9:00 - 10:00
35-225

MIT ID# (last four digits) SOLUTIONS (pmw)

Unified Quiz 4M

November 17, 2006

- Put the last four digits of your MIT ID # on each page of the exam.
- Read all questions carefully.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit cannot be given without intermediate results.
- Show the logical path of your work. Explain clearly your reasoning and what you are doing. *In some cases, the reasoning is worth as much (or more) than the actual answers.*
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units. Intermediate answers and final answers are not correct without the units.
- Report significant digits only.
- Box your final answers.
- ***Calculators and handwritten "crib sheets" are allowed.***
- ***Unified Handout entitled "Review of Stress, Strain, and Elasticity" allowed.***

EXAM SCORING

#1M (25%)	
#2M (25%)	
#3M (25%)	
#4M (25%)	
FINAL SCORE	

Unified Quiz 4 - Materials and Structures (Q4M)
November 17, 2006

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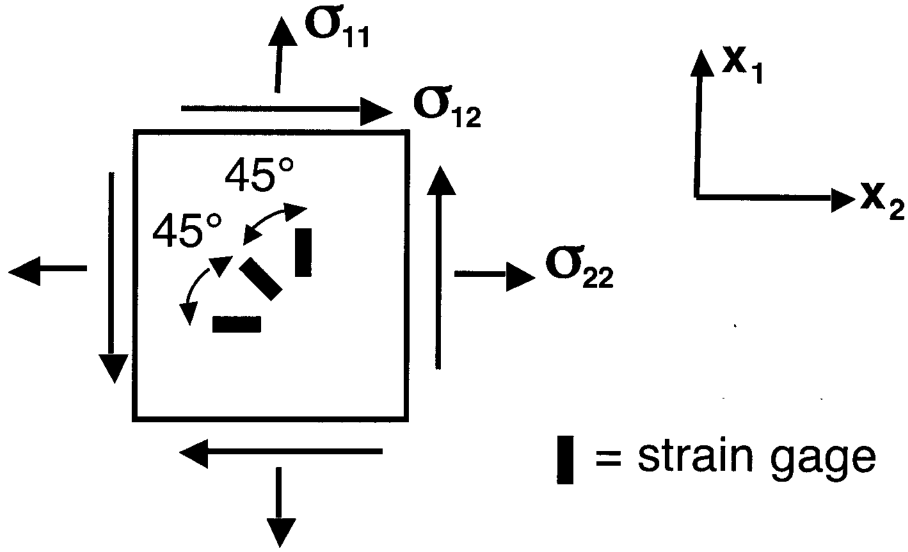
PROBLEM #1M (25%)

An *orthotropic* material has a general loading applied resulting in stresses of σ_{11}, σ_{22}, and σ_{12}, as indicated in the accompanying figure. The stresses may not be aligned with the main axes of the material. A three-gage strain gage rosette is placed on the material and aligned in such a way that the strain in the x_1- direction, ε_{11}, and in the x_2- direction, ε_{22}, are measured along with the strain at a 45° angle to the x_1- direction, $\varepsilon(45)$.



Determine the principal in-plane strains or **clearly** indicate why they cannot be determined.

To get the principal in-plane strains we need to know the full strain state (ϵ_{11}, ϵ_{22}, ϵ_{12}) as the operative equation is:

$$\tau^2 - \tau(\epsilon_{11} + \epsilon_{22}) + (\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2) = 0 \quad (*)$$

and solving for τ (same form as for stress)

We know that three strains allow us to determine the full in-plane strain state.

→ Use the strain transformation equations:
[NOTE: only first one needed]

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta\sin\theta\, \epsilon_{12}$$

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PROBLEM #1M (continued)

Notes:
- Gage in x_1 direction gives ϵ_{11}
- Gage in x_2 direction gives ϵ_{22}
- Other gage gives $\tilde{\epsilon}_{11}$ for $\theta = 45°$ $[\tilde{\epsilon}_{11} = \epsilon(45)]$

Use equation and find ϵ_{12}:

$$\epsilon(45) = \frac{1}{2}\epsilon_{11} + \frac{1}{2}\epsilon_{22} + \epsilon_{12}$$

$$\Rightarrow \epsilon_{12} = \epsilon(45) - \frac{1}{2}(\epsilon_{11} + \epsilon_{22})$$

Now place in principal equation (*):

$$\tau^2 - \tau(\epsilon_{11} + \epsilon_{22}) + \left\{\epsilon_{11}\epsilon_{22} - \epsilon^2(45) + \epsilon(45)(\epsilon_{11} + \epsilon_{22}) - \frac{1}{4}(\epsilon_{11} + \epsilon_{22})^2\right\} = 0$$

