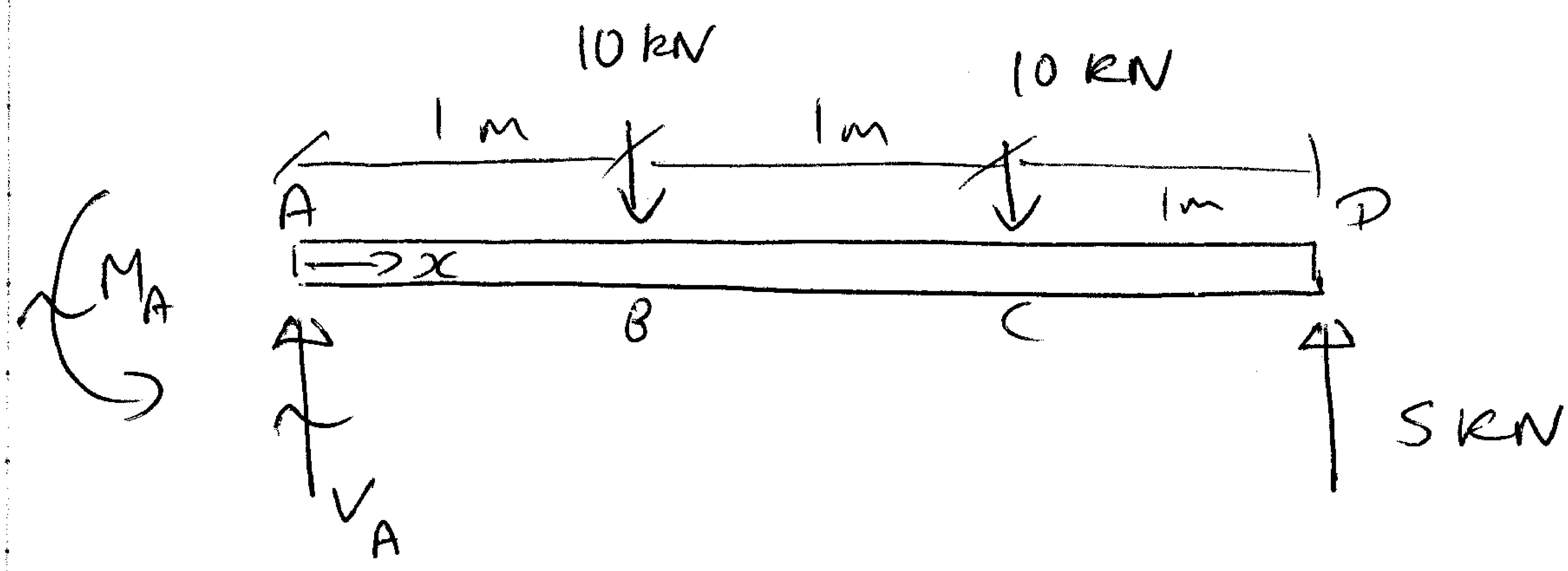
M3

a)



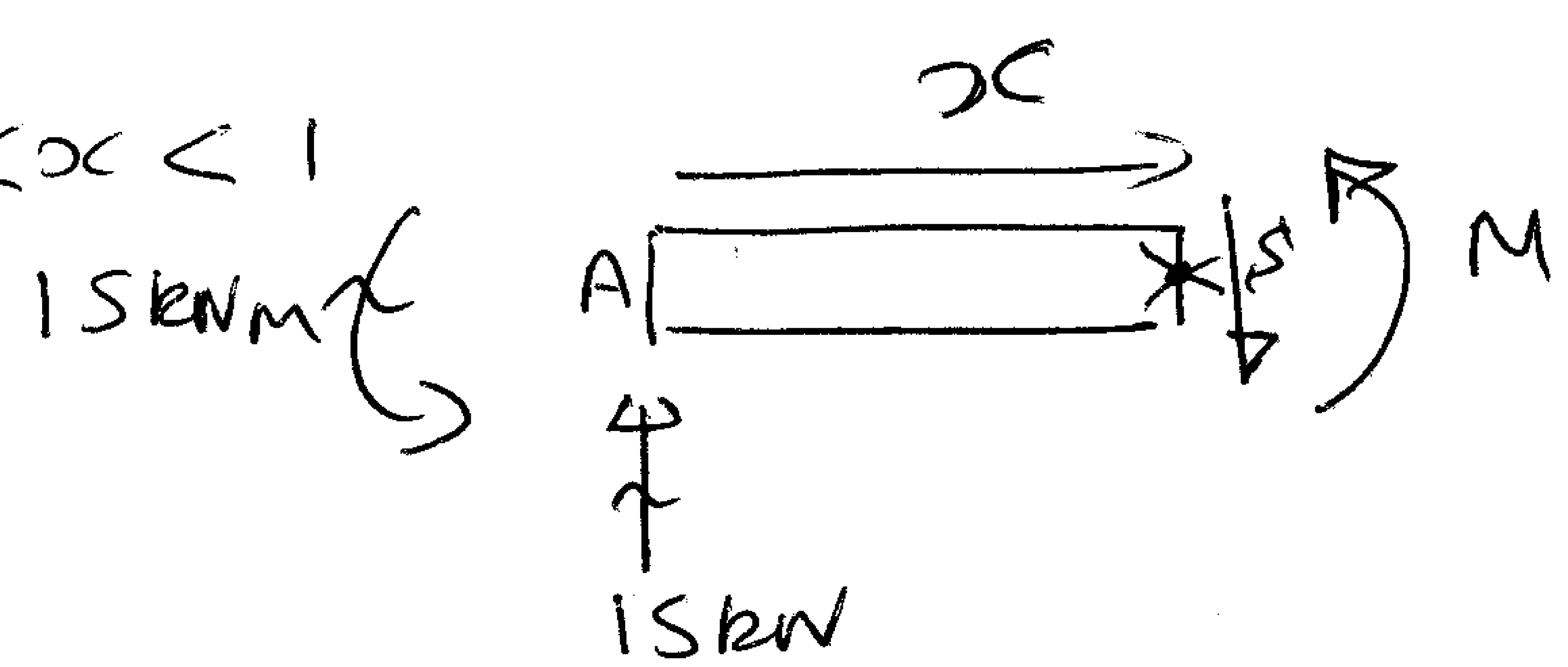
$$\sum F_y \uparrow = 0 \qquad V_A - 10 - 10 + 5 = 0$$

