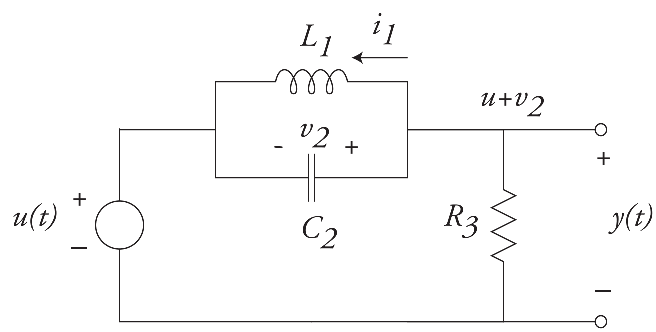
Unified Engineering I **Fall 2006**

Problem S13 (Signals and Systems) SOLUTION

It's easiest to use the node method for this problem. Label the states i_1, and v_2, and the nodes as below:



The state vector is defined as

$$\underline{x} = \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

So we need to find di_1/dt, dv_2/dt. Start with the inductor, with

$$\frac{di_1}{dt} = \frac{v_1}{L_1} = \frac{v_2}{L_1}$$

since $v_2 = v_1$. For the capacitor,

$$\frac{dv_2}{dt} = \frac{i_2}{C_2}$$

To find i_2, apply KCL at the $u + v_2$ node:

$$i_1 + i_2 + \frac{u + v_2}{R_3} = 0$$

Solving for i_2,

$$i_2 = -i_1 - \frac{u + v_2}{R_3}$$

Therefore,

$$\frac{dv_2}{dt} = -\frac{i_1}{C_2} - \frac{u + v_2}{R_3 C_2}$$

In state-space form,

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L_1 \\ -1/C_2 & -1/R_3C_2 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/R_3C_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + [1]u$$

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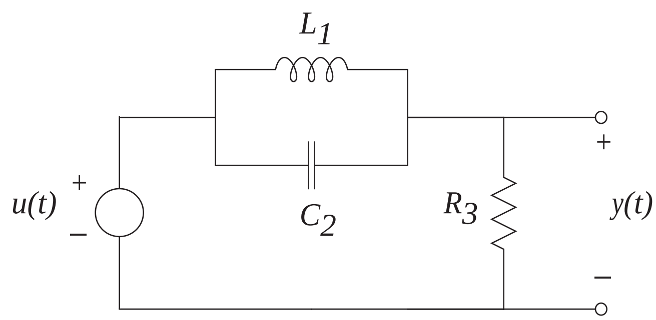
Problem S14 (Signals and Systems) SOLUTION

Consider the RLC circuit of Problem S13, shown below:



1. From S13, we found a state-space description to be

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L_1 \\ -1/C_2 & -1/R_3C_2 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/R_3C_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + [1]u$$

Therefore,

$$sI - A = \begin{bmatrix} s & -1/L_1 \\ 1/C_2 & s + 1/R_3C_2 \end{bmatrix}$$

Inverting,

$$(sI - A)^{-1} = \frac{1}{s^2 + s/R_3C_2 + 1/L_1C_2}\begin{bmatrix} s + 1/R_3C_2 & +1/L_1 \\ -1/C_2 & s \end{bmatrix}$$

Multiplying by C,

$$C(sI - A)^{-1} = \frac{1}{s^2 + s/R_3C_2 + 1/L_1C_2}\begin{bmatrix} -1/C_2 & s \end{bmatrix}$$

Multiplying by B,

$$C(sI - A)^{-1}B = \frac{-s/R_3C_2}{s^2 + s/R_3C_2 + 1/L_1C_2}$$

Adding D,

$$G(s) = C(sI - A)^{-1}B + D = \frac{-s/R_3C_2}{s^2 + s/R_3C_2 + 1/L_1C_2} + 1 = \frac{s^2 + 1/L_1C_2}{s^2 + s/R_3C_2 + 1/L_1C_2}$$

2. To find the transfer function using impedance methods, note that the inductor and capacitor have parallel impedance

$$Z_{LC} = Z_L || Z_C = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{Ls\left(\frac{1}{Cs}\right)}{Ls + \frac{1}{Cs}} = \frac{Ls}{LCs^2 + 1}$$