Solve this via: $a\tau^2 + b\tau + c = 0$

$$\tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \boxed{\epsilon_I, \epsilon_{II} = \frac{(\epsilon_{11} + \epsilon_{22})}{2} \pm \frac{1}{2}\left\{(\epsilon_{11} + \epsilon_{22})^2 - 4\epsilon_{11}\epsilon_{22} + 4\epsilon^2(45) - 4\epsilon(45)(\epsilon_{11} + \epsilon_{22}) + (\epsilon_{11} + \epsilon_{22})^2\right\}^{1/2}}$$

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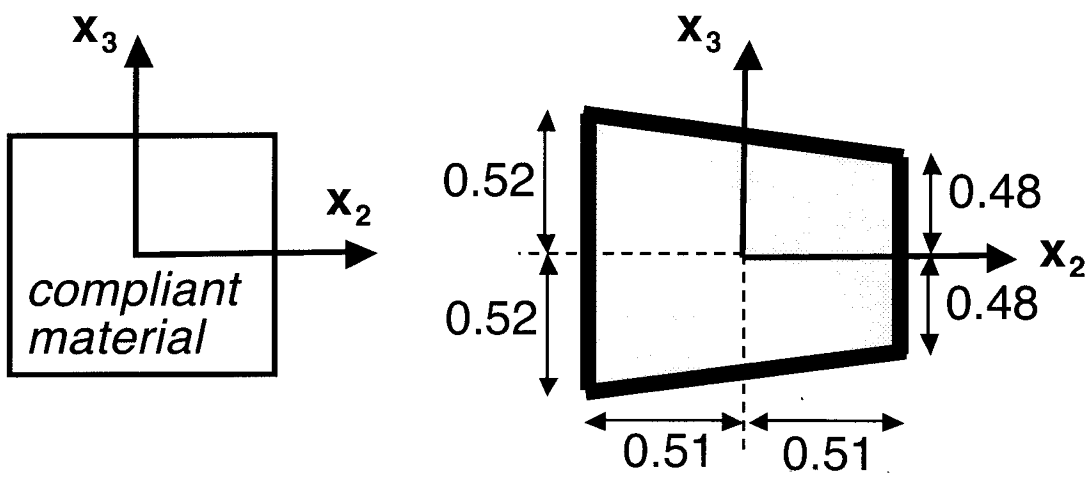
PROBLEM #2M (25%)

A square block of a very compliant material is in the x_2- x_3 plane as pictured below to the left. This block initially has *unit dimensions* on each side. The slab is then placed into a form so that it undergoes deformation consistent with the boundaries of the form, as pictured below to the right. The dimensions are shown in values *normalized to unit dimensions*.



An engineer performs an analysis and indicates that the following strain field should result:

$$\varepsilon_{22} = 0.02 \qquad \varepsilon_{33} = -0.08\,x_2 \qquad \varepsilon_{23} = -0.04\,x_3$$

It is known that the strain does not vary with x_1.

(a) Determine whether this strain field is consistent with the deformation described. Clearly give your reasoning. Use equations as appropriate.

- We have: $\epsilon_{22} = 0.02$, $\epsilon_{33} = -0.08\,x_2$, $\epsilon_{23} = -0.04\,x_3$
- use the strain-displacement relations:

$$\epsilon_{22} = \partial u_2 / \partial x_2$$
$$\epsilon_{33} = \partial u_3 / \partial x_3$$
$$\epsilon_{23} = \frac{1}{2}\left(\partial u_2/\partial x_3 + \partial u_3/\partial x_2\right)$$

→ Now integrate remembering that with $\partial/\partial x_1 = 0$, we have the 2-D case and u_2 and u_3 are functions of x_2 and x_3. We will thus have <u>functions of integration</u>. Thus:

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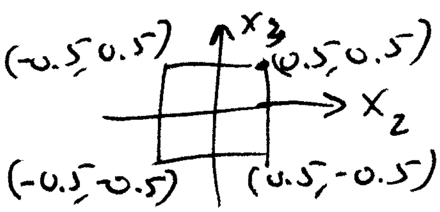
PROBLEM #2M (continued)

$$\epsilon_{22} = 0.02 = \frac{\partial u_2}{\partial x_2} \Rightarrow u_2(x_2, x_3) = \int 0.02\, dx_2$$
$$= 0.02 x_2 + f_1(x_3) \qquad (\#1)$$

$$\epsilon_{33} = -0.08 x_2 = \frac{\partial u_3}{\partial x_3} \Rightarrow u_3(x_2, x_3) = \int (-0.08 x_2)\, dx_3$$
$$= -0.08 x_2 x_3 + f_2(x_2) \qquad (\#2)$$

$$\epsilon_{23} = -0.04 x_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$
$$\Rightarrow -0.08 x_3 = \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_2} - 0.08 x_3 \qquad (\#3)$$

