Unit M4.2
Rods: Stresses and Deflections

Readings:

CDL 3.1

16.003/004 -- "Unified Engineering"
Department of Aeronautics and Astronautics
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LEARNING OBJECTIVES FOR UNIT **M4.2**

Through participation in the lectures, recitations, and work associated with Unit M4.2, it is intended that you will be able to………

- ….**describe** the key aspects composing the model of a rod (bar) and **identify** the associated limitations
- ….**apply** the basic equations of elasticity to **derive** the solution for the general case
- ….**explain** St. Venant's Principle and **apply** it to structural configurations
- ….**analyze** more complex structures (such as a truss) using the basic model of a rod (bar)

We have looked at the rod at various times in the first term. But let's pull that together and present it as one "package".

We start off with definitions…

Definition of a rod

Let's look at how a rod is defined:

"A rod (or bar) is a structural member which is long and slender and is capable of carrying loads along its axis via elongation."

Note: elongation can be positive (tension) or negative (compression)

These "definitions" are basically "assumptions" that allow us to model a structural member -- in this case as a rod (sometimes also known as a bar).

Modeling Assumptions

These flow from the definition

a) Geometry

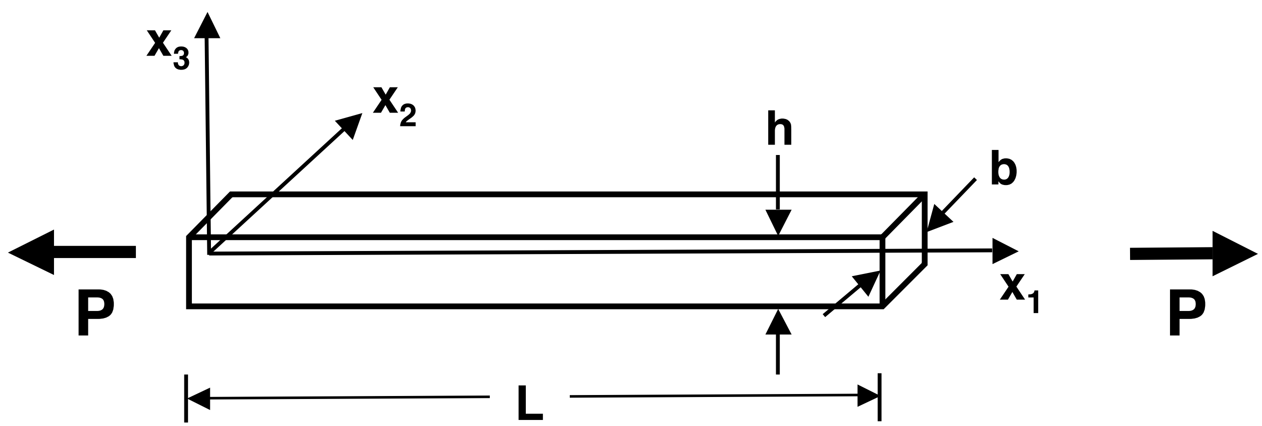
Normal Convention:

- L = length (x_1 - dimension)
- b = width (x_2 - dimension)
- h = thickness (x_3 - dimension)

Assumption: "long" in x_1 - direction

$\Rightarrow$ L >> b, h (slender member)

Figure M4.2-1 **Illustration of geometry of a rod/bar**



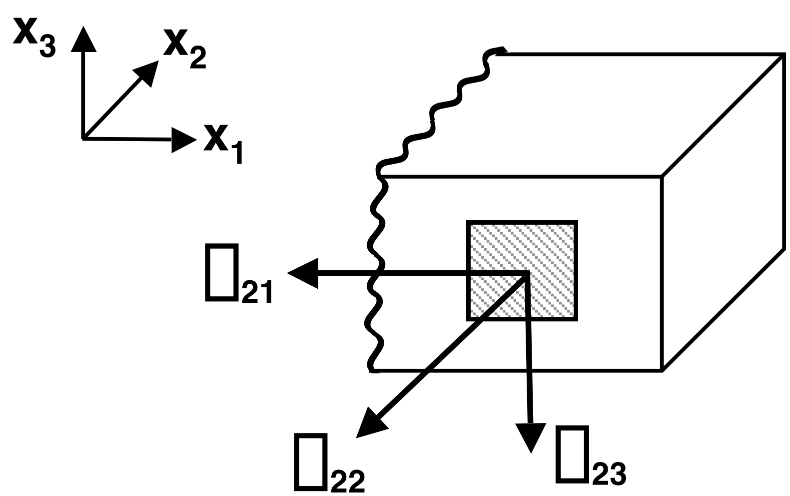
b) Loading

Assumption: Loaded in x_1 - direction only:

This results in a number of assumptions in the boundary conditions and values of the stresses

--> Consider the 2 (y) face $(x_2 = \pm\ b/2)$

Figure M4.2-2 **x_2 (y) face of bar**



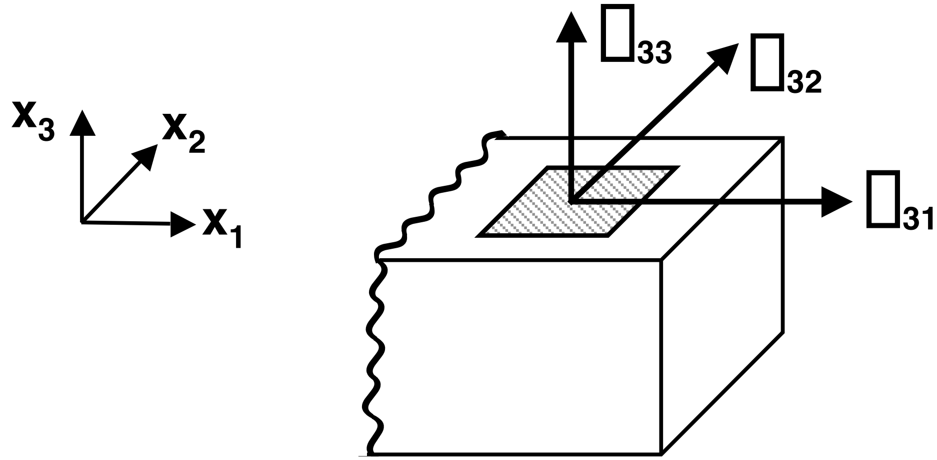
No forces => $\sigma_{21} = 0$

$\sigma_{22} = 0$

$\sigma_{23} = 0$

--> Consider the 3 (z) face $(x_3 = \pm h/2)$

Figure M4.2-3 **x_3 (z) face of bar**



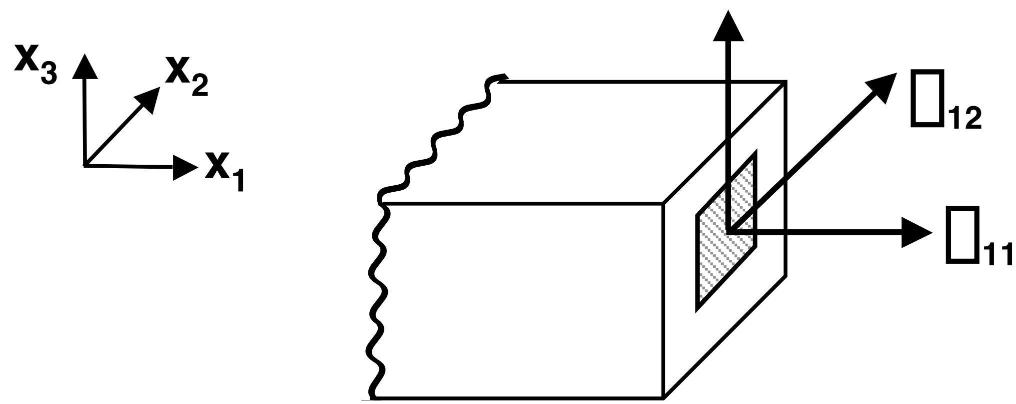
No forces => $\sigma_{31} = 0$

$\sigma_{32} = 0$

$\sigma_{33} = 0$

Consider the 1 (x) - face $(x_1 = 0, L)$

Figure M4.2-4 **x_3 (x) - face of bar**



The one force is P which is in the x_1-direction => $\sigma_{12} = 0$

$\sigma_{13} = 0$

and: $$\iint \sigma_{11}\, dA = P$$

with: $$dA = dx_2\, dx_3$$

Assumption: there are no variations in x_2 and x_3:

$$\Rightarrow \quad \frac{\partial \sigma_{11}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{11}}{\partial x_3} = 0$$

This means the cross-section behaves as a "unit" (assumption on deformation upcoming)

Thus $$\boxed{\sigma_{11} = \frac{P}{A}} \qquad (1)$$

where:

A = cross - sectional area

[Note: for next level of model, A can be function of x_1, e.g., $A(x_1) \Rightarrow$ tapered rod (no longer 1-D)]

A = bh for constant cross-section case

Finally look at

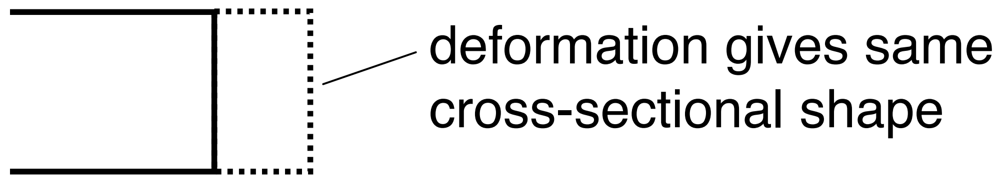
c) Deformation

Assumption: rod cross-section deforms uniformly:

$u_1 = u_1(x_1) \Rightarrow$ only a function of x_1

is this justified? yes, no shear stresses $\Rightarrow$ no changes in angle

Figure M4.2-5 **Assumed deformation for rod**



Now that we've <u>defined</u> the rod (i.e., made the modeling assumptions), we need the

<u>Governing Equations</u>

We always go back to our 15 equation of elasticity.

--> <u>Equilibrium</u> <u>Equations</u>

$$\frac{\partial \sigma_{mn}}{\partial x_m} + f_n = 0$$

All stresses but σ_{11} are zero, so:

$$\frac{\partial \sigma_{11}}{\partial x_1} + f_1 = 0$$

with: f_1 = body force

If there is no body force:

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0$$

$$\Rightarrow \quad \sigma_{11} = \text{constant} = \frac{P}{A}$$ (as found before)

--> Stress-Strain Equations (use compliance force)

$$\varepsilon_{mn} = S_{mnpq} \, \sigma_{pq}$$

Since only σ_{11} exists, we are left with:

$$\varepsilon_{11} = S_{1111} \sigma_{11}$$

$$\varepsilon_{22} = S_{2211} \sigma_{11}$$

$$\varepsilon_{33} = S_{3311} \sigma_{11}$$

$$\left.\begin{aligned} \gamma_{12} &= 2S_{1211}\,\sigma_{11} \\ \gamma_{23} &= 2S_{2311}\,\sigma_{11} \\ \gamma_{13} &= 2S_{1311}\,\sigma_{11} \end{aligned}\right\} = 0 \quad \text{for all but fully anisotropic materials}$$

For <u>orthotropic</u> material (see unit M3.2)

$$S_{1111} = \frac{1}{E_1} \qquad S_{2211} = -\frac{\nu_{12}}{E_1} \qquad S_{3311} = -\frac{\nu_{13}}{E_1}$$

$$\Rightarrow \left\{\begin{aligned} \varepsilon_{11} &= \frac{1}{E_1}\,\sigma_{11} && \text{(2)} \\ \varepsilon_{22} &= -\frac{\nu_{12}}{E_1}\,\sigma_{11} && \text{(3)} \\ \varepsilon_{33} &= -\frac{\nu_{13}}{E_1}\,\sigma_{11} && \text{(4)} \end{aligned}\right.$$

Finally consider the:

--> Strain - Displacement Equations

$$\varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)$$

For the three extensional strains:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad (5)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad (6)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \qquad (7)$$

(others are zero…will check).

So now do the…..

Solution for General Case

Use (2) in (5):

$$\Rightarrow \quad \frac{\sigma_{11}}{E_1} = \frac{\partial u_1}{\partial x_1}$$

Using (1):

$$\frac{P}{AE_1} = \frac{\partial u_1}{\partial x_1}$$

Integrating this gives:

$$u_1 = \frac{P x_1}{AE_1} + g(x_2,\ x_3)$$

function of integration

Let $u_1 = 0$ @ $x_1 = 0$ (constant for x_2 and x_3) gives $g(x_2, x_3) = 0$

So:

$$u_1 = \frac{P x_1}{A E_1} \qquad (8)$$

Similarly for ε_{22} and ε_{33} using (3) and (4):

$$u_2 = -\frac{\nu_{12} P}{A E_1} x_2 \qquad (9)$$

$$u_3 = -\frac{\nu_{13} P}{A E_1} x_3 \qquad (10)$$

measured from centerline
($u_2 = 0$ @ $x_2 = 0$,
$u_3 = 0$ @ $x_3 = 0$)

Check shear strains:

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = 0$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = 0$$

✓OK

Thus, this solution solves all the boundary conditions and the 15 Equations of Equilibrium.