$$V_A = 15 \text{ kN}$$

$$\sum M_A = 0: \quad M_A - 10 \times 1 - 10 \times 2 + 5 \times 3 = 0$$

$$M_A = 15 \text{ kNm}$$

for $0 < x < 1$

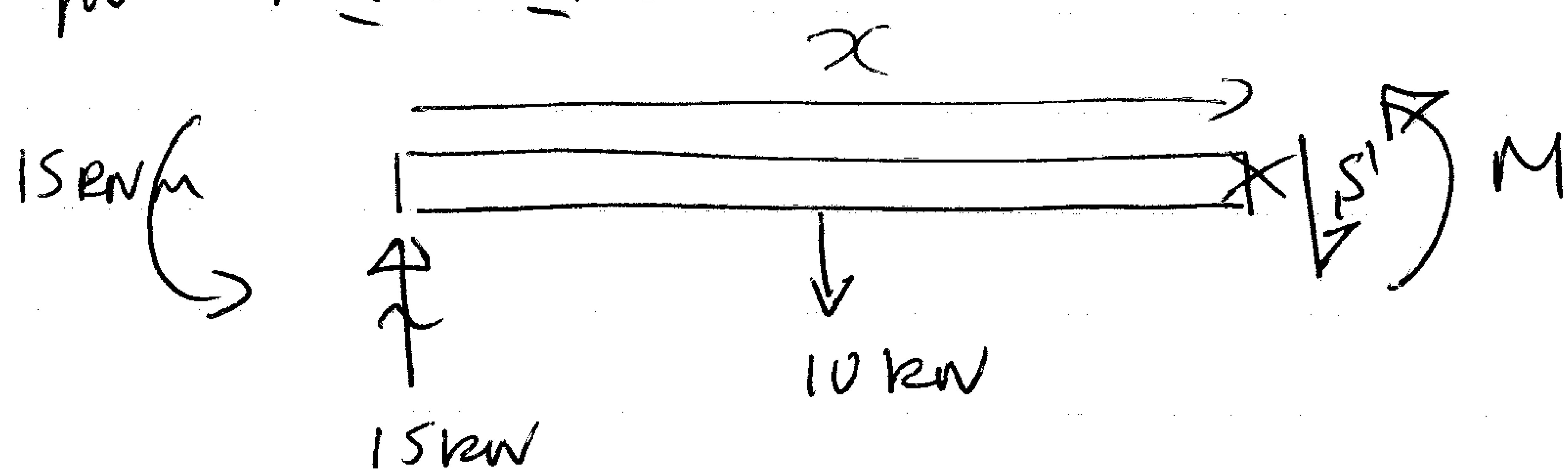


$$\sum F_y \uparrow = 0 \qquad 15 - S = 0 : \quad S = 15 \text{ kN}$$

$$\sum M_{\cancel{A}\,x} = 0: \quad M - 15x + 15 = 0$$

$$M = 15x - 15 \ (\text{kNm})$$

for $1 \le x \le 2$



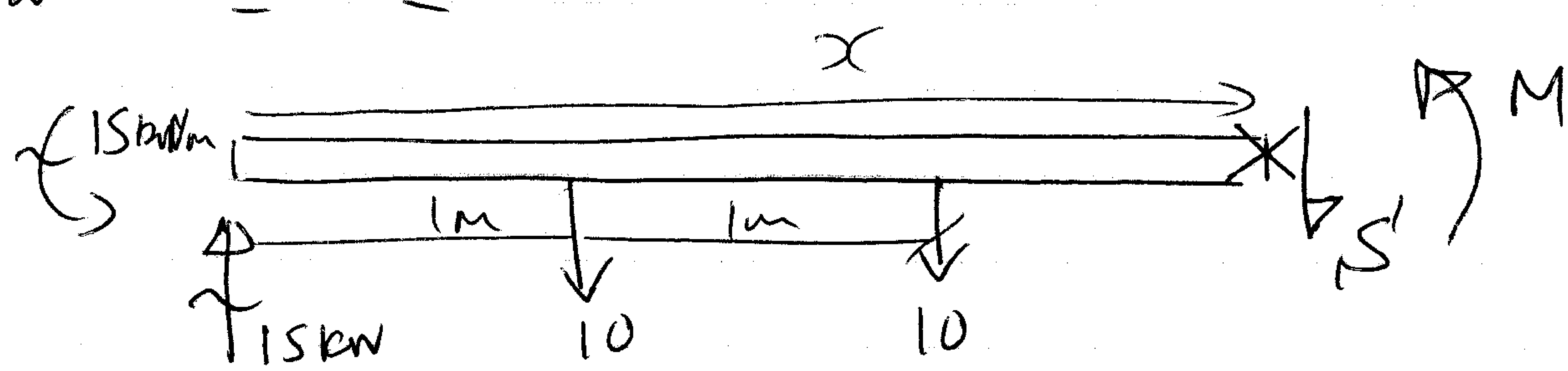
$\sum F_y \uparrow = 0 \qquad 15 - 10 - S' = 0 \qquad S' = 5\text{kN}$

$\sum M_x = 0 \qquad M - 15x + 10(x-1) + 15 = 0$

$\qquad\qquad -5x + 5$

$M = 5x - 5$

for $2 \le x \le 3$



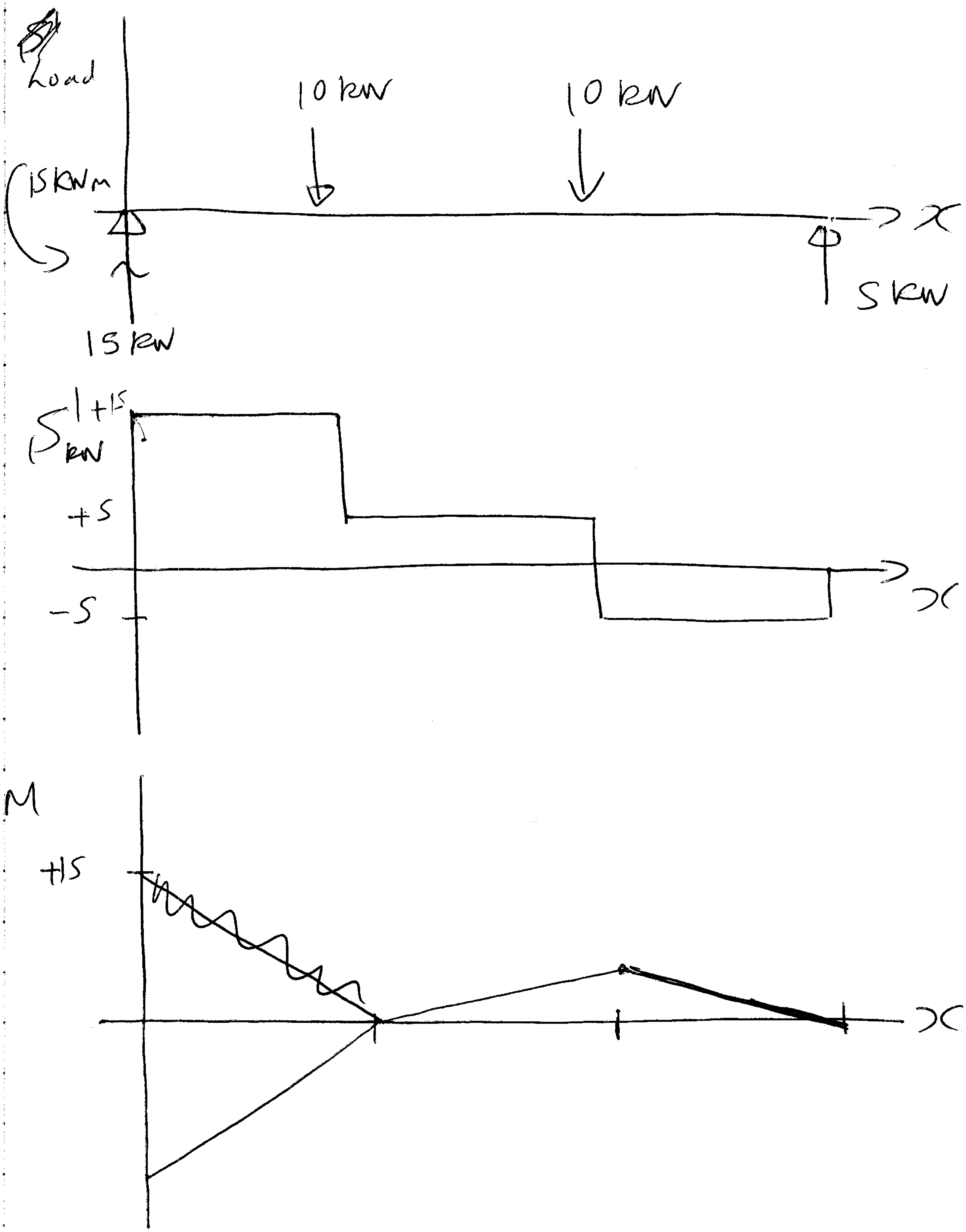
$\sum F_y \uparrow = 0 \qquad 15 - 10 - 10 - S' = 0 \qquad S' = -5\text{kN}$

$\sum M_x = 0 : \quad M + 15 - 15x + 10(x-1) + 10(x-2) = 0$

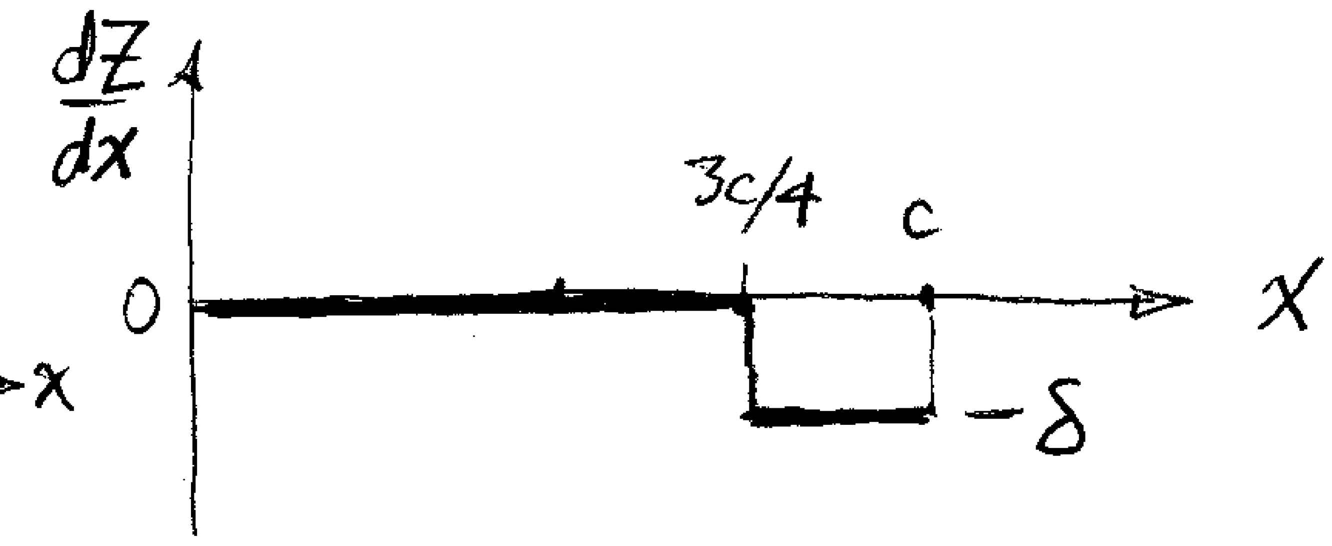
$\quad +15 \quad -15x \quad +10x \quad +10x \quad -10 \quad -20$