→ Now look at the deformation defined via the corners of the block:



$(0.5, 0.5) \rightarrow (0.51, 0.48) \Rightarrow u_2 = 0.01$, $u_3 = -0.02$

$(-0.5, 0.5) \rightarrow (-0.51, 0.52) \Rightarrow u_2 = -0.01$, $u_3 = 0.02$

$(-0.5, -0.5) \rightarrow (-0.51, -0.52) \Rightarrow u_2 = -0.01$, $u_3 = -0.02$

$(0.5, -0.5) \rightarrow (0.51, 0.48) \Rightarrow u_2 = 0.01$, $u_3 = -0.02$

→ finally, are the derived displacement functions consistent with this?

u_2 is if $f_1(x_3) = 0$ ✓

$\Rightarrow f_2(x_2) = 0$ (via #3)

u_3 now is ✓

$\Rightarrow$ $\boxed{\text{CONSISTENT}}$

shown via strain-displacement relations

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PROBLEM #2M (continued)

(b) The form is now changed so that the deformations are increased by a factor of ten. How will this affect the values of the in-plane strains? Describe clearly using equations as necessary.

If the deformations increase by a factor of 10, we get:

$$u_2 = 0.20\, x_2$$

$$u_3 = -0.80\, x_2 x_3$$

The extensional strains would then, if all remain "small and linear" also increase by a factor of 10 giving: $\epsilon_{22} = 0.20$, $\epsilon_{33} = -0.80 x_2$. Even this is rather large, but it becomes clearer that the assumption of small displacements and small angles is breaking down by looking at ϵ_{23}. We get $\epsilon_{23} = -0.80 x_3$. For $x_3 = 0.5$, this gives $\epsilon_{23} = -0.40$. For 0.40 radians, the cos θ is 0.921 ($\theta \approx 23°$). So we are nearly 8% off $\cos\theta = 1$. Thus, there is coupling between extension and shear that needs to be taken into account. The strain - displacement equations used in part (a) cannot be used.

Unified Quiz 4 - Materials and Structures (Q4M)
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PROBLEM #3M (25%)

As an engineer, you have been presented with the statement:

"The modulus of this unidirectional carbon/epoxy material is 300 GPa"

(a) **List** any assumptions and limitations in the use of this information. Be sure to clearly **describe** the source(s) of these assumptions and limitations and their ramifications.

- Hooke's law is just a model. The behavior is not exactly linear.
- There is a point where the stress-strain behavior often deviates from linear and becomes nonlinear. This depends on the material.
- This is an averaged behavior that is only good at lengthscales above the diameters of the individual fibers where there is a difference between fiber and matrix modulus.
- This depends on the assumption of combined action and relies on the matrix and fiber being perfectly bonded and thereby undergoing the same strain.
- A unidirectional composite is directional and this modulus will apply to only one direction (most likely along the fibers).
- The modulus changes with temperature, so this is valid at and around room temperature (and some possible larger region depending on the material).

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PROBLEM #3M (continued)

(b) **List** the factors that contribute at lower levels of lengthscale to this statement. Clearly **indicate** the relative importance of each, the lengthscale(s) at which they operate, and any associate limitations.

There are three basic factors acting at lower lengthscales. They are listed in order of importance:

1. Atomic Bonding: Most basic and most important is the stiffness of the atomic bond(s) within the material. This depends on the type(s) of bond(s) operative and associated material.

LENGTHSCALE: On order of basic atomic distances ($10^{-10} - 10^{-9}$ m)

2. Atomic/Molecular structure/arrangement: Atoms and molecules set themselves in various arrangements depending on the material (and the bond(s)). At these lengthscales, this "structural" arrangement affects overall stiffness at higher lengthscales.

