Problem 1 (15%) ID number (last four digits) SOLUTION

A causal, LTI system, G, has impulse response $g(t)$. The Laplace transform of $g(t)$ is

$$G(s) = \frac{4}{s(s+2)^2}$$

1. What is the region of convergence of the Laplace transform? Explain.
2. Find $g(t)$.
3. Is the system BIBO stable or unstable?

1. Since G is causal, the R.O.C. is to the right of the rightmost pole, which is at $s=0$. Therefore, the R.O.C. is

$$\mathrm{Re}[s] > 0$$

2. Expand $G(s)$ by partial fraction expansion, using the coverup method:

$$G(s) = \frac{4}{s(s+2)^2} = \frac{1}{s} + \frac{a}{s+2} + \frac{-2}{(s+2)^2}$$

Note that a cannot be determined directly, since the pole at $s=-2$ is 2nd order. Having found the other terms, solve:

Problem 1 ID number (last four digits) SOLUTION

$$\frac{a}{s+2} = \frac{4}{s(s+2)^2} - \frac{1}{s} + \frac{2}{(s+2)^2}$$

$$= \frac{4-(s+2)^2+2s}{s(s+2)^2}$$

$$= \frac{-s^2-2s}{s(s+2)^2} = \frac{-1}{s+2}$$

Therefore,

$$g(t) = \left(1 - e^{-2t} - 2t\,e^{-2t}\right)\sigma(t)$$

3. Unstable, since the R.O.C. does <u>not</u> include $\mathrm{Re}[s] = 0$.

Problem 2 (10%) ID number (last four digits) SOLUTION

A causal, LTI system, G, has impulse response $g(t)$ given by

$$g(t) = \frac{1}{1+t}\sigma(t)$$

Is the system BIBO stable? Explain.

G is BIBO stable if and only if

$$M = \int_0^\infty |g(t)|\, dt < \infty$$

But

$$M = \int_0^\infty \frac{1}{1+t}\, dt = \ln(1+t)\Big|_0^\infty = \infty$$

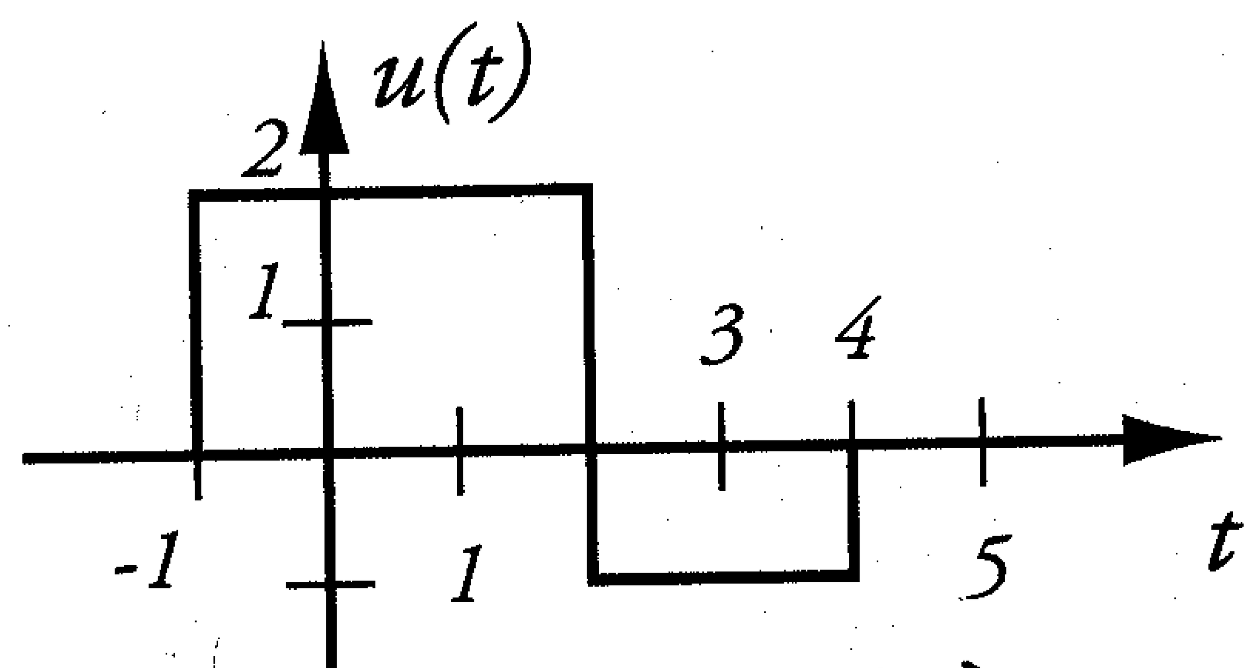
So G is <u>unstable</u>.