$M = 15 - 5x$

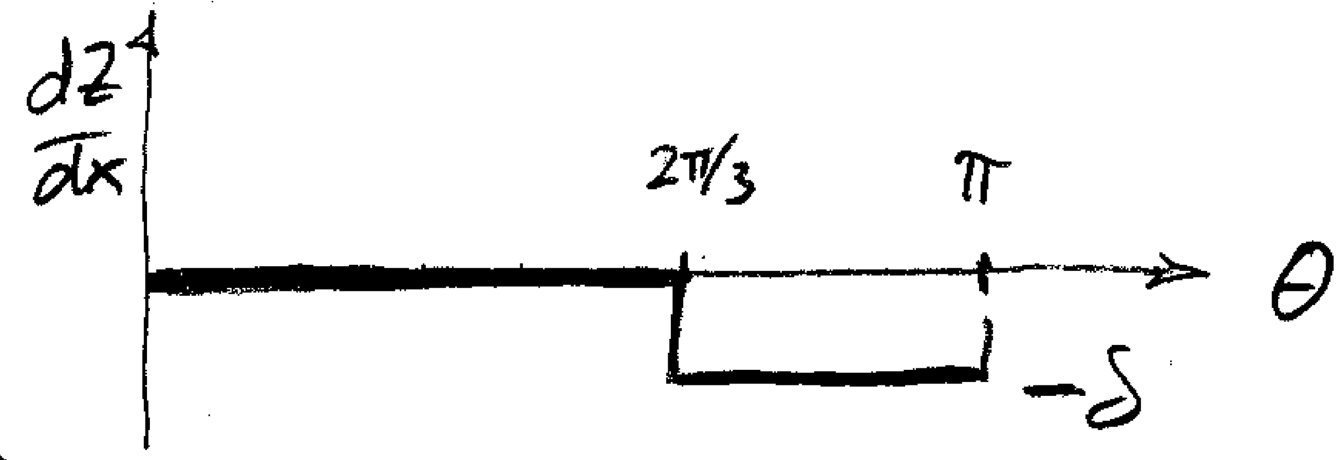
Plotting



a)



$$\frac{x_h}{c} = 0.75, \quad \theta_h = \arccos\left(1 - 2\frac{x_h}{c}\right) = \frac{2\pi}{3}$$



b)
$$A_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\, d\theta = \alpha - \frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi} -\delta \, d\theta = \alpha - \frac{1}{\pi}\left(\pi - \frac{2\pi}{3}\right)(-\delta) = \alpha + \frac{1}{3}\delta$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos\theta \, d\theta = \frac{2}{\pi}\int_{\frac{2\pi}{3}}^{\pi} -\delta \cdot \cos\theta \, d\theta = \frac{2}{\pi}(-\delta)\left(\sin\theta\Big|_{\frac{2\pi}{3}}^{\pi}\right) = 0.551\,\delta$$

$$A_2 = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos 2\theta \, d\theta = \frac{2}{\pi}\int_{\frac{2\pi}{3}}^{\pi} -\delta \cos 2\theta \, d\theta = \frac{2}{\pi}(-\delta)\left(\frac{\sin 2\theta}{2}\Big|_{\frac{2\pi}{3}}^{\pi}\right) = -0.276\,\delta$$

$$\boxed{\begin{aligned} C_\ell &= \pi(2A_0 + A_1) = 2\pi(\alpha + 0.609\,\delta) \\ C_{m,c/4} &= \frac{\pi}{4}(A_2 - A_1) = -0.649\,\delta \end{aligned}}$$

c)
$$\boxed{\begin{aligned} \frac{\partial c_\ell}{\partial \delta} &= 2\pi \cdot 0.609 = 3.826 \\ \frac{\partial c_m}{\partial \delta} &= -0.649 \end{aligned}}$$

a) Assuming L' is uniform across the span,

$$D_i' = L'\alpha_i = L'\frac{w}{V_\infty}$$

Given: $w = \dfrac{L}{2\rho V_\infty b^2}$

$$D_i = D_i' b = L' b \frac{w}{V} = L\frac{w}{V} = \frac{L^2}{2\rho V_\infty^2 b^2} = \frac{L^2}{\frac{1}{2}\rho V^2}\frac{1}{4b^2}$$

$$\boxed{C_{D_i} \equiv \frac{D_i}{\frac{1}{2}\rho V_\infty^2 S} = \frac{L^2}{(\frac{1}{2}\rho V_\infty^2)^2 S^2}\cdot\frac{S}{4b^2} = \frac{C_L^2}{4AR}}$$

b) $C_L = 2\pi\alpha_{eff} = 2\pi(\alpha - \alpha_i)$ ignore $\alpha_{L=0}$, no effect on $\frac{dC_L}{d\alpha}$

but we have $C_{D_i} = \alpha_i C_L$, or $\alpha_i = \dfrac{C_{D_i}}{C_L} = \dfrac{C_L}{4AR}$

$$\Rightarrow C_L = 2\pi\left(\alpha - \frac{C_L}{4AR}\right)$$

$$C_L\left(1 + \frac{\pi}{2AR}\right) = 2\pi\alpha$$

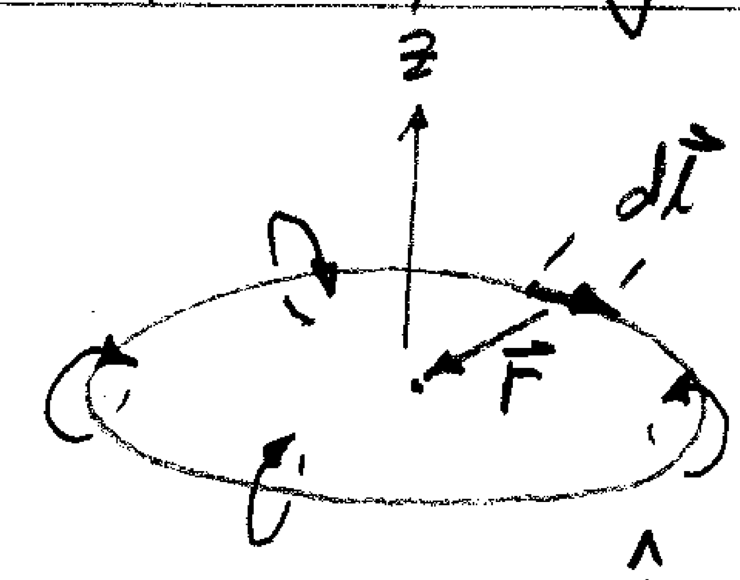
$$\boxed{C_L(\alpha) = \frac{2\pi}{1 + \frac{\pi}{2AR}}\alpha}$$

$$\boxed{\frac{dC_L}{d\alpha} = \frac{2\pi}{1 + \frac{\pi}{2AR}}}$$

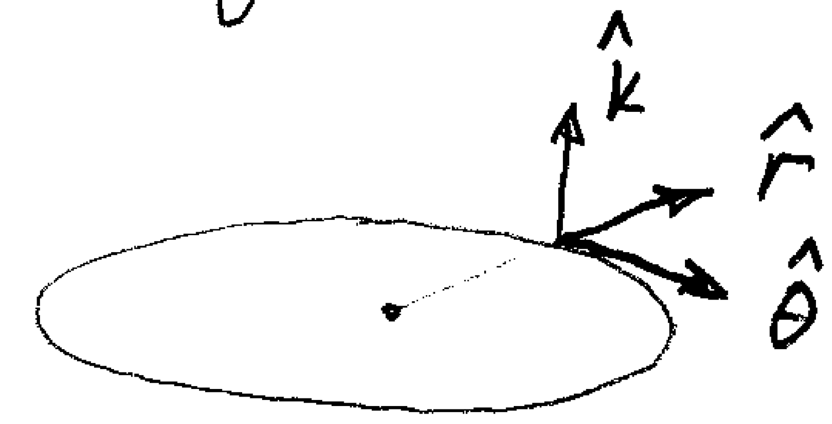
small than 2-D value of 2π by factor of $\dfrac{1}{1 + \frac{\pi}{2AR}}$