(Note that I have dropped the numerical subscripts, since no confusion can arise.) Then the circuit is a voltage divider, with transfer function

$$G(s) = \frac{R}{Z_{LC} + R} = \frac{R}{\frac{Ls}{LCs^2+1} + R} = \frac{R\,(LCs^2 + 1)}{R\,(LCs^2 + 1) + Ls} = \frac{RLCs^2 + R}{RLCs^2 + +RLs + R}$$

Dividing numerator and denominator by RLC gives

$$G(s) = \frac{s^2 + 1/L_1C_2}{s^2 + s/R_3C_2 + 1/L_1C_2}$$

as in part (1).

3. For component values

$$L_1 = 1 \text{ H}, \quad C_2 = 0.25 \text{ F}, \quad R_3 = 10\ \Omega$$

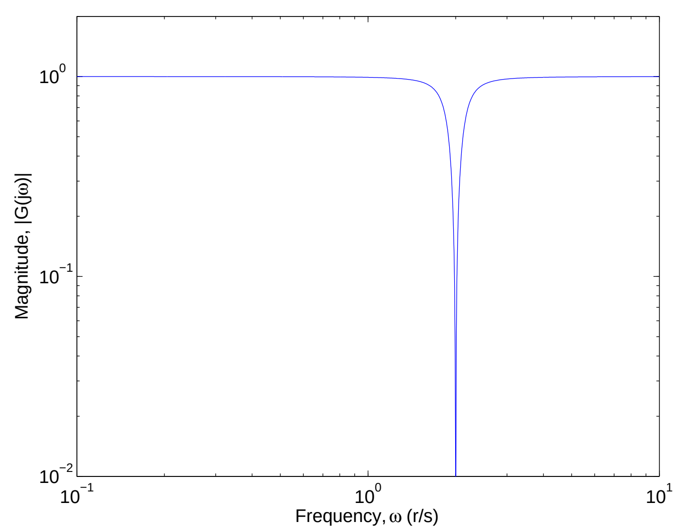
the transfer function is

$$G(s) = \frac{s^2 + 4}{s^2 + 0.4s + 4}$$

Therefore,

$$G(j\omega) = \frac{-\omega^2 + 4}{-\omega^2 + 0.4j\omega + 4}$$

You are then to plot the magnitude of this function. This could be done in a spread sheet, by calculating the transfer function frequency by frequency, or by using the Matlab command `bode`. I used Matlab. (Most of you will probably use a spreadsheet, since we haven't used Matlab this term.) My plot is shown below:



Note that I've plotted the magnitude on a log-log scale. (This is how experts always plot transfer functions.) Most of you will probably plot on a linear-linear scale — that's OK. On either scale, it's clear why this is a "notch filter" — graphically, there's a notch in the plot which is otherwise a straight line.

Problem S15 (Signals and Systems) SOLUTION

In Problem S14, we found that

$$G(s) = \frac{s^2 + 1/L_1C_2}{s^2 + s/R_3C_2 + 1/L_1C_2}$$

For component values

$$L_1 = 1 \text{ H}, \quad C_2 = 0.25 \text{ F}, \quad R_3 = 10 \ \Omega$$

the transfer function is

$$G(s) = \frac{s^2 + 4}{s^2 + 0.4s + 4}$$

1. For the input

$$u(t) = \cos t$$

the complex frequency is $s = j\omega$, where $\omega = 1$, and the complex amplitude is $U = 1$. Therefore,

$$Y = G(s)U = G(1j) \cdot 1 \approx 0.9825 - 0.1310j$$

The real and imaginary parts of the amplitude correspond to the cosine and (minus) sine amplitudes. Therefore,

$$u(t) \approx 0.9825 \cos t + 0.1310 \sin t$$

2. For the input

$$u(t) = 3 \sin 2t$$

the complex frequency is $s = j\omega$, where $\omega = 2$, and the complex amplitude is $U = 3$. Therefore,

$$Y = G(s)U = G(2j) \cdot 3 = 0$$

The amplitude of y is identically zero, because the frequency of the input signal is at exactly the notch frequency. Therefore,

$$u(t) = 0$$

3. For the input

$$u(t) = 2 \cos 4t + \sin 4t$$

the complex frequency is $s = j\omega$, where $\omega = 4$, and the complex amplitude is $U = 2 - j$. Therefore,

