increased. As the heat is transferred from the upper chamber, work continues to be done on the upper chamber since the piston is free to move (and the potential energy of the weight continues to decrease). When the processes are over, the surroundings have provided energy (from the change in potential energy of the weight). The two chambers have received the energy, and it appears as increased internal energy of each of the chambers.

b) What processes will you use to model this system? Why?
(LO's #2, #4, #5)

The first process is not quasi-static but it is adiabatic because we expect pressure to equilibrate faster than the time it takes for appreciable heat transfer to occur. Therefore the upper chamber should be modeled as adiabatic with the work determined by considering the external pressure times the change in volume. After this first relatively fast process, we can assume that the heat transfer takes place more slowly. It is no longer adiabatic (both chambers have heat transfer with one anther). But the upper chamber undergoes constant pressure cooling since the piston is free to move, but the weight remains on it. Since the cooling is relatively slow we can assume that the process in the upper chamber is quasi-static in terms of evaluating the work as pdv. The process in the lower chamber is constant volume heating (no work).

c) What is the temperature the gas in the upper chamber comes to shortly after the instantaneous dropping of the weight? (LO #4)

The process is an impulsive compression (not quasi-static). You are given the initial state. You are told it is an adiabatic process, and you know the external pressure that is applied. The first law with q=0 becomes:

$\Delta u = -w = -p_{ext}(v_2 - v_1)$ or $c_v(T_2 - T_1) = -p_{ext}(v_2 - v_1)$

You know T_1, v_1, but not T_2 and v_2. However you know from the ideal gas law that

$v_2 = \frac{RT_2}{p_2} = \frac{RT_2}{p_{ext}}$ since $p_2 = p_{ext}$ when the system comes to pressure equilibrium. Therefore,

$$c_v(T_2 - T_1) = -p_{ext}\left(\frac{RT_2}{p_{ext}} - v_1\right) = -p_{ext}\left(\frac{RT_2}{p_{ext}} - \frac{RT_1}{p_1}\right) = p_{ext}\frac{RT_1}{p_1} - RT_2$$

so $c_v T_2 - c_v T_1 = p_{ext}\frac{RT_1}{p_1} - RT_2$ or $(c_v + R)T_2 = p_{ext}\frac{RT_1}{p_1} + c_v T_1$

$(c_v + c_p - c_v)T_2 = T_1\left(\frac{Rp_{ext}}{p_1} + c_v\right)$ or $T_2 = \frac{T_1}{c_p}\left(\frac{Rp_{ext}}{p_1} + c_v\right)$

$$T_2 = \frac{300\text{K}}{1003.5\text{J/kgK}}\left(\frac{287\text{J/kgK}(1000\text{x}10^3\text{Pa})}{100\text{x}10^3\text{Pa}} + 716.5\text{J/kgK}\right) = 1072\text{K}$$

d) What is the temperature the gas in the lower chamber comes to when the whole system eventually reaches thermodynamic equilibrium? (LO #4)

The upper chamber slowly cools in a constant pressure cooling process. The lower chamber slowly heats in a constant volume heating process. Other than the rigid copper wall separating the two chambers everything else is thermally-insulated, so the heat transferred from the upper chamber is equal to the heat transferred to the lower chamber.

The easiest way to do this is to use two different forms of the first law for the two chambers.

$du = \delta q - pdv$ is convenient for the constant volume process in the lower chamber

$dh = \delta q + vdp$ is convenient for the constant pressure process in the upper chamber

So for the lower chamber $c_v\left(T_{3_{lower}} - T_{2_{lower}}\right) = \delta q$ since volume is constant

and for the upper chamber $c_p\left(T_{3_{upper}} - T_{2_{upper}}\right) = \delta q$ since pressure is constant

Here I have assigned state 2 as the state right after the upper chamber comes to pressure equilibrium. For the lower chamber, this is the same as state 1 (for my model, I assumed that no appreciable heat is transferred to the lower chamber during the first process).