(Anderson p 416)

1. $\vec{V} = \dfrac{\Gamma}{4\pi}\displaystyle\int \frac{d\vec{l} \times \vec{r}}{r^3}$

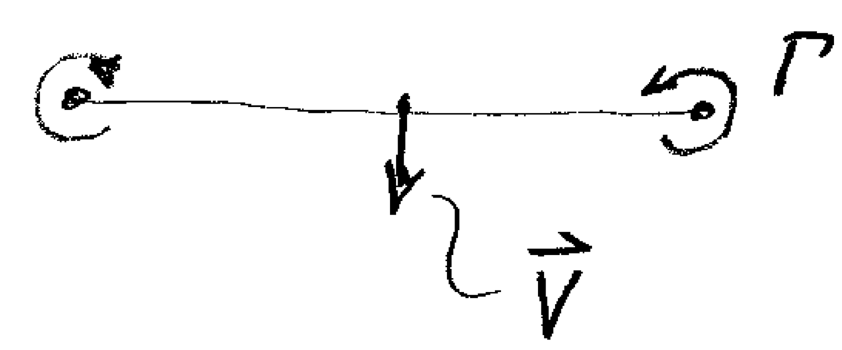


$d\vec{l} = R\,d\theta\,\hat{\theta}$, $\vec{r} = -R\hat{r}$, $r = R$

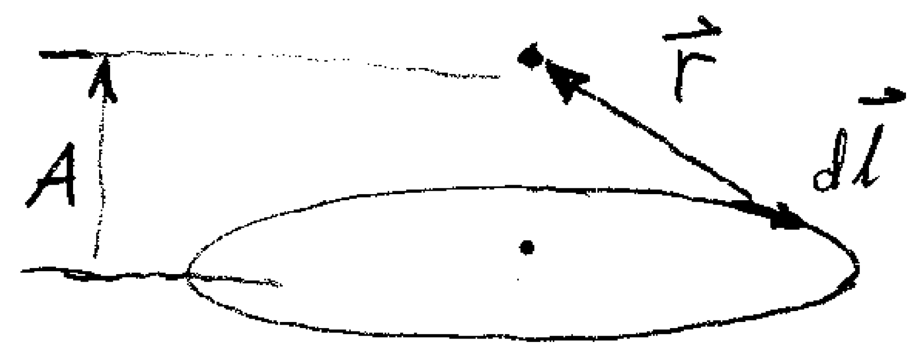


$d\vec{l} \times \vec{r} = -R^2 d\theta\ \hat{\theta} \times \hat{r} = -R^2 d\theta\ \hat{k}$

$$\boxed{\vec{V} = \frac{\Gamma}{4\pi}\int_0^{2\pi} \frac{-R^2\,d\theta}{R^3}\,\hat{k} = -\frac{\Gamma}{2R}\,\hat{k}}$$



2. Now we have



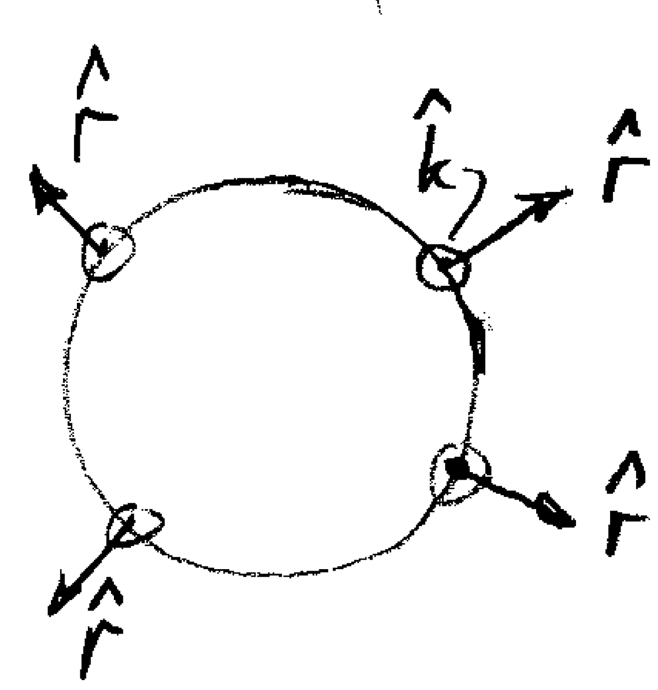
$d\vec{l} = R\,d\theta\,\hat{\theta}$, $\vec{r} = -R\hat{r} + A\hat{k}$

$r^3 = \left(R^2 + A^2\right)^{3/2}$

$$d\vec{l} \times \vec{r} = \left(-R^2(\hat{\theta}\times\hat{r}) + RA(\hat{\theta}\times\hat{k})\right) d\theta = \left(-R^2\hat{k} - RA\hat{r}\right) d\theta$$

Note that $\hat{k}$ is constant, but $\hat{r}$ depends on θ (varies around circle)

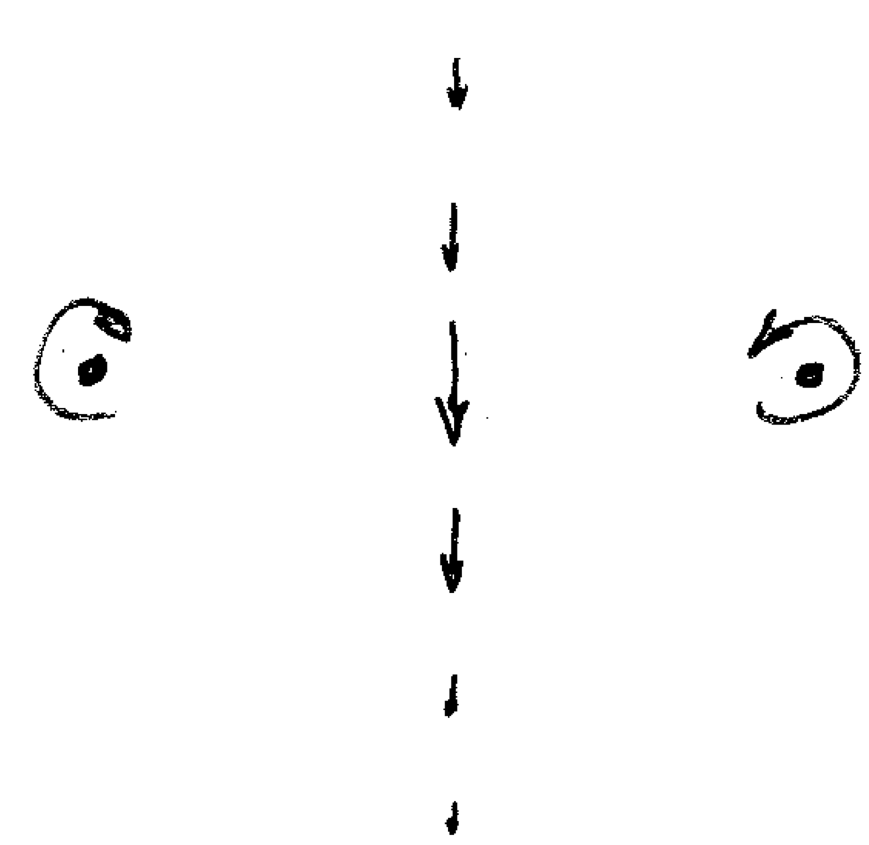
$$\vec{V} = \frac{\Gamma}{4\pi}\int_0^{2\pi} \frac{-R^2\hat{k} - RA\hat{r}}{(R^2+A^2)^{3/2}}\,d\theta$$