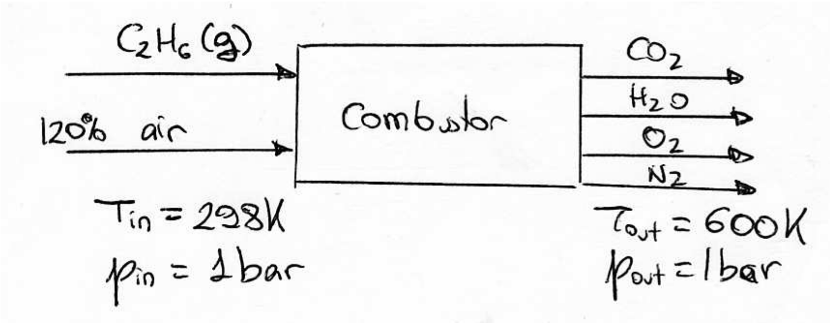
$$Y = G(s)U = G(4j) \cdot (2 - j) \approx (0.9825 + 0.1310j)(2 - j) \approx 2.0961 - 0.7205j$$

Finally,

$$u(t) \approx 2.0961 \cos 4t + 0.7205 \sin 4t$$

Problem 4

GIVEN:



ASSUMPTIONS:
- Steady flow.
- Neglect kinetic and potential energy effects.

CONCEPTS:
- Stoichiometry.
- First Law in CV form.
- Enthalpy of formation.

SOLUTION:

a) The stoichiometric reaction is:

$$C_2H_6 + 3.5(O_2 + 3.76N_2) \rightarrow 2CO_2 + 3H_2O + 3.5 \times 3.76N_2$$

With 20% excess air:

$$C_2H_6 + 4.2(O_2 + 3.76N_2) \rightarrow 2CO_2 + 3H_2O + 0.7O_2 + 4.2 \times 3.76N_2$$

Then, the air/fuel ratio is:

$$\frac{1}{f} = \frac{m_{\text{air}}}{m_{\text{fuel}}} = \frac{4.2(M_{O_2} + 3.76M_{N_2})}{M_{C_2H_6}} = 19.22$$

b) Per kmol of fuel, the products will be:

- 2 kmol of CO_2.
- 3 kmol of H_2O.
- 0.7 kmol of O_2.
- 15.79 kmol of N_2.

c) The enthalpy of a reaction at the standard conditions is:

$$\begin{aligned}\Delta\overline{h}_{\text{reaction}} = 2\overline{h}^0_{f,CO_2} + 3\overline{h}^0_{f,H_2O} + 0.7\overline{h}^0_{f,O_2} + 15.79\overline{h}^0_{f,N_2} - \\ -\overline{h}^0_{f,C_2H_6} - 4.2\overline{h}^0_{f,O_2} - 15.79\overline{h}^0_{f,N_2} = \\ = 2\overline{h}^0_{f,CO_2} + 3\overline{h}^0_{f,H_2O} - \overline{h}^0_{f,C_2H_6} = -1.428 \times 10^6 \text{ kJ/kmol}\end{aligned}$$

d) The amount of heat transfer per kmol of fuel is obtained from the first law:

$$\bar{q} = \sum_R n_R \bar{h}_R - \sum_P n_P \bar{h}_P$$

The enthalpy of the reactants is:

$$\sum_R n_R \bar{h}_R = \bar{h}_{C_2H_6} + 4.2\bar{h}_{O_2} + 15.79\bar{h}_{N_2} = \bar{h}^0_{f,C_2H_6} + 4.2\bar{h}^0_{f,O_2} + 15.79\bar{h}^0_{f,N_2} = \bar{h}^0_{f,C_2H_6}$$

