9:00 - 10: 00
35-225

MIT ID# (last four digits) SOLUTIONS - TAL

Unified Quiz 7M

May 4, 2005

- Put your MIT ID# (last four digits) on each page of the exam.
- Read all questions carefully.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit <u>cannot</u> be given without intermediate results.
- Show the logical path of your work. Explain <u>clearly</u> your reasoning and what you are doing. *In some cases, the reasoning is worth as much (or more) than the actual answers.*
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units. Intermediate answers and final answers are not correct without the units.
- Report significant digits only.
- Box your final answers.
- ***Calculators, handwritten "crib sheets", and Unified Handout "CONCEPT REVIEW SHEET for Unified Q7M" allowed.***

EXAM SCORING

#1M (25%)	
#2M (35%)	
#3M (20%)	
#4M (20%)	
FINAL SCORE	

Unified Quiz Materials and Structures 7 (Q7M)
May 4, 2005

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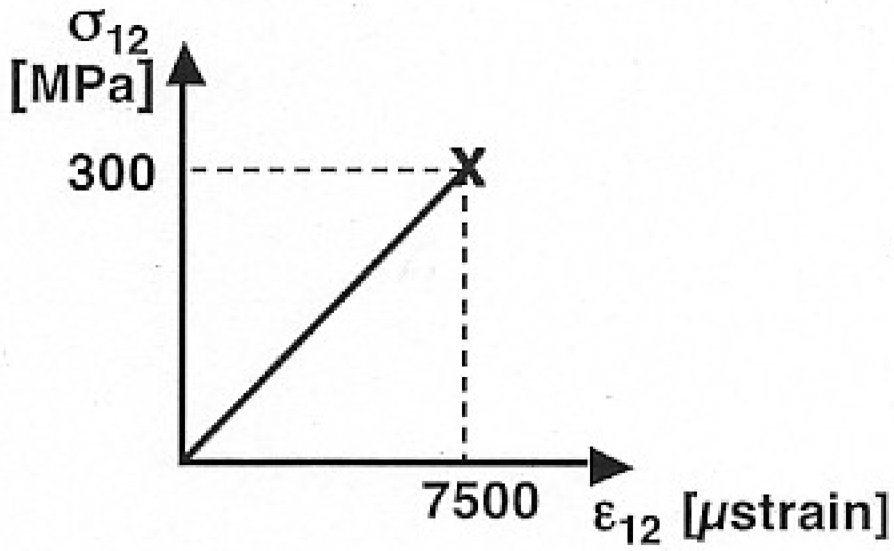
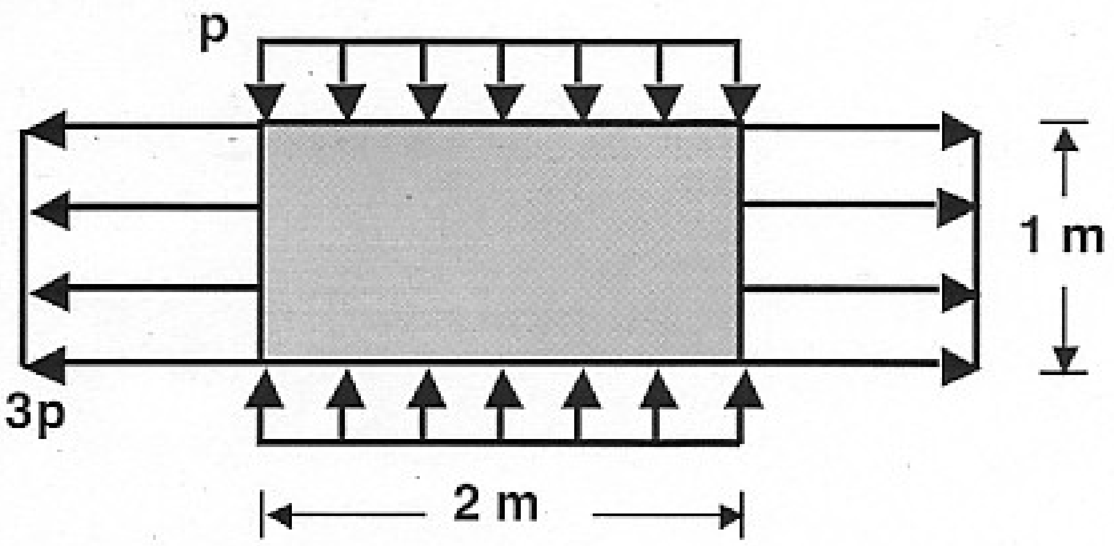
PROBLEM #1M (25%)

