Massachusetts Institute of Technology
Department of Aeronautics and Astronautics
Cambridge, MA 02139

Unified Engineering
Spring 2005

Problem Set #11
Solutions

PAL
4/13/05
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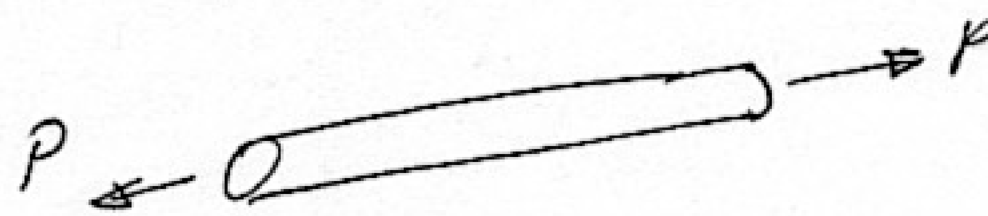
UNIFIED ENGINEERING

Problem Set #11 -- SOLUTIONS

M13. Spring modeled as a rod in uniaxial tension



(a) Total energy of "rod" (spring) in terms of stress and strain

$$u = \int_0^\epsilon \sigma \, \delta\epsilon \qquad \text{per } \underline{\text{unit}} \ \underline{\text{volume}}$$

stress prior to yielding (and assuming linear behavior:

$$\sigma = E\epsilon$$

Combine these two equations to get:

$$u = \int_0^\epsilon E\epsilon \, \delta\epsilon$$

$$\Rightarrow u = \tfrac{1}{2} E\epsilon^2 \Big]_0^\epsilon = \tfrac{1}{2} E\epsilon^2$$

Place this in terms of σ and E using the stress-strain relation in the form:

$$\epsilon = \frac{\sigma}{E}$$

This gives:

$$u = \frac{1}{2} E \left(\frac{\sigma}{E}\right)^2$$

$$\Rightarrow \boxed{u = \frac{1}{2} \frac{\sigma^2}{E}}$$ per unit volume

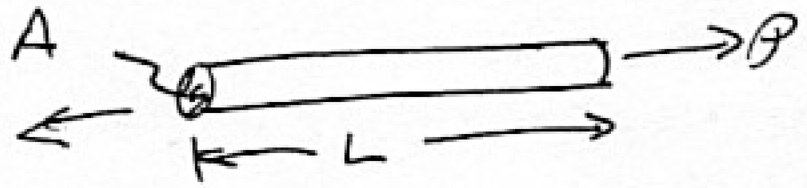
(b) Need to place the total energy in terms of the pertinent parameters

(i) For a given volume, from (a) we see it depend on σ and E. We maximize σ to the point of yielding (σ_y)

$$u = \frac{1}{2} \frac{\sigma_Y^2}{E}$$ per unit volume

For a rod, we know that:

$$\sigma = \frac{P}{A}$$



and volume = AL = constant

So for a given volume, $\boxed{\text{maximize } \frac{\sigma_Y^2}{E}}$

Note that $\frac{1}{2}$ is just a constant

(ii) for a given mass of material....
this means that the volume will change.
In (i) noted that volume = LA.
Knowing density ρ, then:

$$\text{Mass} = \rho L A = \text{Constant} \qquad (1)$$

To determine total energy, need:

(Energy per unit volume)(Volume) = Energy

Use equation (1) to get an expression for the volume, LA:

$$\text{Volume} = LA = \frac{\text{Constant}}{\rho}$$

Use with the expression for U per unit volume:

$$\Rightarrow \text{Energy} = \frac{1}{2}\frac{\sigma_Y^2}{E}\frac{\text{Constant}}{\rho}$$

disregarding the constants

$$\Rightarrow \boxed{\text{maximize } \frac{\sigma_Y^2}{E\rho}}$$

(iii) for a given cost of material....
this (again) means that the volume will change.
Operate with the material parameter of cost per mass, c. So:

total cost = (cost per mass)(mass)

with the total cost as a constant. We must also use an expression for mass involving material parameters as we did in (ii)

$$\text{Mass} = \rho L A$$

so:

$$\text{Cost} = (c)(\rho L A) = \text{constant}$$

we can thus, once again, get an expression for the volume in terms of pertinent material parameters:

$$\text{Volume} = LA = \frac{\text{constant}}{\rho c}$$

Use this in the expression for total energy:

$$\text{Energy} = \frac{1}{2}\frac{\sigma_Y^2}{E}\frac{\text{constant}}{\rho c}$$

Again, disregarding constants.