The enthalpy of the products is:

$$\sum_P n_P \bar{h}_P = 2\bar{h}_{CO_2} + 3\bar{h}_{H_2O} + 0.7\bar{h}_{O_2} + 15.79\bar{h}_{N_2},$$

where

$$\begin{aligned}
\bar{h}_{CO_2} &= \bar{h}^0_{f,CO_2} + \Delta\bar{h}_{CO_2}|_{T=600\text{ K}} \\
\bar{h}_{H_2O} &= \bar{h}^0_{f,H_2O} + \Delta\bar{h}_{H_2O}|_{T=600\text{ K}} \\
\bar{h}_{O_2} &= \bar{h}^0_{f,O_2} + \Delta\bar{h}_{O_2}|_{T=600\text{ K}} = \Delta\bar{h}_{O_2}|_{T=600\text{ K}} \\
\bar{h}_{N_2} &= \bar{h}^0_{f,N_2} + \Delta\bar{h}_{N_2}|_{T=600\text{ K}} = \Delta\bar{h}_{N_2}|_{T=600\text{ K}}
\end{aligned}$$

Then:

$$\sum_P n_P \bar{h}_P = (2\bar{h}^0_{f,CO_2} + 3\bar{h}^0_{f,H_2O}) + (2\Delta\bar{h}_{CO_2} + 3\Delta\bar{h}^0_{H_2O} + 0.7\Delta\bar{h}_{O_2} + 15.79\Delta\bar{h}_{N_2})|_{T=600\text{ K}}$$

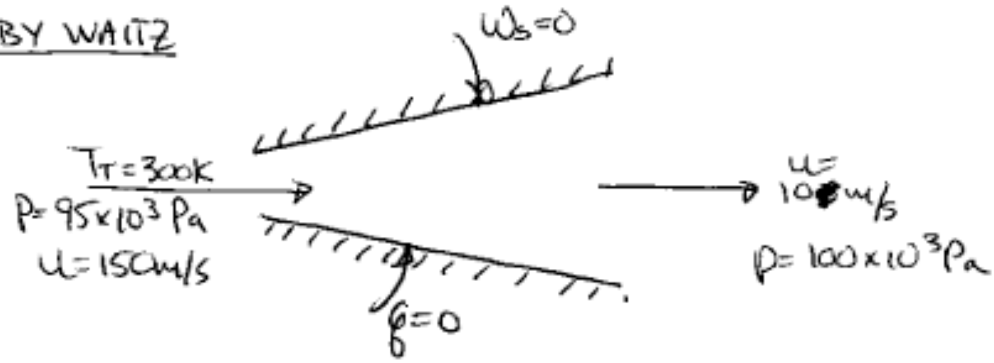
$\Delta\bar{h}_{CO_2}$, $\Delta\bar{h}_{H_2O}$, $\Delta\bar{h}_{O_2}$ and $\Delta\bar{h}_{N_2}$ are the enthalpy change between the reference state and the substance at the exit conditions ($T_{out} = 600$ K, $p_{out} = 1$ bar). These $\Delta\bar{h}$ can be found in tables:

$$\begin{aligned}
\Delta\bar{h}_{CO_2} &= 12906 \text{ kJ/kmol} \\
\Delta\bar{h}_{H_2O} &= 10499 \text{ kJ/kmol} \\
\Delta\bar{h}_{O_2} &= 9245 \text{ kJ/kmol} \\
\Delta\bar{h}_{N_2} &= 8894 \text{ kJ/kmol}
\end{aligned}$$

Adding the different enthalpy of the different products and reactants, and using the definition of $\Delta\bar{h}_{reaction}$:

$$\bar{q} = -\Delta\bar{h}_{reaction} - 2\Delta\bar{h}_{CO_2} - 3\Delta\bar{h}^0_{H_2O} - 0.7\Delta\bar{h}_{O_2} - 15.79\Delta\bar{h}_{N_2} = 1.224 \times 10^6 \text{ kJ/kmol}$$