You are asked to consider the characteristics and failure of an unknown isotropic material. You are provided with stress-strain data determined from a torsional test on a rod of the material. The shear stress versus tensorial shear strain diagram that results from this test is shown below.



Use this data to determine the failure of a rectangular plate made of the same material. The plate is 2 meters long by 1 meter wide and 10 mm thick. The plate is loaded in a special machine that applies a line load (force per unit length) along each side. The line load is tensile with a magnitude of 3p along the width and is compressive with a magnitude of p along the length. Determine the value of the loading parameter p at which this plate will fail (under perfect load distribution conditions).



The point of failure in shear is 300 MPa

$\Rightarrow \tau_{yield} = \tau_{ult} = 300$ MPa

we also know that:

$\sigma_{ult} = 2\tau_{ult}$

$\Rightarrow \boxed{\sigma_{ult} = 600 \text{ MPa}}$

You can use either the Tresca or the von Mises criterion.

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PROBLEM #1M (continued)

For either case, need the principal stresses.
There are no applied shear stresses, so this is a principal stress state with $\sigma_{III} = 0$.
For the in-plane stresses, there is a line load [force/length] applied over a thickness. So:

$$\sigma = \frac{\text{line load}}{\text{thickness}}$$

thickness = 0.01m

$$\Rightarrow \sigma_I = 3p/0.01\,m = 300p\ [\tfrac{1}{m}]$$

$$\sigma_{II} = -p/0.01\,m = -100p\ [\tfrac{1}{m}]$$

Note: negative sign due to compression

For units, say all stresses are in [N/m²] and the line load is in [N/m], so drop the units based on this.

Tresca Criterion

$|\sigma_I - \sigma_{II}| = \sigma_{yield}$ or σ_{ult} $\Rightarrow |300p - (-100p)| = \sigma_{ult}$
$\Rightarrow 400p = 600 \times 10^6 \Rightarrow p = 1.5 \times 10^6$

$|\sigma_{II} - \sigma_{III}| = \sigma_y$ or σ_{ult} $\Rightarrow |-100p - 0| = \sigma_{ult}$
$\Rightarrow 100p = 600 \times 10^6 \Rightarrow p = 6 \times 10^6$

$|\sigma_{III} - \sigma_I| = \sigma_Y$ or σ_{ult} $\Rightarrow |0 - 300p| = \sigma_{ult}$
$\Rightarrow 300p = 600 \times 10^6 \Rightarrow p = 2 \times 10^6$

worst case is the first, so:

$$\boxed{p = 1.5 \times 10^6\ \frac{N}{m}}$$

PROBLEM #1M (continued)

If one used the von Mises Criterion

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\sigma_{ult}^2$$

$$\Rightarrow (300p - (-100p))^2 + (-100p - 0)^2 + (0 - 300p)^2 = 2\sigma_{ult}^2$$

$$(400p)^2 + (100p)^2 + (300p)^2 = 2\sigma_{ult}^2$$

gives:

$$(160{,}000 + 10{,}000 + 90{,}000)p^2 = 2\sigma_{ult}^2$$

$$\Rightarrow 130{,}000\, p^2 = (\sigma_{ult})^2$$

giving

$$p = \frac{\sigma_{ult}}{361}$$

using $\sigma_{ult} = 600 \times 10^6\ N/m^2$

finally:

$$\boxed{p = 1{,}662{,}000\ \frac{N}{m}}$$

$$= 1.662 \times 10^6\ \frac{N}{m}$$

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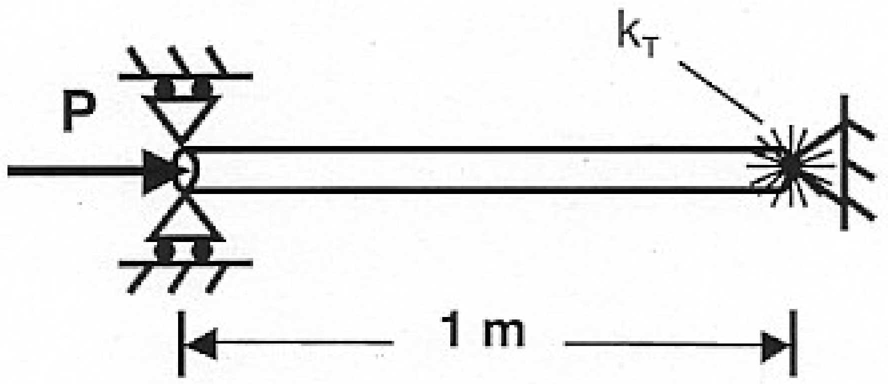
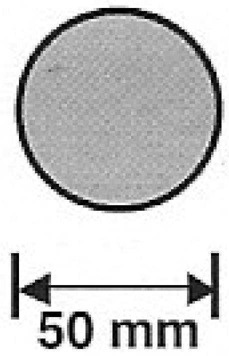
PROBLEM #2M (35%)

A component of a drive system of a piece of international heavy machinery has a circular cross-section. This piece can be modeled as a component that is connected to a roller support at one end, where the load is applied, and is attached via a torsional spring of stiffness k_T at the other end. The component is sized so that it will not yield or crush and is 1 meter long with a diameter of 50 mm. The component is made of machinery-grade steel with a modulus of 200 GPa.



Cross-Section



(a) Set up the equation(s) needed to determine the response of this component assuming that manufacturing, alignment, and loading are "perfect". This includes any deformation prior to instability. Describe how you would use the resulting equation(s) to determine the response but **DO NOT SOLVE**. As much as possible, describe the *nature* of the solution that results. Use figures if/as appropriate.

Start with the general solution to the governing relationship for out-of-plane deflection (u_3) of a column:

$$u_3 = A \sin\sqrt{\frac{P}{EI}}\, x_1 + B\cos\sqrt{\frac{P}{EI}}\, x_1 + C + Dx_1$$

where x_1 is the dimension along the column. Place $x_1 = 0$ at the torsional spring.

To find the constants and the eigenvalue (P_{cr}) we need to use the boundary conditions.

For the case of a torsional spring:

@ $x_1 = 0$, $u_3 = 0$ (a)

$$M = -k_T \frac{du_3}{dx_1} \Rightarrow -k_T \frac{du_3}{dx_1} = EI\frac{d^2u_3}{dx_1^2} \quad (b)$$

For a roller support:

@ $x_1 = L = 1\text{ m}$, $u_3 = 0$ (c)

$$M = 0 \Rightarrow EI\frac{d^2u_3}{dx_1^2} = 0 \quad (d)$$

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PROBLEM #2M (continued)

Now find the derivatives of the general solution so that the B.C.'s can be used to setup the equations to be solved.