The heat that leaves the upper chamber is the same magnitude, but opposite in sign from that which is added to the lower chamber. So with the addition of a negative sign, we can equate the two first law expressions.

$$c_p\left(T_{3_{upper}} - T_{2_{upper}}\right) = -c_v\left(T_{3_{lower}} - T_{2_{lower}}\right)$$

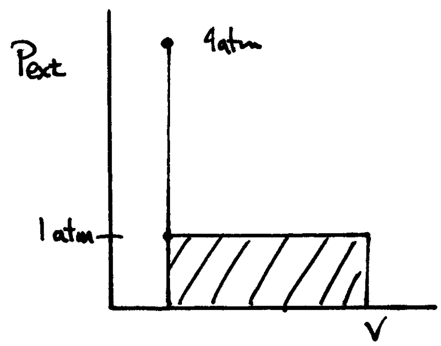
and we know that $T_{2lower} = T_1$=300K, and T_{2upper}=1072K from above, and T_{3upper}=T_{3lower} since the system comes to thermal equilibrium!

$$1003.5\,\mathrm{J/kgK}\left(T_{3_{lower/upper}} - 1072\mathrm{K}\right) = -716.5\,\mathrm{J/kgK}\left(T_{3_{lower/upper}} - 300\mathrm{K}\right)$$

So $T_{3_{upper}} = T_{3_{lower}} = 750.5\mathrm{K}$

SOLUTIONS TO T5 BY WAITZ

a) All weights removed instantaneously



$P_{initial} = 4\text{atm}$, $T_i = 300K$

$$V_i = \frac{287 \cdot 300}{405.3\ kPa} = 0.21 m^3/kg$$

WE KNOW $p_{final} = 1\text{atm}$ BUT WE DO NOT KNOW T_f OR V_f.

* NOTE THAT WE EXPECT $T_f < T_i$ SINCE IT IS A THERMALLY-INSULATED (ADIABATIC) CYLINDER AND WE ARE GETTING WORK OUT SO ΔU SHOULD BE NEGATIVE.

CONSIDER 1st LAW FOR ADIABATIC PROCESS:

$$\Delta u = \cancel{q}^{0} - w \quad \text{so} \quad C_v \Delta T = -P_{ext} \Delta V$$

$$\left.\begin{aligned} C_v(T_f - T_i) &= -p_{ext}(V_f - V_i) \\ \text{ideal gas: } p_f V_f &= RT_f \end{aligned}\right\} \begin{array}{l} 2 \text{ eqns} \\ 2 \text{ unknowns} \\ (V_f \,\&\, T_f) \end{array}$$

LET $V_f = \frac{RT_f}{p_f}$ $\quad C_v(T_f - T_i) = -p_{ext}\left(\frac{RT_f}{p_f} - V_i\right)$

AT THE FINAL STATE $p_f = p_{ext}$ (it comes to thermodynamic equilibrium)

SO $\quad C_v T_f - C_v T_i = -RT_f + p_{ext} V_i$

$$(C_v + R) T_f = p_{ext} V_i + C_v T_i$$

$$\boxed{T_f = \frac{P_{ext} V_i + C_v T_i}{C_v + R}} \; *$$

PLUG IN SOME NUMBERS

$$T_f = \frac{(101325)(0.21) + (716.5)(300)}{716.5 + 287} = 235.4\text{ K}$$

$$P_f = P_{ext} = 101325\ N/m^2$$

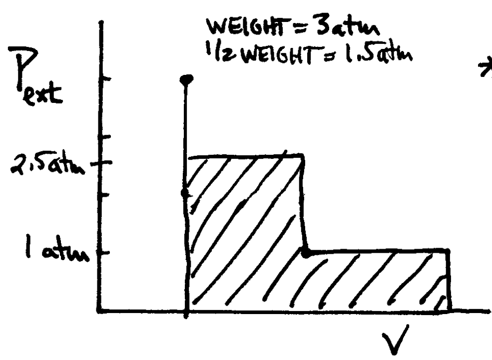
$$PV = RT \Longrightarrow V_f = 0.67\ m^3/kg$$

$$\Delta u = -w$$

$$w = -C_v(235.4 - 300) = +46.3\ kJ/kg$$

work by system

b) Two step process (expect more work out of system!)