T12 SOLUTIONS BY WAITZ



a) $$q - w_s = \left(C_p T_{OUT} + \frac{C_{OUT}^2}{2}\right) - \left(C_p T_{IN} + \frac{C_{IN}^2}{2}\right)$$

$$C_p T_T = C_p T + \frac{C^2}{2} \Longrightarrow \frac{1003.5\,(300K)}{1003.5} - \frac{(150)^2}{2(1003.5)} = \boxed{T_{IN} = 288.8K}$$

$$\frac{T_T}{T} = \left(\frac{P_T}{P}\right)^{\frac{\gamma-1}{\gamma}} \quad \left(\frac{300}{288.8}\right)^{\gamma/\gamma-1} \cdot 95\times10^3 = \boxed{P_{T_{IN}} = 108546 \text{ Pa}}$$

b) SINCE $q = 0$ & $w_s = 0$, $h_{T_{OUT}} = h_{T_{IN}}$

$$\therefore \boxed{T_{T_{OUT}} = T_{T_{IN}} = 300K} \qquad T_{OUT} = T_{T_{OUT}} - \frac{C_{OUT}^2}{2C_p}$$

$$T_{OUT} = 299.95K$$

$$\left(\frac{T_{T_{OUT}}}{T_{OUT}}\right)^{\gamma/\gamma-1} \cdot P_{OUT} = P_{T_{OUT}}$$

$$\left(\frac{300}{299.95}\right)^{\gamma/\gamma-1} \cdot 100\times10^3 = \boxed{P_{T_{OUT}} = 100058 \text{ Pa}}$$

c) $$\Delta s_{2-1} = C_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) = 1003.5 \ln\left(\frac{299.95}{288.8}\right) - 287 \ln\left(\frac{100\times10^3}{95\times10^3}\right)$$

$$\Delta s = 23.3 \text{ J/kg-K}$$

d) IF EXPANDED IN A REVERSIBLE MANNER (q-s, adiabatic)

$$\left(\frac{P_{OUT}}{P_{IN}}\right)^{\gamma-1/\gamma} = \left(\frac{T_{OUT}}{T_{IN}}\right) \quad \therefore \quad T_{OUT} = 288.8\left(\frac{100}{95}\right)^{\gamma-1/\gamma} = 293.06K$$

T_T = const if adiabatic & no external work

so $\boxed{T_{T_{OUT}} = 300\,K}$

$T_{T_{OUT}} = T_{OUT} + \frac{C^2}{2C_p} \Longrightarrow C_{OUT} = 118 \frac{m}{s}$

$\left(\frac{T_{T_{OUT}}}{T_{OUT}}\right)^{\gamma/\gamma-1} = \frac{P_{T_{OUT}}}{P_{OUT}} \quad \left(\frac{300}{293.06}\right)^{\gamma/\gamma-1} \cdot 100\times10^3 = 108546\,Pa$ ✓

$\boxed{P_{T_{OUT}} = P_{T,IN} = 108546\,Pa}$ ✓

T13 SOLUTIONS BY WAITZ

a) $s_2 - s_1 = C_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{V_2}{V_1}\right) = 10085 \ln\left(\frac{270}{300}\right) + 4124 \ln\left(\frac{200}{100}\right)$

$\Delta s = 1796 \frac{J}{kg\text{-}K}$

$w = 32550\ J/kg$

$\Delta u = \cancel{q}^{0} - w \qquad w = -C_v \Delta T = -10085(270-300)$

b) FOR QUASI-STATIC ADIABATIC $\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = \left(\frac{100}{200}\right)^{\left(\frac{14209}{10085}-1\right)} = 0.753$

$T_2 = 226\ K$

$w = -10085(226-300) = 746290\ J/kg$

$\Delta s = 0$

c) IRREVERSIBLE PROCESS: $T_1 = 300K, \quad T_2 = 288K$

$V_1 = 0.0861 \frac{m^3}{kg}, \quad V_2 = 0.1722 \frac{m^3}{kg}$

$\Delta s = C_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{V_2}{V_1}\right) = 716.5 \ln\left(\frac{288}{300}\right) + 287 \ln\left(\frac{0.1722}{0.0861}\right)$

$\Delta s_{irrev.} = 169.7 \frac{J}{kg\text{-}K}$

REVERSIBLE PROCESS: $T_1 = 300K \quad T_2 = 227.4K$

$V_1 = 0.0861 \frac{m^3}{kg} \quad V_2 = 0.1722\ m^3/kg$

$\Delta s_{rev.} = 716.5 \ln\left(\frac{227.4}{300}\right) + 287 \ln\left(\frac{0.1722}{0.0861}\right) = 0$ ✓