Note: for ease of writing use $\lambda = \sqrt{\frac{P}{EI}}$

So: $u_3 = A \sin \lambda x_1 + B \cos \lambda x_1 + C + D x_1$

$$\frac{du_3}{dx_1} = A\lambda \cos \lambda x_1 - B\lambda \sin \lambda x_1 + D$$

$$\frac{d^2 u_3}{dx_1^2} = -A\lambda^2 \sin \lambda x_1 - B\lambda^2 \cos \lambda x_1$$

Apply the B.C.'s, one at a time:

(a) $\Rightarrow B + C = 0$

(b) $\Rightarrow -k_T (A\lambda + D) = EI(-B\lambda^2) \Rightarrow B = \frac{k_T}{\lambda^2 EI}(A\lambda + D)$

$\Rightarrow B = \frac{k_T}{P}(A\lambda + D)$

or: $A \frac{k_T \lambda}{P} - B + D\frac{k_T \lambda}{P} = 0$

(c) $\Rightarrow A \sin \lambda L + B \cos \lambda L + DL = 0$

(d) $\Rightarrow -A\lambda^2 \sin \lambda L - B\lambda^2 \cos \lambda L = 0$

gives: $-A \sin \lambda L - B \cos \lambda L = 0$

Assembling this:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ \frac{k_T \lambda}{P} & -1 & 0 & \frac{k_T}{P} \\ \sin \lambda L & \cos \lambda L & 0 & 1 \\ \sin \lambda L & \cos \lambda L & 0 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0$$

Solve this to find the eigenvalues (lowest = P_{cr}) and the associated eigenmodes (buckling shape/deflection)

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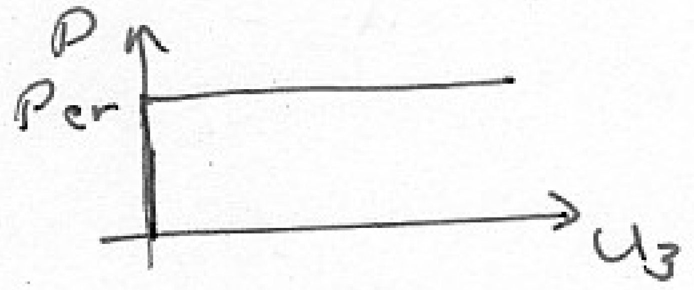
PROBLEM #2M (continued)

Prior to P_{cr} for a perfect case the only deflection is along the load (u_1) as $u_3 = 0$. Thus:

$$u_1 = \frac{P x_1}{AE}$$

So total motion at tip with $x_1 = L$ is $\boxed{u_1 = \frac{PL}{AE}}$

Out of plane deflection is:



and $I = \frac{\pi R^2}{4} = \frac{\pi (50 mm)^2}{4}$

$A = \pi R^2 = \pi (50 mm)^2$

$L = 1 m$

$E = 200 GPa$

(b) **Describe** the differences in the approach to determine the response when imperfections must be considered. Set up any needed equations but **DO NOT SOLVE**. As before, as much as possible, describe the *nature* of the solutions that result. Use figures if/as appropriate. In this overall description, do not use numbers, but only describe in a generic sense.

For the imperfect case, use the same basic solution to work from. However, the boundary conditions change at $x = L$ due to this eccentricity (e.g. load off-line). So, if the load is applied off the centerline by an eccentricity, e then still have $u_3 = 0$, but now:

$$M = EI \frac{d^2 u_3}{dx_1^2} = -Pe \quad \text{(a finite value)}$$

The first three boundary conditions are the same, so the resulting equations from (a), (b), and (c) do not change. However, condition (d) becomes:

$$-A\lambda^2 \sin \lambda L - B\lambda^2 \cos \lambda L = -Pe$$

So the overall set of equations become:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ k_T\lambda/P & -1 & 0 & k_T/P \\ \sin\lambda L & \cos\lambda L & 0 & 1 \\ \lambda^2 \sin\lambda L & \lambda^2 \cos\lambda L & 0 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ Pe \end{Bmatrix}$$