* Define subscript "m" as middle condition (1/2 weight removed)

USE SAME EQUATIONS

$$T_m = \frac{P_{ext①} V_i + C_v T_i}{C_v + R} = \frac{253312 \cdot 0.21 + 716.5 \cdot 300}{716.5 + 287} = 267.2\text{K}$$

$P_m = 253312\ N/m^2$

$V_m = 0.30\ m^3/kg$ $\qquad W_{①} = 23.5\ kJ/kg$

$$T_f = \frac{P_{ext②} V_m + C_v T_m}{C_v + R} = \frac{101325 \cdot 0.30 + 716.5 \cdot 267.2}{716.5 + 287}$$

$T_f = 221\ \text{K}$, $p_f = 101325\ \text{Pa}$, $V_f = 0.63\ \text{m}^3/\text{kg}$

$w_{②} = -716.5\,(221 - 267.2) = 33.1\ \text{kJ/kg}$

$w_{total} = w_{①} + w_{②} = 56.6\ \text{kJ/kg}$

c) Quasi-static process (expect the <u>most</u> work)
adiabatic

$\Rightarrow pv^{\gamma} = \text{const.}$ $\qquad \dfrac{T_f}{T_i} = \left(\dfrac{p_f}{p_i}\right)^{\gamma-1/\gamma} = \left(\dfrac{1}{4}\right)^{0.4/1.4}$

$= 0.673$

$\therefore\ T_f = 202\ \text{K}$

$p_f = 101325\ \text{Pa}$

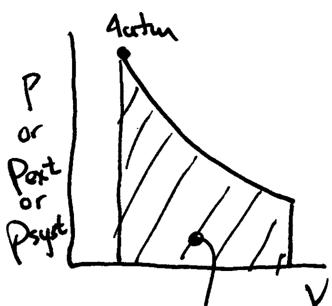
$V_f = 0.572\ \text{m}^3/\text{kg}$

$\Delta u = \cancelto{0}{q} - w \qquad C_v \Delta T = -w$

$w = -716.5\,(202 - 300) = 70.2\ \text{kJ/kg}$

MESSAGES:

1. THE AMOUNT OF WORK EXTRACTED DEPENDS ON PATH
2. IN CALCULATING WORK FOR <u>NON</u>-QUASI-STATIC PROCESSES, NEED TO USE p_{ext}.
3. THE CLOSER WE APPROACH A QUASI-STATIC PROCESS, THE MORE WORK WE GET OUT OF THE SYSTEM. THEREFORE IN DESIGNING AN ENGINE WE WANT TO LIMIT NON-QUASI-STATIC PROCESSES TO EVERY EXTENT POSSIBLE.



we get the most area under the curve

T6 SOLUTIONS BY WAITZ

a)

	Q	W	PROCESS
LEG 1-2	0	−	ADIABATIC COMPRESSION
LEG 2-3	+	+	CONST. PRESSURE EXPANSION
LEG 3-4	+	+	ISOTHERMAL EXPANSION
LEG 4 1	−	0	CONST VOLUME COOLING

b) LEG 1-2 ADIABATIC COMPRESSION, $pv^{\gamma} = \text{CONSTANT} \Rightarrow \frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1}$

$$\frac{P_2}{P_1} = 20 = \left(\frac{T_2}{300}\right)^{1.4/0.4} \Rightarrow T_2 = 706\,K$$

$$\Delta u = \cancel{q}^{\,0} - w \qquad -w = C_v(T_2 - T_1) \Rightarrow W = -291\,kJ/kg$$

$$\Delta h = C_p(T_2 - T_1) = 1003.5(706-300)$$

$$\boxed{W = -291\,kJ/kg,\quad q = 0,\quad \Delta u = 291\,kJ/kg,\quad \Delta h = 407\,\frac{kJ}{kg}}$$