LENGTHSCALE: One to three orders above atomic distance ($10^{-9} - 10^{-7}$ m)

3. Material (Composite) Structure: Overall material can be made up of different materials. For case of composites, fibers (or particulates) are embedded in a matrix. The direction and nature of these materials and how they interact affect the stiffness of the overall material

LENGTHSCALE: four to five orders above atomic distances (up to and above microns -- fiber diameters) ($10^{-6} - 10^{-5}$ m)

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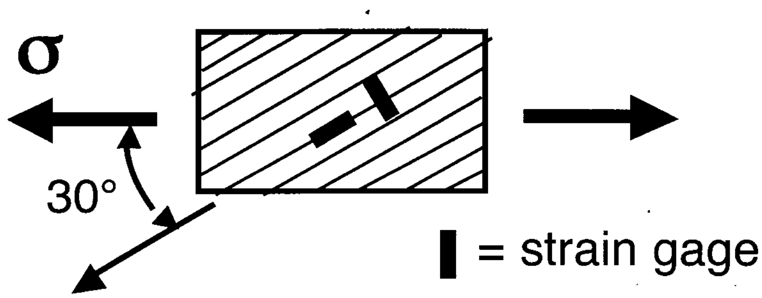
PROBLEM #4M (25%)

A unidirectional composite ply has the following elastic constants:

E_1 = 120 GPa	ν_{12} = 0.30	G_{12} = 35 GPa
E_2 = 20 GPa	ν_{21} = 0.05	

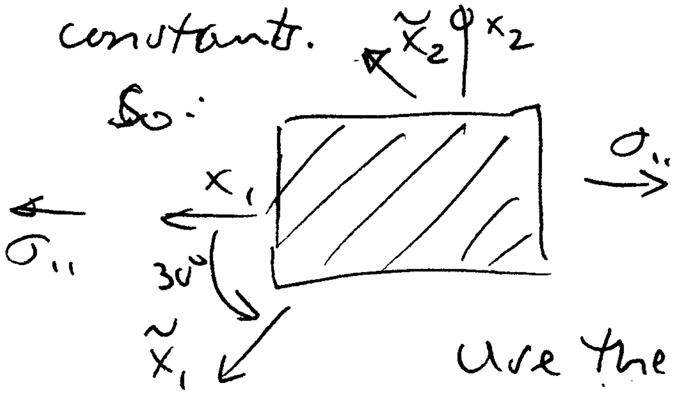
These are referenced with the x_1 direction along the fibers and the x_2 direction perpendicular to the fibers. The unidirectional ply is loaded by a stress of 100 MPa at an angle 30° off the fiber direction as pictured. Strain gages are placed on the surface of the ply parallel and perpendicular to the fiber direction (also shown).



Can the strain readings be determined? If so, do so. If not, clearly explain why not, using equations as appropriate.

- The elastic (engineering) constants describe behavior in axes aligned with the fiber direction (where the composite is orthogonal in behavior)
- If we transform the applied stress to stresses in the fiber axis system, we can use the elastic constants.

So:



$\sigma_{11} = 100 MPa$

$\sigma_{22} = 0 \quad \sigma_{12} = 0$

Use the transformation equations with $\theta = 30°$

PROBLEM #4M (continued)

$\tilde{\sigma}_{11} = \cos^2\theta\, \sigma_{11}$

$\tilde{\sigma}_{22} = \sin^2\theta\, \sigma_{11}$

$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\, \sigma_{11}$

(Note: other terms not needed since $\sigma_{22} = 0$, $\sigma_{12} = 0$)

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2} = 0.866$

$\Rightarrow \tilde{\sigma}_{11} = \frac{3}{4}(100\,MPa) = 75\,MPa$

$\tilde{\sigma}_{22} = \frac{1}{4}(100\,MPa) = 25\,MPa$

$\tilde{\sigma}_{12} = -\left(\frac{1}{2}\right)(0.866)(100\,MPa) = -43\,MPa$

We are looking for ϵ_{11} and ϵ_{22} in the fiber axis system (call these $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{22}$).

Use the basic stress-strain relations for orthotropic materials (in fiber axes, the unidirectional ply is an orthotropic material)

$\tilde{\epsilon}_{11} = \frac{1}{E_1}\left[\tilde{\sigma}_{11} - \nu_{12}\tilde{\sigma}_{22}\right]$

$\tilde{\epsilon}_{22} = \frac{1}{E_2}\left[-\nu_{21}\tilde{\sigma}_{11} + \tilde{\sigma}_{22}\right]$

$\Rightarrow \tilde{\epsilon}_{11} = \frac{1}{120\,GPa}\left[75\,MPa - 0.3(25\,MPa)\right] = 563 \times 10^{-6} = 563\,\mu\text{strain}$

$\tilde{\epsilon}_{22} = \frac{1}{20\,GPa}\left[(-0.05)(75\,MPa) + 25\,MPa\right] = 1063 \times 10^{-6} = 1063\,\mu\text{strain}$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{strain parallel to fibers} = 563\,\mu\text{strain} \\ \text{strain perpendicular to fibers} = 1063\,\mu\text{strain}\end{array}}$