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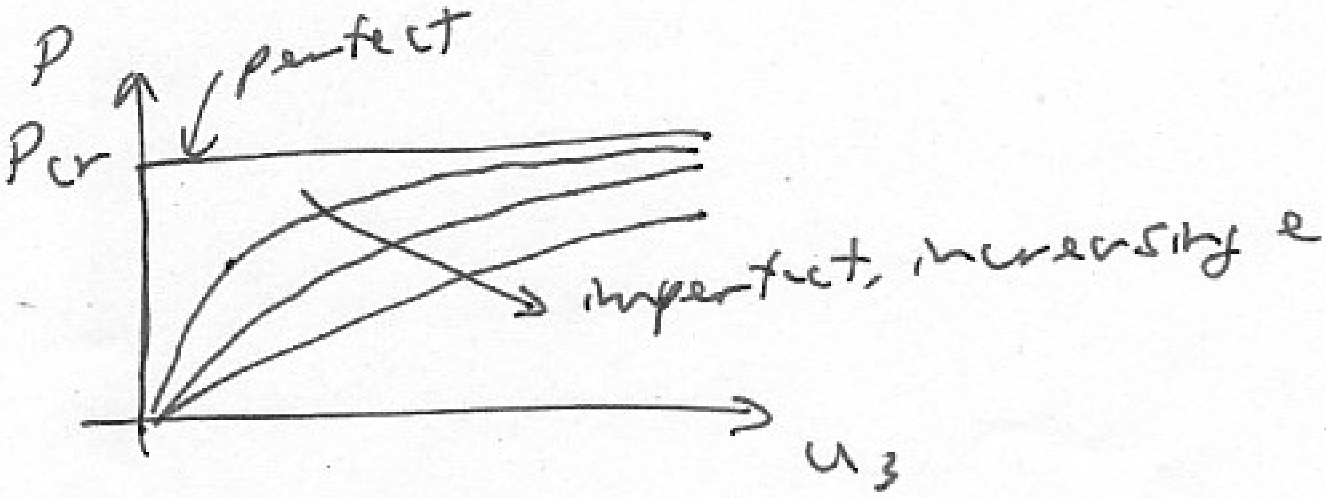
PROBLEM #2M (continued)

So the homogeneous solution and thus P_{cr} are still valid. But now there is also a particular solution.

Overall this looks like:



All else is similar

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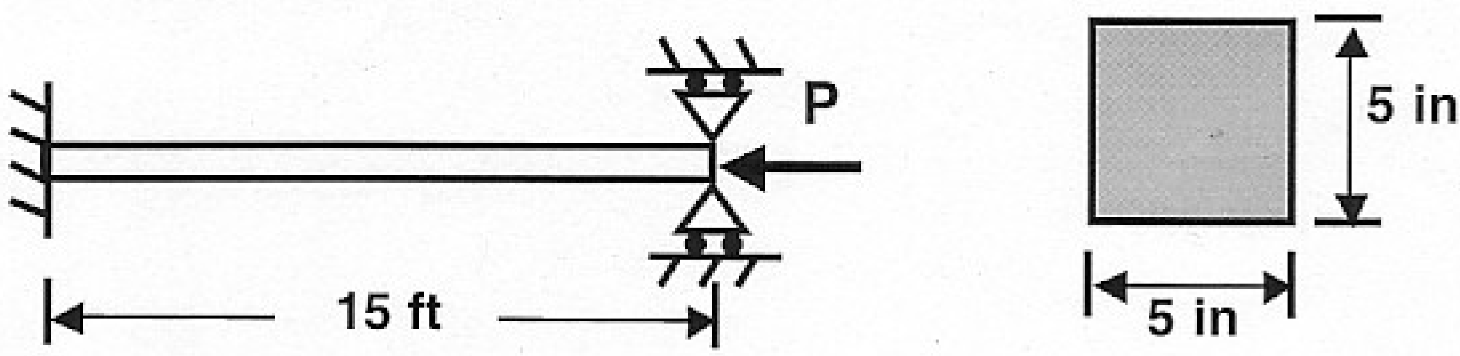
PROBLEM #3M (20%)

A structural component of a bridge is loaded in compression along its longitudinal axis. The component is ~~pinned~~ clamped at one end and has a roller support at the other end where the compressive load is applied. The component is 15 feet long, has a square cross-section that is 5 inches to a side, and is made of steel (E = 30 Msi, ν = 0.3, σ_{ult} = 200 ksi).