LEG 2-3 CONSTANT PRESSURE EXPANSION, $p = \text{const.}$

$$dh = \delta q + v\cancel{dp}^{\,0}$$

$$\therefore \Delta h = C_p(T_3 - T_2) = q = 1003.5(1800-706) = 1098\,\frac{kJ}{kg}$$

$$\Delta u = C_v(T_3 - T_2) = 716.5(1800-706) = 784\,\frac{kJ}{kg}$$

$$\Delta u = q - w \quad \therefore w = q - \Delta u = 1098 - 784 = 314\,\frac{kJ}{kg}$$

$$\boxed{W = 314\,\frac{kJ}{kg},\quad q = 1098\,\frac{kJ}{kg},\quad \Delta u = 784\,\frac{kJ}{kg},\quad \Delta h = 1098\,\frac{kJ}{kg}}$$

LEG 3-4 ISOTHERMAL EXPANSION, $T = \text{const.} \therefore \Delta u = 0$

$$\cancel{\Delta u}^{\,0} = q - w \quad \therefore q = w \quad ; \quad W = RT_3 \ln \frac{V_4}{V_3}$$

$$P_3 = 20 \cdot 100 \times 10^3\,Pa,\ P_3 = 2 \times 10^6\,Pa$$

$$T_3 = 1800K \qquad V_3 = \frac{RT_3}{P_3} = 0.258\,\frac{m^3}{kg}$$

$$V_4 = V_1 = \frac{RT_1}{P_1} = \frac{287 \cdot 300}{100 \times 10^3} = 0.861\,\frac{m^3}{kg}$$

$$W = 287(1800)\ln\left(\frac{0.861}{0.258}\right) = 623\,kJ/kg$$

$$\boxed{W = 623\,\frac{kJ}{kg},\quad q = 623\,\frac{kJ}{kg},\quad \Delta u = 0,\quad \Delta h = 0}$$

LEG 4-1 CONSTANT VOLUME COOLING, V= const.

$du = \delta q - \cancel{\delta w}^{0}$ const volume $\therefore \Delta u = C_v(T_1 - T_4) = q$

$\Delta u = 716.5(300-1800) = -1075\ kJ/kg = q$

$\Delta h = 1003.5(300-1800) = -1505\ kJ/kg$

$\boxed{W=0,\ q = -1075\,\frac{kJ}{kg},\ \Delta u = -1075\ kJ/kg,\ \Delta h = -1505\ kJ/kg}$

c) $\boxed{\text{WORK OF CYCLE} = -291 + 314 + 623 = 646\ kJ/kg}$

d) $\eta = \dfrac{W_{cycle}}{Q_{ADDED}} = \dfrac{646}{1098 + 623}$ $\boxed{= 0.375}$

e) IN REVERSE, WORK CYCLE = $-646\ kJ/kg$ AND ALL SIGNS ON HEAT ARE REVERSED. SO THE HEAT FLOWS INTO THE SYSTEM (FROM FOOD), HEAT IN WOULD BE $1075\,\frac{kJ}{kg}$.

SO FOR EACH JOULE OF WORK IN COULD REMOVE $\dfrac{1075}{646} = 1.66\ J$ OF HEAT FROM FOOD.

Problem S1 (Signals and Systems) SOLUTION

1. Consider the system of equations

$$\begin{array}{rcrcrcr} 1x & + & 2y & + & 2z & = & 3 \\ 3x & + & 2y & + & 1z & = & 2 \\ 1x & + & 3y & + & 3z & = & 4 \end{array}$$

Solve for x, y, and z, in three separate ways.

(a) Determine x, y, and z using (symbolic) elimination of variables.

