A causal, LTI system, $G$, has impulse response $g(t)$. The Laplace transform of $g(t)$ is

$$G(s) = \frac{4}{s(s + 2)^2}$$

1. What is the region of convergence of the Laplace transform? Explain.

2. Find $g(t)$.

3. Is the system BIBO stable or unstable?

1. Since $G$ is causal, the R.O.C. is to the right of the rightmost pole, which is at $s=0$. Therefore, the R.O.C. is \[ \text{Re}[s] > 0 \]

2. Expand $G(s)$ by partial fraction expansion, using the coverup method:

$$G(s) = \frac{4}{s(s+2)^2} = \frac{1}{s} + \frac{a}{s+2} + \frac{-2}{(s+2)^2}$$

Note that $a$ cannot be determined directly, since the pole at $s=-2$ is 2nd order. Having found the other terms, solve:
\[
\frac{a}{s+2} = \frac{4}{s(s+2)^2} - \frac{1}{s} + \frac{2}{(s+2)^2} \\
= \frac{4 - (s+2)^2 + 2s}{s(s+2)^2} \\
= \frac{-s^2 - 2s}{s(s+2)^2} = \frac{-1}{s+2}
\]

Therefore,

\[g(t) = \left(1 - e^{-2t} - 2t e^{-2t}\right) \delta(t)\]

3. Unstable, since the R.O.C. does not include \(\text{Re}[s] = 0\).
A causal, LTI system, $G$, has impulse response $g(t)$ given by

$$g(t) = \frac{1}{1+t} \sigma(t)$$

Is the system BIBO stable? Explain.

$G$ is BIBO stable if and only if

$$M = \int_0^\infty |g(t)| \, dt < \infty$$

But

$$M = \int_0^\infty \frac{1}{1+t} \, dt = \ln(1+t) \bigg|_0^\infty = \infty$$

so $G$ is unstable.
Problem 3

Given the signals \( g(t) \) and \( u(t) \) as plotted below, find the signal \( y(t) \) given by

\[
y(t) = g(t) * u(t)
\]

Sketch the result in the grid below, as accurately as possible. Explain your reasoning on the page that follows. The grid squares do not have to represent 1 unit — you can chose the units as appropriate to plot the result. Be sure to label the axes of the grid.
Use graphical convolution. Note that this problem is a little like the smoothing problem — \( g(t) \) looks a little like the smoother from class, but has area 2.

Also, because \( u \) has step discontinuities only, and \( g \) is continuous, \( y(t) \) will be continuous and have continuous slope. In fact, because \( u \) is piecewise constant and \( g \) is piecewise linear, \( y \) will be piecewise quadratic, i.e., little segments of parabolas.

\( t \leq -2 \): For \( t \leq -2 \), \( g(t - \tau) \) and \( u(\tau) \) don't overlap, so \( y(t) = 0 \).

\( t = -1 \):

\[
\begin{array}{c}
\text{Area} = 2
\end{array}
\]

\[ \Rightarrow y(-1) = 2 \]
This process can be continued for each \( t \). I got the following table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

Which, together with remarks above, gives \( y(t) \) plotted on 1st page.
Consider an LTI system $G$ with input signal $u(t)$ and output signal $y(t)$.

1. What is the definition of the transfer function, $G(s)$?

2. Explain why the transfer function is the Laplace transform of the impulse response.

1. For an LTI system, an exponential input $u(t) = e^{st}$ should produce an exponential output, $y(t) = G(s) e^{st}$. The amplitude of the output, $G(s)$, is the transfer function.

2. For an arbitrary input, $u(t)$, the response is

$$y(t) = g(t) * u(t) = u(t) * g(t)$$

For the exponential input,

$$y(t) = e^{st} * g(t)$$

$$= \int_{-\infty}^{\infty} e^{s(t-\tau)} g(\tau) \, d\tau$$

$$= e^{st} \left[ \int_{-\infty}^{\infty} e^{-s\tau} g(\tau) \, d\tau \right]$$

$$\equiv \mathcal{L}[g(t)]$$
So the output is

\[ y(t) = \mathcal{L}[g(t)] e^{st} \]

Comparing this result and the definition, we must have that the transfer function is

\[ G(s) = \mathcal{L}[g(t)] \]
Problem 5 (25%)  

Find the step response of the circuit below. The component values are \( R = 4 \Omega, \ L = 2 \text{ H} \).

Using impedance methods, we can reduce the circuit to

which we then treat as a static circuit. The inductor and resistor are in parallel, with equivalent impedance

\[
L_s/|R = \frac{L_s R}{L_s + R} = \frac{8s}{2s + 4} = \frac{4s}{s + 2}
\]

For this simple circuit, all the current \( U(s) \) goes through this single impedance, so
\[ Y(s) = \frac{4s}{s+2} U(s) \]

(This is just \( U = iR \). For a step input, \( U(s) = \frac{1}{s} \), \( \text{Re}(s) > 0 \). Also, the circuit is causal, so the transfer function is valid for \( \text{Re}(s) > -2 \). Therefore,)

\[ Y(s) = \frac{4s}{s+2} \cdot \frac{1}{s} = \frac{4}{s+2}, \quad \text{Re}(s) > -2 \]

(Note cancellation of factor \( s \)). Therefore,

\[ y(t) = y_s(t) = 4e^{-2t} \sigma(t) \]