Cross-Section



A replacement for such components is being considered to be made out of aluminum (E = 10 Msi, ν = 0.3, σ_{ult} = 50 ksi) due to corrosion concerns and resulting maintenance and lifetime issues. Determine the cross-sectional dimension of the square component needed to mimic the behavior exhibited by the steel component. Include both perfect and imperfect considerations.

The most important structural parameter for a column is bending stiffness = EI

This is true whether the loading is perfect or imperfect.

So to mimic the behavior of the steel case, EI must be a constant:

$$(EI)_{steel} = (EI)_{Al}$$

For a square cross-section: $I = \frac{a^4}{12}$

$$\text{So: } (30 \times 10^6 \text{ psi})\left(\frac{(5 \text{ in})^4}{12}\right) = (10 \times 10^6 \text{ psi}) \frac{a_{Al}^4}{12}$$

$$\Rightarrow 3\,(5 \text{ in})^4 = a_{Al}^4$$

$$\Rightarrow \boxed{a_{Al} = 6.6 \text{ in}}$$

BUT, one must check that squashing doesn't occur first. If it does, it is the area times the ultimate stress that must stay constant

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PROBLEM #3M (continued)

The basic buckling load equation is:

$$P_{cr} = c\frac{\pi^2 EI}{l^2}$$

Use a higher estimate for the coefficient of edge fixity, c. If the stress (σ_{cr}) associated with the buckling load is less than σ_{ult}, then buckling will occur even for lower values of c prior to squashing.

Approximate the clamped-clamped case with c = 4:

$$P_{cr} = \frac{4\pi^2 EI}{(15\,ft \times 12\,in/ft)^2}$$

for steel:
$$P_{cr} = \frac{4\pi^2 (30\times10^6 \frac{lbs}{in^2})\left(\frac{(5in)^4}{12}\right)}{(15\,ft \times 12\,in/ft)^2}$$

$$\Rightarrow P_{cr} = 1.90\times10^6\,lbs$$

$$\sigma_{cr} = \frac{P_{cr}}{Area} = \frac{1.90\times10^6\,lbs}{25\,in} = 76\,ksi$$
less than $\sigma_{ult} = 200\,ksi$

for aluminum:
$$P_{cr} = \frac{4\pi^2 (10\times10^6 \frac{lbs}{in^2})\left(\frac{(6.6in)^4}{12}\right)}{(15\times12\,in/ft)^2}$$

$$\Rightarrow P_{cr} = 1.91\times10^6\,lbs$$ (the same since EI the same)

$$\sigma_{cr} = \frac{P_{cr}}{Area} = \frac{1.91\times10^6\,lbs}{(6.6in)^2} = 44\,ksi$$
less than $\sigma_{ult} = 50\,ksi$

So failure by buckling $\boxed{a_{Al} = 6.6\,in}$

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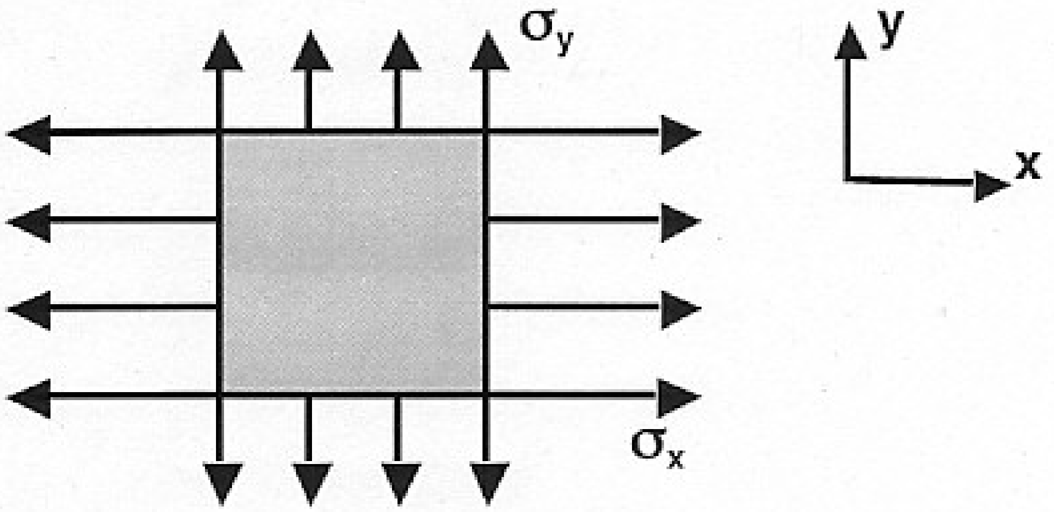
PROBLEM #4M (20%)