To solve by eliminating variables, solve for x in terms of y and z, using the first equation, so that

$$x = 3 - 2y - 2z \tag{1}$$

Plug this into the second and third equations, and simplify, to obtain

$$\begin{array}{rcrcr} -4y & - & 5z & = & -7 \\ y & + & z & = & 1 \end{array}$$

Solve the first of these for y, so that

$$y = \frac{7}{4} - \frac{5}{4}\,z \tag{2}$$

Plug this into the second to obtain

$$-\frac{1}{4}\,z = -\frac{3}{4}$$

so that $z = 3$. Plug this back into Equation (2) to obtain $y = -2$. Finally, plug both back into Equation (1) to obtain $x = 1$. This result is correct, and even organized, but a little bit prone to error, especially as the variable names are written over and over.

(b) Determine x, y, and z by Gaussian reduction.

Form the array that describes the problem:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 1 & 3 & 3 & 4 \end{array}\right]$$

Subtract 3 times the first row from the second; subtract the first row from the third to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -4 & -5 & -7 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Divide the second row by -4:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{5}{4} & \frac{7}{4} \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Subtract the second row from the third:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{5}{4} & \frac{7}{4} \\ 0 & 0 & -\frac{1}{4} & -\frac{3}{4} \end{array}\right]$$

Divide the last row by $-1/4$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{5}{4} & \frac{7}{4} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Back substitute to obtain $z = 3$, $y = -2$, $x = 1$.

In practice, we don't write the array over and over — we instead modify the rows in place, and cross out the old rows.

(c) Determine x, y, and z using Cramer's rule.

Cramer's rule says that the solution for x is given by

$$x = \frac{\begin{vmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 4 & 3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 3 \end{vmatrix}} = \frac{3 \cdot 2 \cdot 3 + 2 \cdot 1 \cdot 4 + 2 \cdot 2 \cdot 3 - 4 \cdot 2 \cdot 2 - 3 \cdot 1 \cdot 3 - 3 \cdot 2 \cdot 2}{1 \cdot 2 \cdot 3 + 2 \cdot 1 \cdot 1 + 2 \cdot 3 \cdot 3 - 1 \cdot 2 \cdot 2 - 3 \cdot 1 \cdot 1 - 3 \cdot 3 \cdot 2} = 1$$

Similarly,

$$y = \frac{\begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 1 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 3 \end{vmatrix}} = -2$$

$$z = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 1 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 3 \end{vmatrix}} = 3$$

Of course, this solution hides the pain of finding the last two determininants.

(d) Which method is fastest?

For this problem, with $n = 3$, row reduction and Cramer's rule are about the same speed. However, for larger n, row reduction is much better.

2. Consider the system of equations

$$\begin{array}{ccccccc} 2x & + & 3y & + & 9z & = & 1 \\ 5x & - & 3y & - & 3z & = & 6 \\ 2x & + & 1y & + & 4z & = & 2 \end{array}$$

This time, solve for x, y, and z, using only Gaussian elimination.

Write the equation in array format:

$$\left[\begin{array}{ccc|c} 2 & 3 & 9 & 1 \\ 5 & -3 & -3 & 6 \\ 2 & 1 & 4 & 2 \end{array}\right]$$

Divide first row by 2:

$$\left[\begin{array}{ccc|c} 1 & 3/2 & 9/2 & 1/2 \\ 5 & -3 & -3 & 6 \\ 2 & 1 & 4 & 2 \end{array}\right]$$

Subtract 5 times first row from second row and 2 times first row from third:

$$\left[\begin{array}{ccc|c} 1 & 3/2 & 9/2 & 1/2 \\ 0 & -21/2 & -51/2 & 7/2 \\ 0 & -2 & -5 & 1 \end{array}\right]$$

Divide second row by $-21/2$:

$$\left[\begin{array}{ccc|c} 1 & 3/2 & 9/2 & 1/2 \\ 0 & 1 & 17/7 & -1/3 \\ 0 & -2 & -5 & 1 \end{array}\right]$$