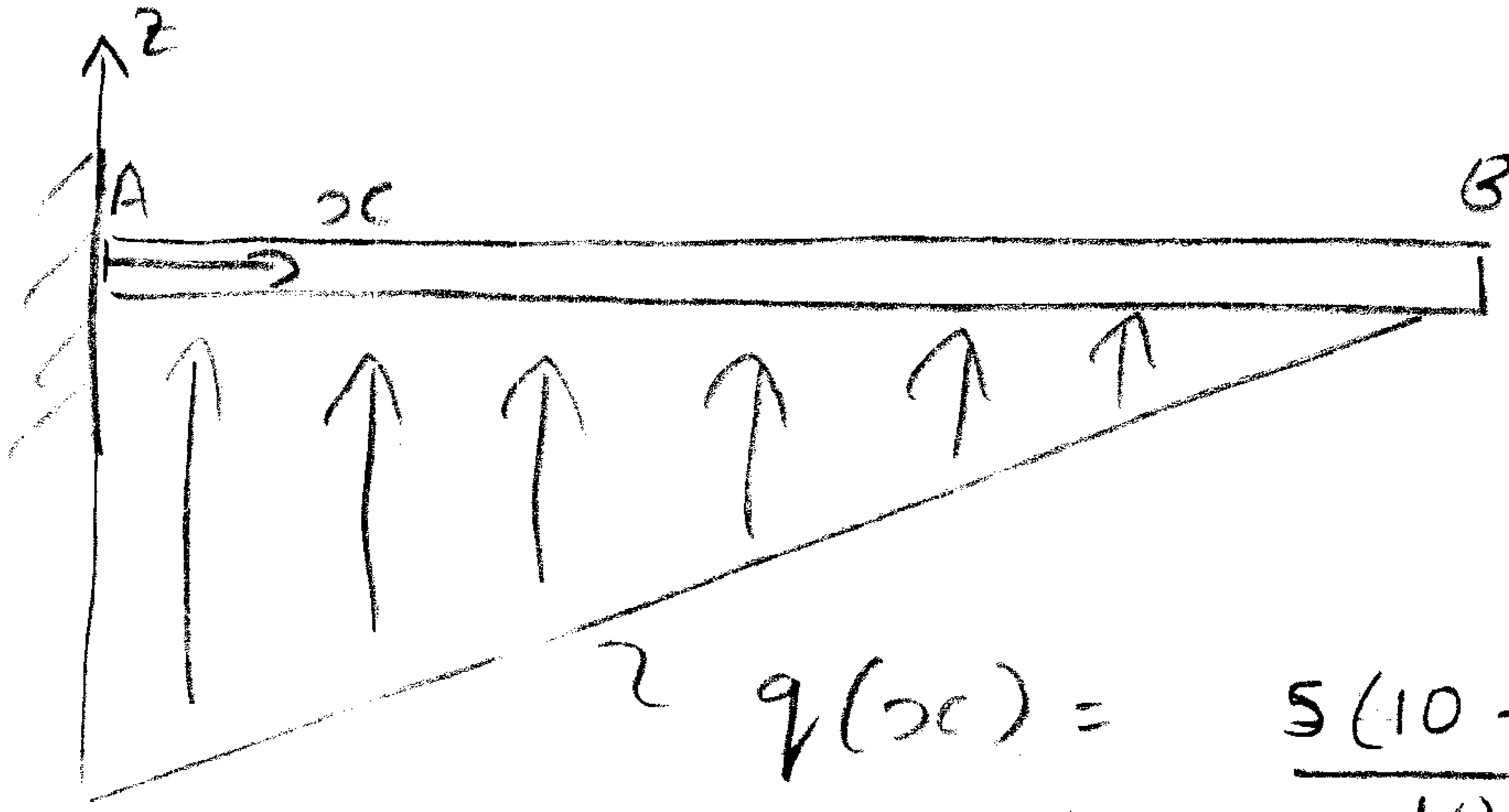
$$\vec{V} = -\frac{\Gamma}{2}\frac{R^2}{(R^2+A^2)^{3/2}}\,\hat{k} + \frac{\Gamma}{4\pi}\frac{-RA}{(R^2+A^2)^{3/2}}\int_0^{2\pi}\hat{r}\,d\theta$$

But we note that $\displaystyle\int_0^{2\pi}\hat{r}\,d\theta = 0$, since $\hat{r}$ cancels when integrated around perimeter.

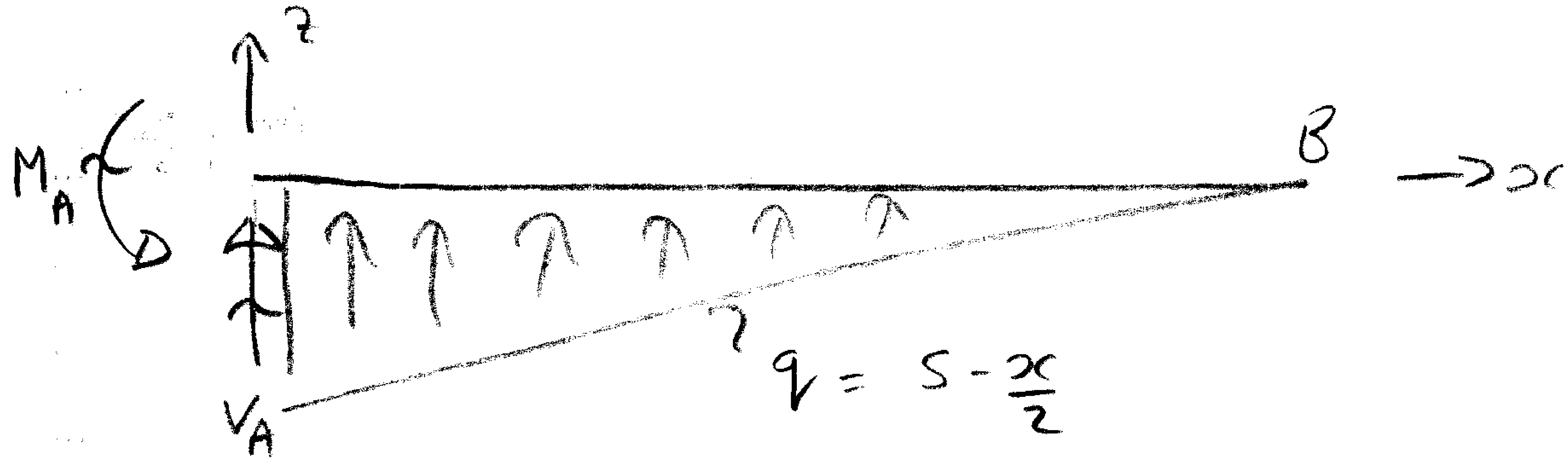
$$\boxed{\vec{V} = -\frac{\Gamma}{2}\frac{R^2}{(R^2+A^2)^{3/2}}\,\hat{k}}$$



M4



$$q(x) = \frac{5(10-x)}{10} = 5 - \frac{x}{2} \text{ N/m}$$



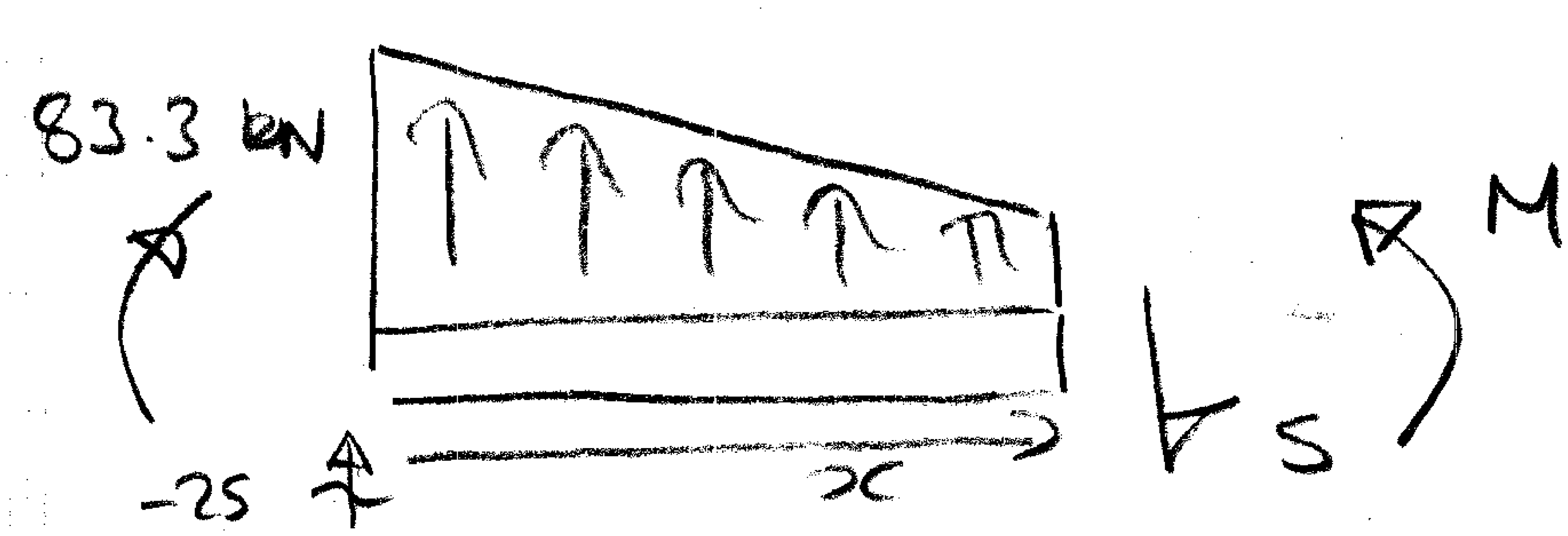
$$\Sigma F_z \uparrow = 0 \qquad V_A + \int_0^{10} 5 - \frac{x}{2}\, dx = 0$$

$$V_A = -\left[5x - \frac{x^2}{4}\right]_0^{10} = -25 \text{ kN}$$

$$\circlearrowleft \Sigma M_A = 0: \quad M_A + \int_0^{10} \left(5 - \frac{x}{2}\right) x\, dx = 0$$

$$M_A = -\left[\frac{5x^2}{2} - \frac{x^3}{6}\right]_0^{10} = -83.3 \text{ kNm}$$