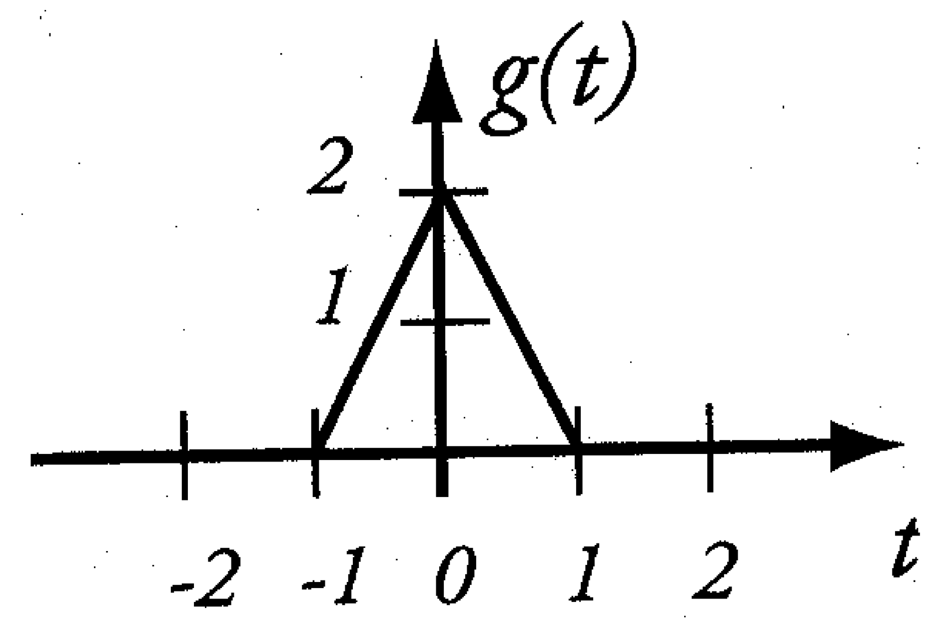
Problem 3 ID number (last four digits) SOLUTION