$$\Rightarrow \boxed{\text{maximize } \frac{\sigma_Y^2}{E\rho c}}$$

(c) Looking at these six materials, calculate the pertinent combinations of parameters (i.e. the criterion for each case) and compare:

Material	Given Volume: maximize (σ_Y^2/E) $[10^6\ Pa]$	Given Mass: maximize $(\sigma_Y^2/E\rho)$ $[Pa/(g/m^3)]$	Given Cost: maximize $(\sigma_Y^2/E\rho c)$
Al alloy	3.57	1.32	0.66
Spring steel	27.43	3.43	1.14
Rubber	18.0	20.0	15.4
Titanium	16.90	3.76	0.37
Nickel	18.69	2.10	0.49
Graphite/Epoxy	4.23	2.82	0.014

Note: Being clear and consistent on units is important.
Look for each case.

Volume
σ_Y^2/E : $\dfrac{[10^6\ Pa]^2}{[10^9\ Pa]} = [10^3\ Pa]$ extra 10^3 comes from numbers

mass
$\sigma_Y^2/E\rho$: Start from $\dfrac{\sigma_Y^2}{E}$ and add $\dfrac{1}{\rho}$

$$[10^6 Pa] \cdot \frac{1}{[10^6 g/m^3]} = [Pa/(g/m^3)]$$

could also go to: $[N/m^2/(g/m^3)] = \left[\dfrac{N\cdot m}{g}\right]$

Cost:
$\sigma_Y^2/E\rho c$: Start from $\sigma_Y^2/E\rho$ and add $\dfrac{1}{c}$

$$\left[Pa/(g/m^3)\right] \cdot \frac{1}{[\$/10^3 g]} = \left[\frac{10^3 Pa \cdot m^3}{\$}\right]$$

could also go to: $\left[\frac{10^3 N \cdot m}{\$}\right]$

Comments:

- For the volume criterion spring steel is the best material by at least 50%
- For the mass criterion, rubber is the best material by almost an order of magnitude
- For the cost criterion, rubber is the best material by ~~almost~~ an order of magnitude
- Overall, rubber is easily the best in two of the three criteria and is second (close) in the other, so it is the most likely choice overall.

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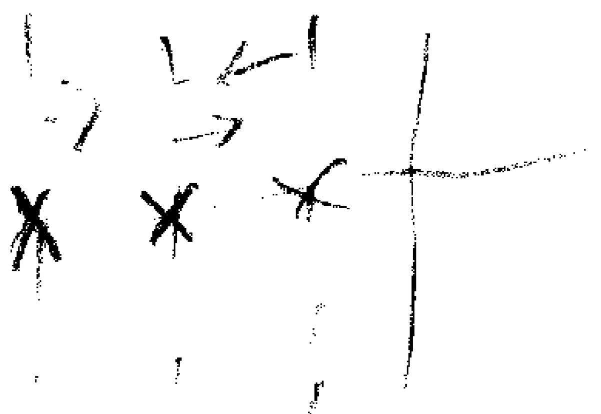
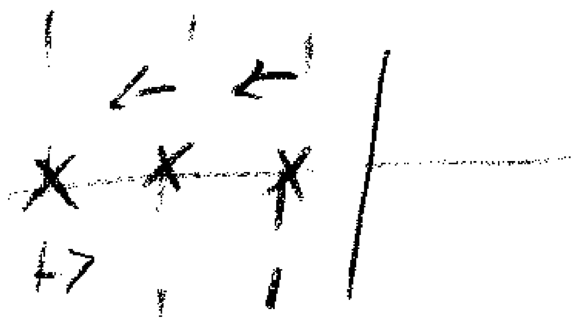
1. $e^{-2t}u(t) - 2e^{2t}u(-t) + 3\delta(t)$

2. $-e^{-2t}u(-t) - 2e^{2t}u(-t) + 3\delta(t)$

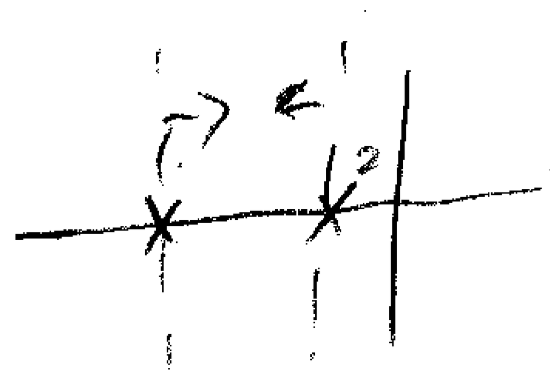
3. $e^{-3t}u(t) + 2e^{-2t}u(t) - 3e^{-t}u(-t)$