An aircraft structure is to be designed using either the basic strength approach or the damage tolerance approach. The component is loaded such that analysis shows that the material is subjected to a biaxial stress state with the stress in the x-direction being twice the stress in the y-direction. There is no shear stress. Titanium is being considered for this piece. The particular titanium has a modulus of 16.4 Msi, a Poisson's ratio of 0.31, a value of the tensile ultimate strength of 135 ksi, and a value of fracture toughness of 59 ksi(in)$^{1/2}$.



Using the numbers given as the design values, determine the size of the crack that must be tolerated via the damage tolerant approach, in order to design for the same loading ability as via the basic strength approach.

In applying both approaches, consider the ultimate load condition. For the basic strength approach, use the von Mises criterion. For the damage tolerance approach, only consider the stress perpendicular to the crack and assume that the geometric factor associated with the assumed crack configuration is equal to 1.

Begin with the von Mises criterion for Basic Strength Approach

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\sigma_{ult}^2$$

Here there is a biaxial stress state so this is a principal stress state. Indicate:

$$\sigma_I = 2\sigma_{II} \quad \text{and} \quad \sigma_{III} = 0$$

Put this in the criterion and solve for the σ_I ($=\sigma_x$) where failure occurs:

$$(\sigma_I - \tfrac{1}{2}\sigma_I)^2 + (\tfrac{1}{2}\sigma_I - 0)^2 + (0 - \sigma_I)^2 = 2\sigma_{ult}^2$$

$$\Rightarrow \quad \tfrac{1}{4}\sigma_I^2 + \tfrac{1}{4}\sigma_I^2 + \sigma_I^2 = 2\sigma_{ult}^2$$

gives: $\tfrac{3}{4}\sigma_I^2 = \sigma_{ult}^2$

with $\sigma_{ult} = 135 \times 10^3$ psi

PROBLEM #4M (continued)

$\sigma_I = \sqrt{4/3}\,(135 \times 10^3 \text{ psi}) \doteq 156 \times 10^3 \text{ psi}$

So failure occurs when $\sigma_x = 156$ ksi
(and $\sigma_y = 78$ ksi)

To use the <u>damage tolerant approach</u>, the applicable equation is:

$$\lambda \sigma \sqrt{\pi a} = K$$

Here $\lambda = 1$ and at fracture $K = K_c = 59 \text{ ksi} (\text{in})^{1/2}$

It is indicated that fracture occurs for this approach ~~by~~ considering the stress perpendicular to the crack.
A crack can be in any direction, so we must determine the maximum normal stress that can arise.
This is a principal stress state, so the maximum stress (σ_I) is the maximum normal stress that can arise.
Via the basic strength approach, failure occurs when $\sigma_x = 156$ ksi.
Using that in the damage tolerance equation:

$$156 \text{ ksi} \sqrt{\pi a} = 59 \text{ ksi} (\text{in})^{1/2}$$

$$\Rightarrow \pi a = \left(\frac{59}{156}\right)^2 \text{ in}$$

So: $a = \frac{1}{\pi}\left(\frac{59}{156}\right)^2$ in

$$\Rightarrow \boxed{2a = 0.092 \text{ in}}$$

Size of crack that can be tolerated