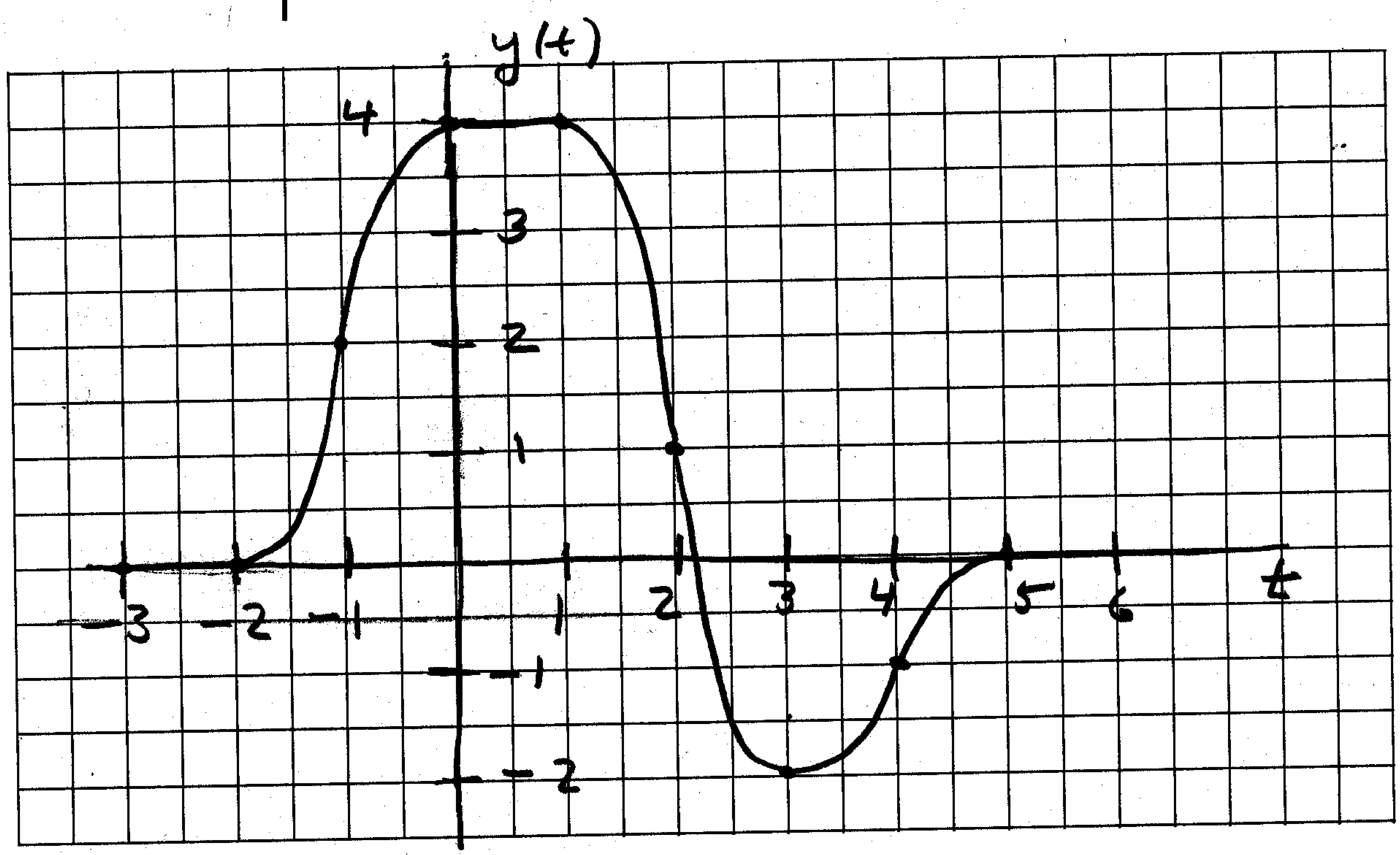
Given the signals $g(t)$ and $u(t)$ as plotted below, find the signal $y(t)$ given by

$$y(t) = g(t) * u(t)$$

Sketch the result in the grid below, as accurately as possible. Explain your reasoning on the page that follows. The grid squares *do not* have to represent 1 unit — you can chose the units as appropriate to plot the result. Be sure to label the axes of the grid.







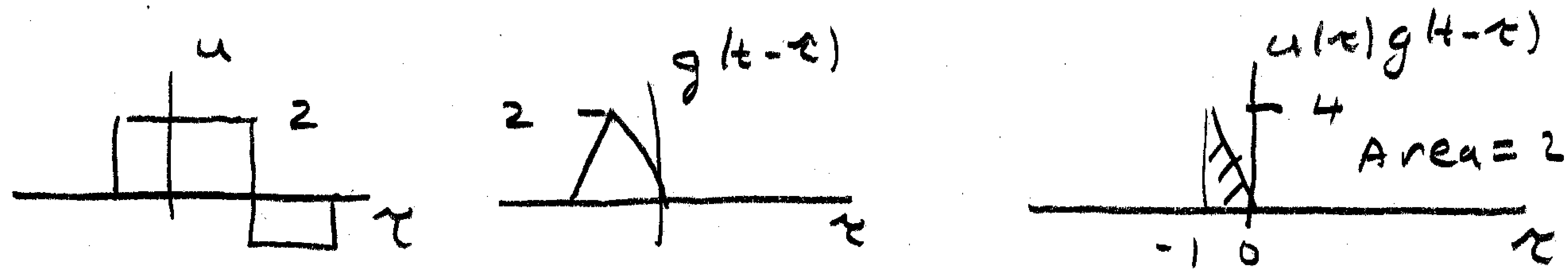
Problem 3 (25%) ID number (last four digits) SOLUTION

Use graphical convolution. Note that this problem is a little like the smoothing problem — $g(t)$ looks a little like the smoother from class, but has area 2.

Also, because u has step discontinuities only, and g is continuous, $y(t)$ will be continuous and have continuous slope. In fact, because u is piecewise constant, and g is piecewise linear, y will be piecewise quadratic, i.e., little segments of parbolas.