Add twice the second row to the third:

$$\left[\begin{array}{ccc|c} 1 & 3/2 & 9/2 & 1/2 \\ 0 & 1 & 17/7 & -1/3 \\ 0 & 0 & -1/7 & 1/3 \end{array}\right]$$

Divide third row by $-1/7$:

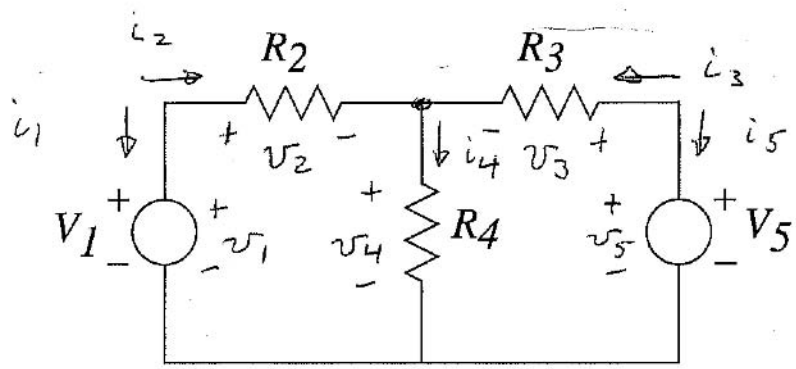
$$\left[\begin{array}{ccc|c} 1 & 3/2 & 9/2 & 1/2 \\ 0 & 1 & 17/7 & -1/3 \\ 0 & 0 & 1 & -7/3 \end{array}\right]$$

Back substitute to obtain $z = -7/3$, $y = -1/3 - (17/7)(-7/3) = 16/3$, $x = 1/2 - (3/2)(16/3) - (9/2)(-7/3) = 3$.

Unified Engineering I **Fall 2006**

Problem S2 (Signals and Systems) SOLUTION

1. Label the circuit as below. Note: For each resistor, labeling of the +/− terminals is arbitrary, but the current *must* be into the + terminal. If you used a different labeling, your final answer may differ by a sign.



2. KVL gives

$$
\begin{aligned}
-v_1 + v_2 + v_4 &= 0 \quad \text{(left loop)} \\
-v_3 - v_4 + v_5 &= 0 \quad \text{(right loop)}
\end{aligned}
$$

3. KCL gives

$$
\begin{aligned}
i_1 + i_2 &= 0 \quad \text{(upper left node)} \\
i_4 - i_2 - i_3 &= 0 \quad \text{(upper middle node)} \\
i_3 + i_5 &= 0 \quad \text{(upper right node)} \\
-i_1 - i_4 - i_5 &= 0 \quad \text{(lower node)}
\end{aligned}
$$

   Note that one of the above equations is redundant — any three form a linearly independent set of equations.

4. The constitutive laws are

$$
\begin{aligned}
v_1 &= V_1 = 7\,\mathrm{V} \\
v_2 &= R_2 I_2 = (1\,\Omega) I_2 \\
v_3 &= R_3 I_3 = (2\,\Omega) I_3 \\
v_4 &= R_4 I_4 = (2\,\Omega) I_4 \\
v_5 &= V_5 = 2\,\mathrm{V}
\end{aligned}
$$

5. There are 10 equations (2 from KVL, 3 from KCL, 5 consitutive relations) and 10 unknowns; hence, we should be able to solve uniquely for each variable. The unknowns can be solved for using, for example, row reduction. The result is

$$\begin{aligned} v_1 &= 7\text{ V}, & i_1 &= -3\text{ A} \\ v_2 &= 3\text{ V}, & i_2 &= 3\text{ A} \\ v_3 &= -2\text{ V}, & i_3 &= -1\text{ A} \\ v_4 &= 4\text{ V}, & i_4 &= 2\text{ A} \\ v_5 &= 2\text{ V}, & i_5 &= 1\text{ A} \end{aligned}$$