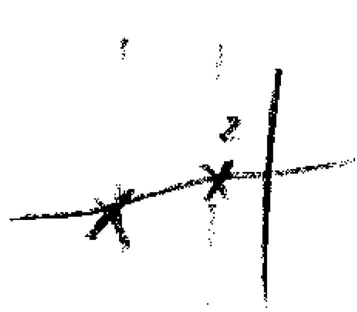
4. $e^{-3t}u(t) - 2e^{-2t}u(-t) - 3e^{-t}u(-t)$



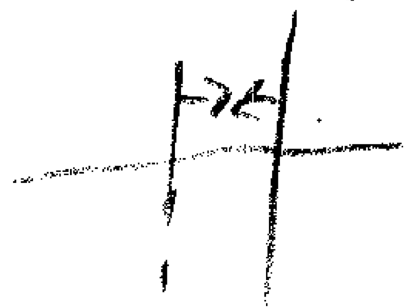
5. $3e^{-2t}u(t) - e^{-t}u(-t) - 2te^{-t}u(-t)$



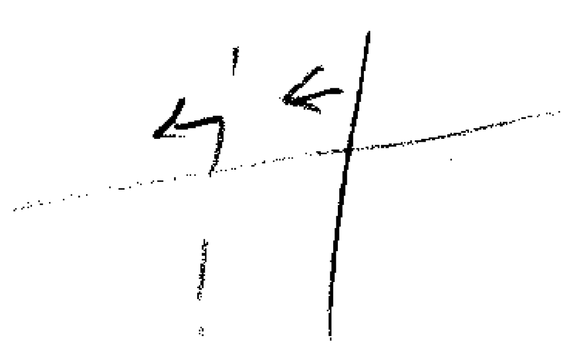
6. $-3e^{-2t}u(-t) - e^{-t}u(-t) - 2te^{-t}u(-t)$



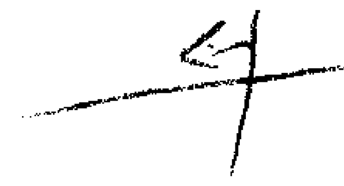
7. $-u(-t) - 2tu(-t) + 3e^{-t}u(t) + 4te^{-t}u(t)$



8. $-u(-t) - 2tu(-t) - 3e^{-t}u(-t) - 4te^{-t}u(-t)$



9. $-\cos(2t)u(-t) - \sin(3t)u(-t)$



Problem S15 Solutions

[redacted]

1.

$$G_1(j\omega) = \frac{-a^2}{(j\omega - a)(j\omega + a)}$$

2.

$$A_1(\omega) = \frac{a^2}{\sqrt{a^2+\omega^2}\sqrt{a^2+\omega^2}} = \frac{a^2}{(a^2+\omega^2)}$$

$$\phi_1(\omega) = -\tan^{-1}\left(-\frac{\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) = 0$$

3.

$$G_2(j\omega) = \frac{a^2}{(j\omega + a)^2}$$

$$A_2(\omega) = \frac{a^2}{(a^2+\omega^2)} = A_1(\omega)$$

[redacted] $\phi(\omega) = \tan^{-1}\left[-\frac{2a\omega}{a^2-\omega^2}\right]$

4.

Write $\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$

$$Y = |G_1(j\omega)|\frac{e^{j\omega t}}{2}\left(e^{j\phi_1(\omega)}\right) + |G_1(-j\omega)|\frac{e^{-j\omega t}}{2}\left(e^{-j\phi_1(\omega)}\right)$$

$$= A_1(\omega)\left(\frac{e^{j(\omega t + \phi_1(\omega))} + e^{-j(\omega t + \phi_1(\omega))}}{2}\right)$$

$$y_1(t) = A_1(\omega)\cos\left(\omega t + \phi_1(\omega)\right)$$

likewise for $y_2(t)$

5.

The filters both have the same effect on the magnitude of the input, $A_1(\omega) = A_2(\omega)$.

6.

The non-causal filter produces no phase shift, while the phase shift of the causal filter is between 0 and -180°, depending on ω.

7.

The non-causal filter produces no phase shift. Therefore, setting the input is easier and the waveform will arrive at the next stage on time. Signals with multiple frequency components would be jumbled due to the variance of pure phase shift at each frequency of the causal filter. The non-causal filter will scale each frequency but produce no phase shift, thereby making an effective multiple frequency low-pass filter.