$t \le -2$: For $t \le -2$, $g(t-\tau)$ and $u(\tau)$ don't ovelap, so $y(t) = 0$.

$t = -1$:



$\Rightarrow y(-1) = 2$

Problem 3 ID number (last four digits) SOLUTION

This process can be continued for each t.
I got the following table:

t	$y(t)$
-2	0
-1	2
0	4
1	4
2	1
3	-2
4	-1
5	0

which, together with remarks above, gives $y(t)$ plotted on 1st page.

Problem 4 (25%) ID number (last four digits) SOLUTION

Consider an LTI system G with input signal $u(t)$ and output signal $y(t)$.

1. What is the definition of the transfer function, $G(s)$?
2. Explain why the transfer function is the Laplace transform of the impulse reponse.

1. For an LTI system, an exponential input $u(t) = e^{st}$ should produce an exponential output, $y(t) = G(s)\, e^{st}$. The amplitude of the output, $G(s)$, is the transfer function.

2. For an arbitrary input, $u(t)$, the response is

$$y(t) = g(t) * u(t) = u(t) * g(t)$$

For the exponential input,

$$y(t) = e^{st} * g(t)$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)} g(\tau)\, d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} e^{-s\tau} g(\tau)\, d\tau}_{\equiv \mathcal{L}[g(t)]}$$

Problem 4 ID number (last four digits) SOLUTION

So the output is

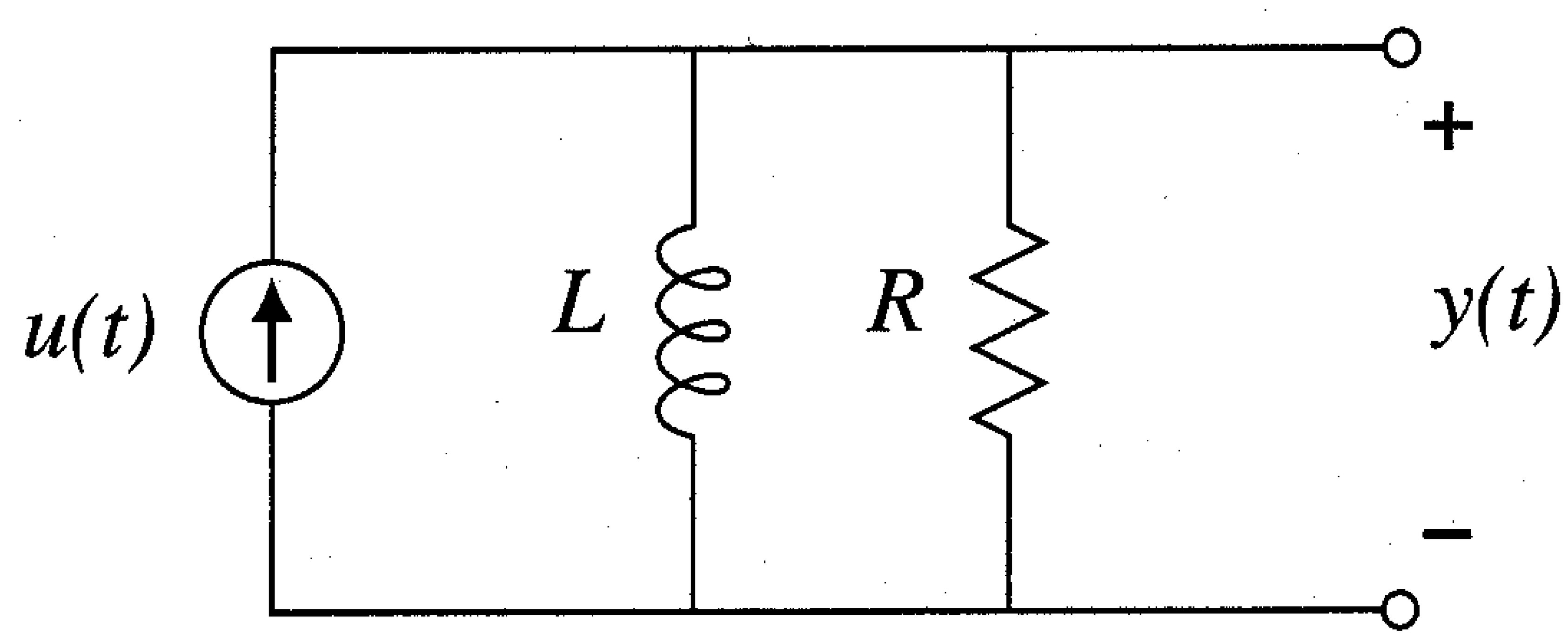
$$y(t) = \mathcal{L}[g(t)]\, e^{st}$$

Comparing this result and the definition, we must have that the transfer function is