Moment distribution



$\sum F_z \uparrow = 0$:

$$-25 - S' + \int_0^x 5 - \frac{x}{2}\, dx = 0$$

$$S' = \left[5x - \frac{x^2}{4}\right]_0^x - 25 = 5x - \frac{x^2}{4} - 25 \Leftarrow$$

$$\sum \circlearrowright M_A = 0: M - 83.3 + \int_0^x \left(5 - \frac{x}{2}\right) x\, dx - S'x = 0$$

$$M - 83.3 + \frac{5x^2}{2} - \frac{x^3}{6} - 5x^2 + \frac{x^3}{4} + 25x = 0$$

$$M = -\frac{1}{12}x^3 + \frac{5}{2}x^2 - 25x + 83.3 \Leftarrow$$

Max bending stress at root $x = 0$ $z = \pm \frac{100\text{mm}}{2}$

$M_{max} = 83.3$ kNm @ $x = 0$

$I = \frac{1}{12} b h^3$ $\qquad \sigma_{max} = \frac{6 M_{max} \frac{h}{2}}{\cancel{\frac{1}{12}} b h^{\cancel{3}\,2}} = \frac{6 M_{max}}{b h^2}$

$$= \frac{6 \times 83.3 \times 10^3}{50 \times 10^{-3} \times (100 \times 10^{-3})^2} = 1 \text{ GPa !! (high)}$$

M4 tip deflection, from moment curvature relation

$$M = EI\frac{d^2w}{dx^2}$$

integrate moment twice →

$$EI\frac{dw}{dx} = -\frac{1}{12}\frac{x^4}{4} + \frac{5}{2}\frac{x^3}{3} - \frac{25x^2}{2} + 83.3x + A$$

$$\frac{dw}{dx} = 0 \text{ @ } x = 0 \Rightarrow A = 0$$

$$EIw = -\frac{1}{12}\frac{x^5}{20} + \frac{5}{2}\frac{x^4}{12} - \frac{25x^3}{6} + \frac{83.3x^2}{2} + B$$

$$w = 0 \text{ @ } x = 0 \Rightarrow B = 0$$

$$EIw = -\frac{x^5}{240} + \frac{5x^4}{24} - \frac{25x^3}{6} + \frac{83.3}{2}x^2$$

at tip $x = 0$

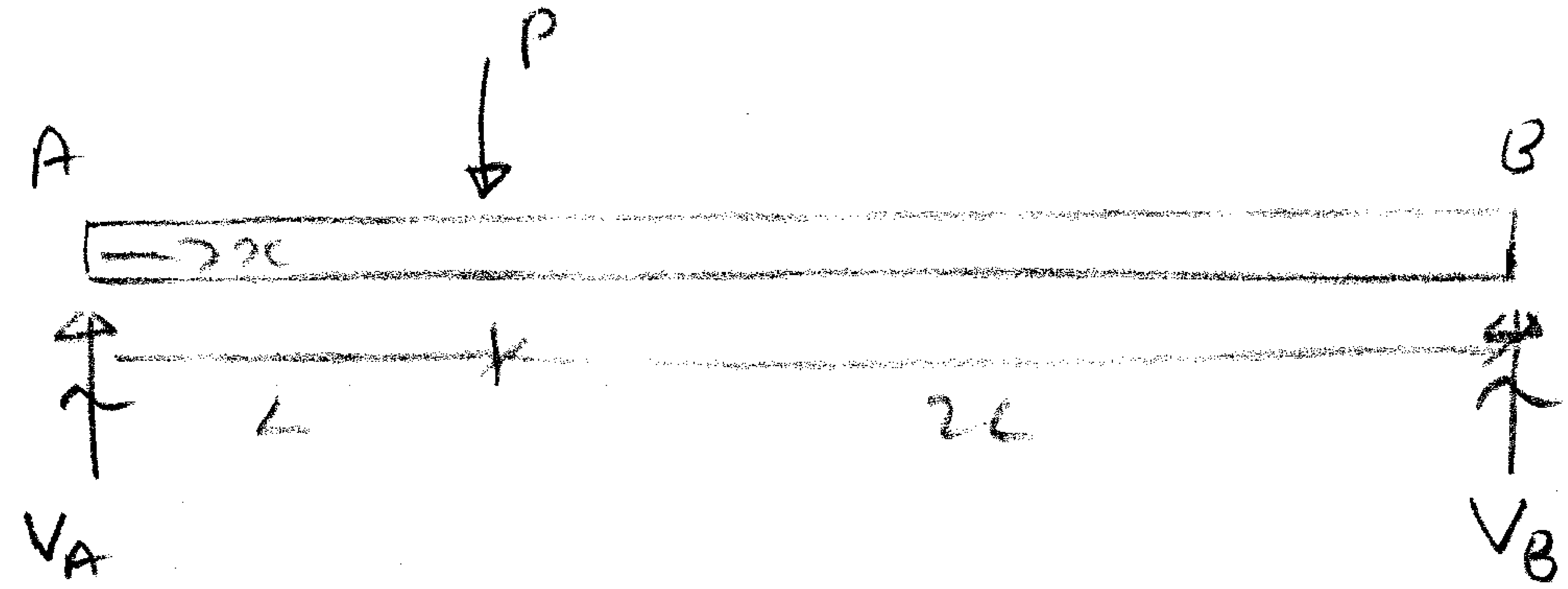
$$\delta = \frac{1}{EI}\left[\frac{-10^5}{240} + \frac{5.10^4}{24} - \frac{25\times10^3}{6} + \frac{83.10^2}{2}\right] = \frac{1650\times10^3}{EI}$$

$E = 70\times10^9$, $I = \frac{1}{12}\times50\times10^{-3}\times(100\times10^{-3})^3 = 4.17\times10^{-6}$

$$\delta = \frac{1650\times10^{3}}{70\times10^{9}\times 4.17\times10^{-6}} = 5.66 \text{ m.}$$

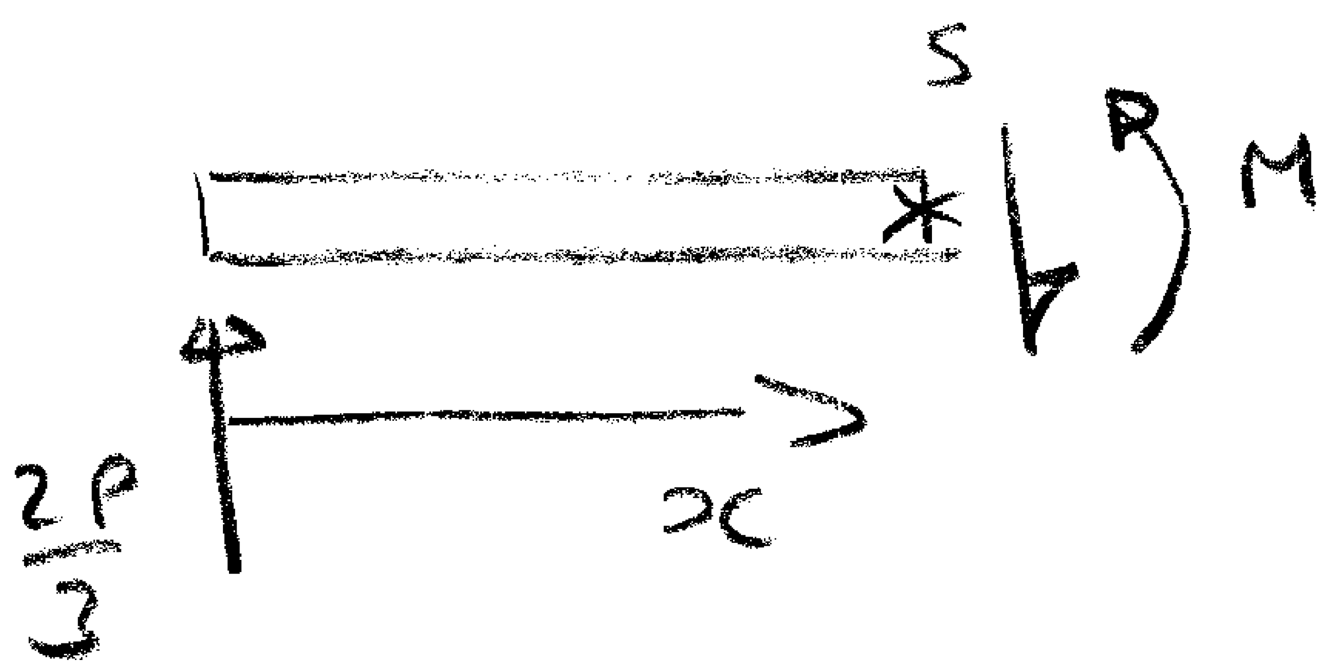
clearly rather large - not small
deflection