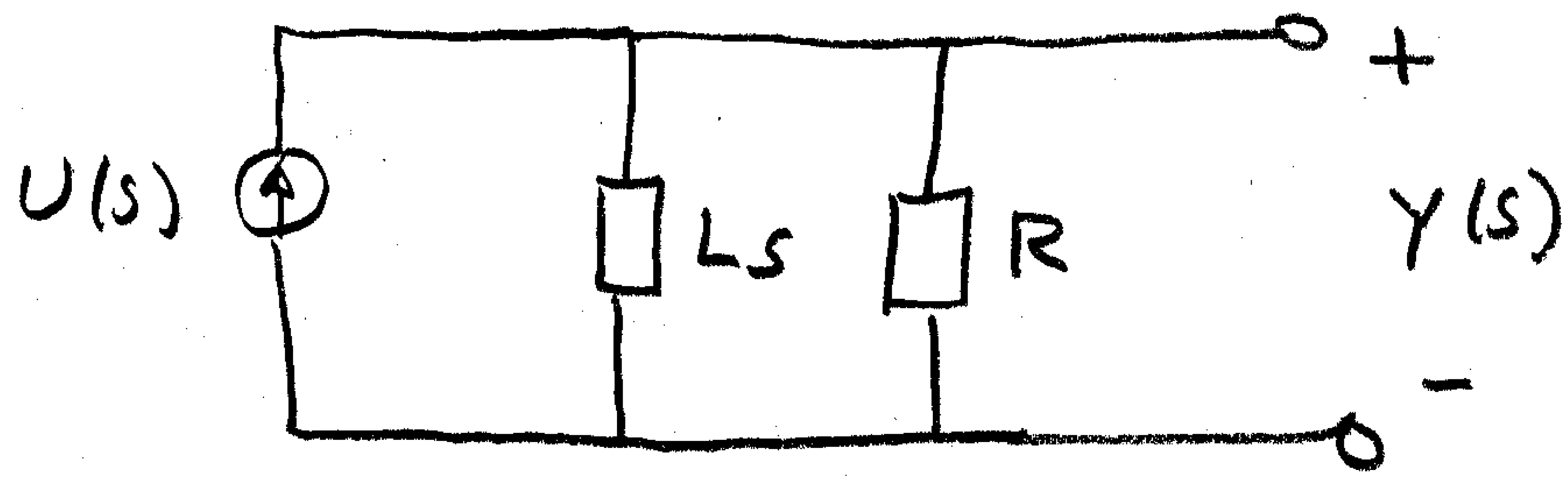
$$G(s) = \mathcal{L}[g(t)]$$

Problem 5 (25%) ID number (last four digits) SOLUTION

Find the step response of the circuit below. The component values are $R = 4\Omega$, $L = 2$ H.



Using impedance methods, we can reduce the circuit to



which we then treat as a static circuit. The inductor and resistor are in parallel, with equivalent impedance

$$Ls \| R = \frac{LsR}{Ls + R} = \frac{8s}{2s + 4} = \frac{4s}{s+2}$$

For this simple circuit, all the current U(s) goes through this single impedance, so

Problem 5 ID number (last four digits) SOLUTION

$$Y(s) = \frac{4s}{s+2} \, U(s)$$

(This is just $v = iR$). For a step input, $U(s) = \frac{1}{s}$, $Re(s) > 0$. Also, the circuit is causal, so the transfer function is valid for $Re(s) > -2$. Therefore,

$$Y(s) = \frac{4s}{s+2} \cdot \frac{1}{s} = \frac{4}{s+2}, \quad Re(s) > -2$$

(Note cancellation of factor s) Therefore,

$$y(t) = g_s(t) = 4e^{-2t} \sigma(t)$$