MS



$\Sigma M_A = 0 : \quad 3L V_B - PL = 0 \quad : \quad V_B = \frac{P}{3}, \quad V_A = \frac{2P}{3}$

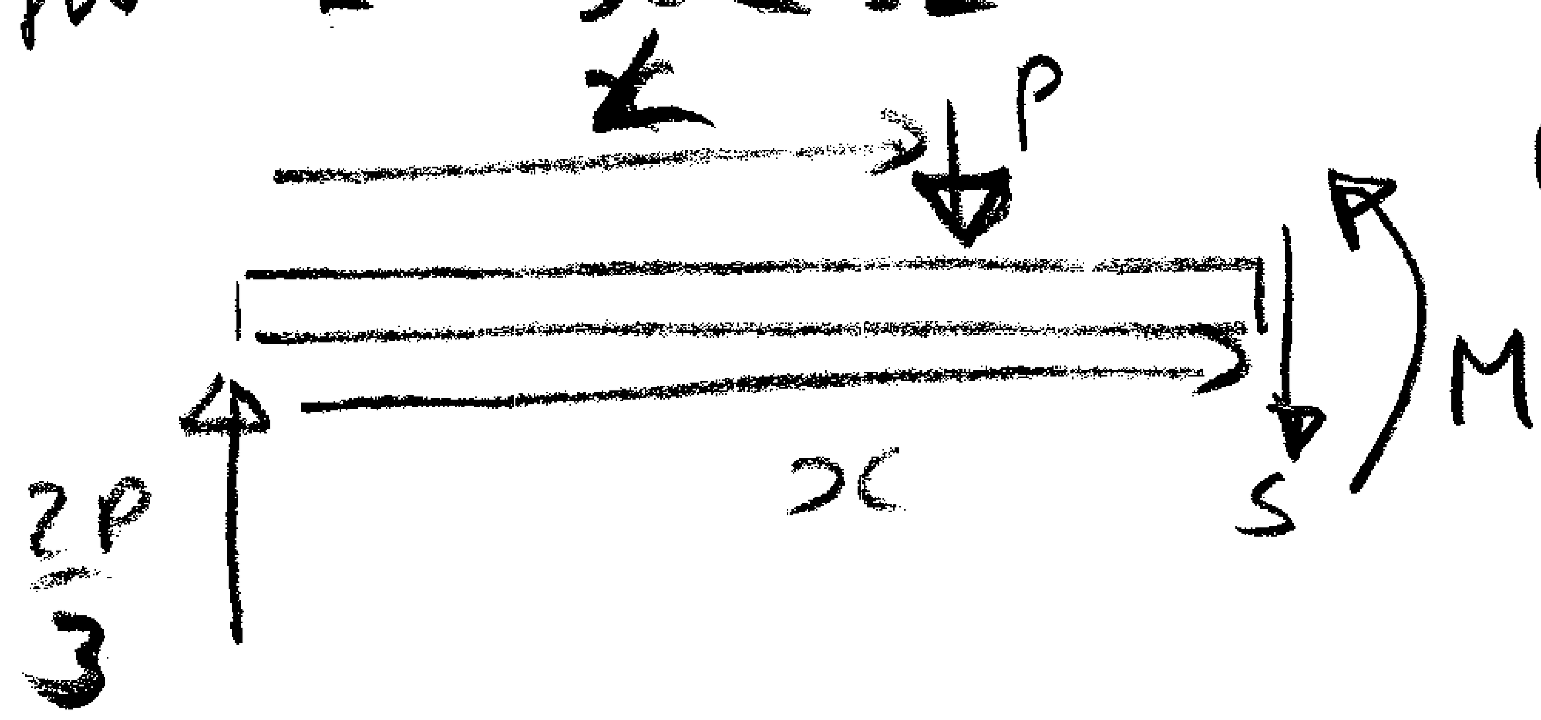
for $\quad 0 < x < L$



$\Sigma M_x = 0 : \quad M - \frac{2P}{3} x = 0$

$M = \frac{2Px}{3}$

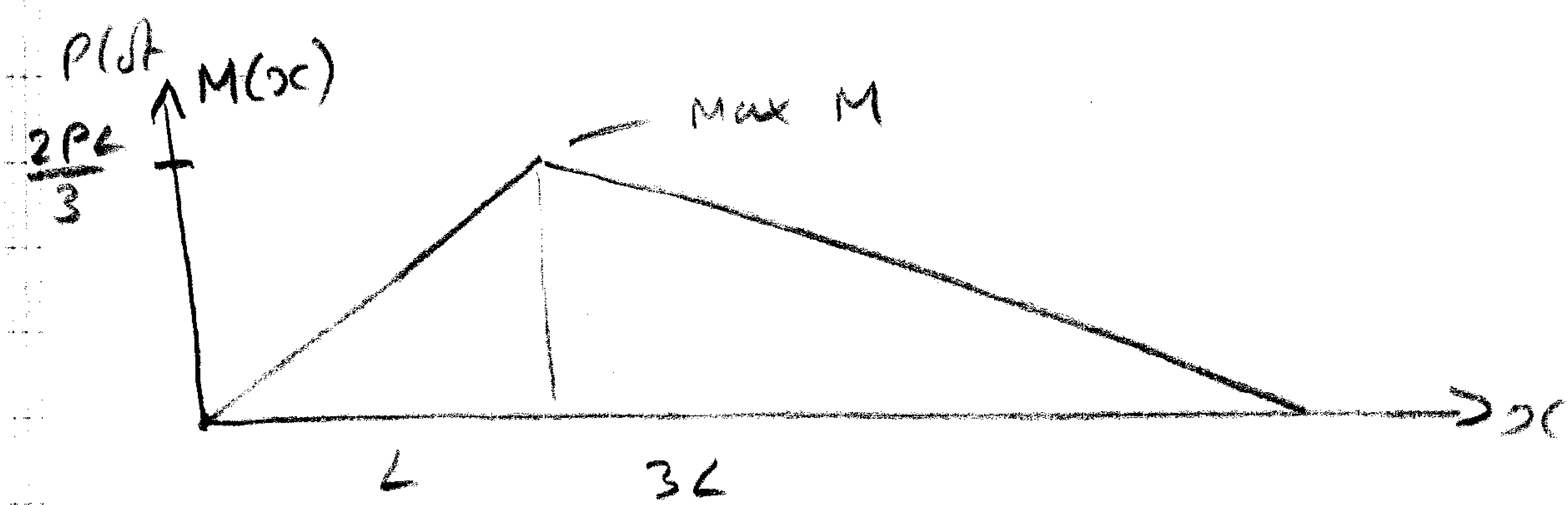
for $\quad L < x < 3L$



$\Sigma M_x = 0 : \quad M + P(x - L) - \frac{2P}{3} x = 0$

$M = -\frac{P}{3} x + PL$

Plot



deflection from: $EI \dfrac{d^2 w}{dx^2} = M$

① $0 < x < L$

$$EI \frac{d^2 w^1}{dx^2} = \frac{2Px}{3}$$

$$EI \frac{dw^1}{dx} = \frac{2Px^2}{6} + A$$

$$EI\, w^1 = \frac{2Px^3}{18} + Ax + B$$

② $L < x < 3L$

$$EI \frac{d^2 w^2}{dx^2} = -\frac{Px}{3} + PL$$

$$EI \frac{dw^2}{dx} = -\frac{Px^2}{6} + PLx + C$$

$$EI\, w^2 = -\frac{Px^3}{18} + \frac{PLx^2}{2} + Cx + D$$

Apply boundary conditions

@ $x = 0$ $\quad w^1 = 0$ $\qquad x = 3L \quad w^2 = 0$

$$\Rightarrow \quad B = 0, \qquad -\frac{27PL^3}{18} + \frac{9PL^3}{2} + 3C\cancel{L} + D = 0$$

$$= 3PL^3 + 3C\cancel{L} + D = 0 \quad ①$$

@ $x = L$

$$\frac{dw^1}{dx} = \frac{dw^2}{dx} \Rightarrow \frac{2PL^2}{6} + A = -\frac{PL^2}{6} + PL^2 + C$$

$$A - C = \frac{PL^2}{2} \quad ②$$

@ $x = L$

$$w^1 = w^2$$

$$\frac{2PL^3}{18} + AL = -\frac{PL^3}{18} + \frac{PL^3}{2} + CL + D$$

$$(A - C)L = \frac{PL^3}{3} + D \quad ③$$

substitute for $A - C = \frac{PL^2}{2}$ ②

$$\Rightarrow \frac{PL^3}{2} = \frac{PL^3}{3} + D \quad \Rightarrow D = \frac{PL^3}{6} \Leftarrow$$

substitute in ① $\quad 3PL^2 + 3C\cancel{L} + \frac{PL^2}{6} = 0$

$$C = \frac{1}{3}\left(-3PL^2 - \frac{PL^2}{6}\right) = -\frac{19}{18}PL^2 \Leftarrow$$

$$\therefore A = \frac{PL^2}{2} + \frac{19}{18}PL^2 = \frac{14}{9}PL^2 \Leftarrow$$

$\therefore$ for $0 < x < L$

$$w = \frac{1}{EI}\left(\frac{2Px^3}{18} + \frac{14PL^2}{9}x\right)$$

for $L < x < 3L$

$$w = \frac{1}{EI}\left(-\frac{Px^3}{18} + \right.$$

for $L < x < 3L$

$$w = \frac{1}{EI}\left(-\frac{Px^3}{18} + \frac{PLx^2}{2} - \frac{19PL^2x}{18} + \frac{PL^3}{6}\right) \checkmark$$

max deflection when $\frac{dw}{dx} = 0$

between $0 < x < L$?

$$\frac{dw}{dx} = \frac{1}{EI}\left(\frac{6Px^2}{18} + \frac{28PL^2x}{9}\right) = 0 \qquad \underline{\text{Not for } 0 < x < L}$$

between $L < x < 3L$ or $1 < \frac{x}{L} < 3$

$$\frac{dw}{dx} = \frac{PL^2}{EI}\left(-\frac{3\cancel{P}x^2}{18L^2} + \frac{\cancel{2P}Lx}{\cancel{x}L} - \frac{19}{18}\cancel{PL^2}\right) = 0$$

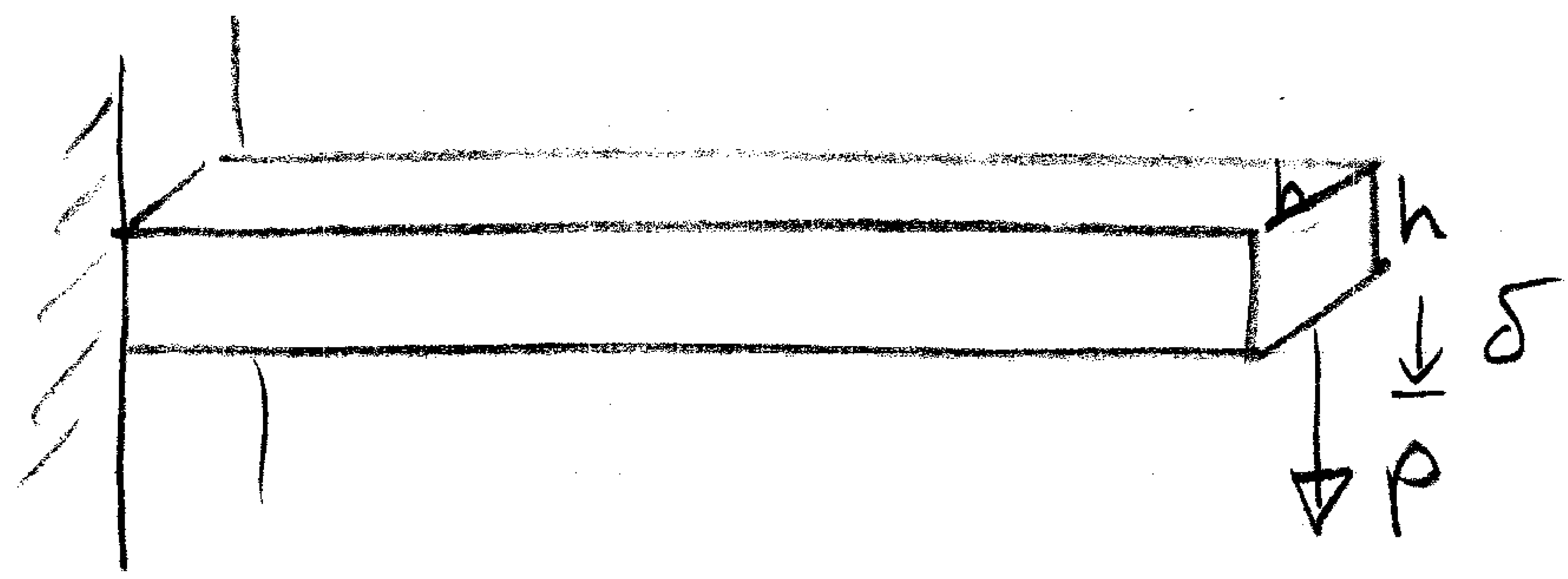
for $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4 \times \frac{1}{6} \times \frac{19}{18}}}{-\frac{2}{6}}$

Root between L and $3L$ = $x = 1.37L$

$$= \overset{w}{\delta} = \frac{PL^2}{EI}\left(\frac{-1}{5}\right) = -\frac{PL^2}{5EI}$$

Note this does not occur at $x = L$ where M = maximum.

M6



For cantilever beam $\delta = \frac{PL^3}{3EI}$, $\sigma_{max} = \frac{PL}{I}\frac{h}{2}$

$I = \frac{1}{12}h^4$

Mass of beam $= \underset{\text{density}}{\rho} LA = \rho L h^2 = M$

~~Max~~ Stiffness of beam $= \frac{P}{\delta} = \frac{3EI}{L^3} = \frac{3}{12}\frac{h^4 E}{L^3}$

eliminate h^2 $\quad h^2 = \frac{M}{\rho L}$

$\therefore$ stiffness $= \frac{P}{\delta} = \frac{1}{4}\left(\frac{M}{\rho L}\right)^2 E = K$

max stiffness for given mass (or vice-versa)

requires max $\frac{E}{\rho^2}$ $\Leftarrow$

For strength: $P_{max} = \dfrac{2\sigma_f I}{L h} = \dfrac{\sigma_f h^{\cancel{4}\,3}}{6L\cancel{h}}$

$$P_{max} = \frac{\sigma_f h^3}{6L}$$

mass of beam, $m = \rho h^2 L$

eliminate h

$$P_{max} = \frac{\sigma_f}{6L}\left(\frac{m}{\rho L}\right)^{3/2}$$

so max P_{max} for given mass requires

max $\dfrac{\sigma_f}{\rho^{3/2}}$ $\Leftarrow$

	E/GPa	σ_f/MPa	ρ/kg/m³	E/ρ^2	$\sigma_f/\rho^{3/2}$
steel	193	1435	7900	3092	2044
Al	71	350	2800	9056	2362
Ti	120	850	4500	5926	2815
CFRP	70	700	1500	31111	12050 $\Leftarrow$
Wood	12	50	600	33333	3402
Si	410	300	3000	45000 $\leftarrow$?	1826

c) a) SiC has the highest E/ρ^2, but may be impractical because of its low toughness

better choices might be wood or CFRP

b) CFRP has the highest $\sigma_f/\rho^{3/2}$

other factors - toughness, environmental durability, cost

d I & box sections are structurally efficient because they move material away from the neutral axis of the beam.

The key issue in selecting the shape of a beam is the ability to manufacture the shape in question.