But there is a….

Slight Inconsistency in Model

Important to check assumptions and look for **consistency**.

We have modeled this as a one-dimensional structure, but the full three-dimensional equations show:

$$\varepsilon_{22} \neq 0 \Rightarrow u_2 \neq 0$$
$$\varepsilon_{33} \neq 0 \Rightarrow u_3 \neq 0$$

=> Cross-section changes shape slightly:

$$b^* = b + \int_{-b/2}^{b/2} \varepsilon_{22}\, dx_2$$

$$h^* = h + \int_{-h/2}^{h/2} \varepsilon_{33}\, dx_3$$

$$A^* = (b^*)(h^*)$$

$\Rightarrow$ σ_{11} changes to: $\sigma_{11}^* = \dfrac{P}{A^*}$

Thus, it seems we cannot solve equations sequentially but must satisfy them simultaneously

--> what does this imply?

* All structures are three-dimensional in nature and the full 3-D equations of elasticity must be solved /satisfied for an exact solution *

But: within the limitations and assumptions of modeling, can "relax" this

Key assumption here:

Cross-section does not change shape

$\Rightarrow$ ε_{22} and ε_{33} are "small"

--> How "small" depends on accuracy needed (can always check after solving)

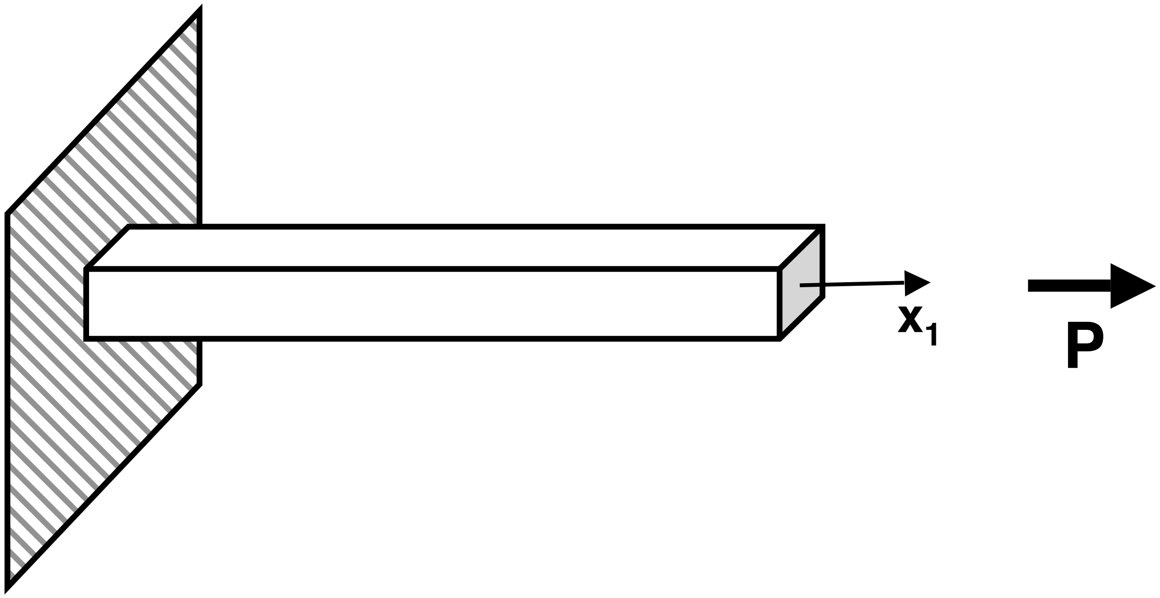
Generally strains less than 1% $\Rightarrow$ *error of 1% - 2% at most*

--> There is one other reason that the model "breaks down" and this is attributable to Boundary Conditions. This is dealt with by using…..

St. Venant's Principle

Consider a rod which is rigidly attached to a wall:

Figure M4.2-6 **Geometry of rod rigidly attached to a wall**



Since the bar is rigidly attached at the wall, there can be no deflection there:

$$\left.\begin{array}{l} u_1 = 0 \\ u_2 = 0 \\ u_3 = 0 \end{array}\right\} @\ x_1 = 0$$

$u_1 = 0$ @ $x_1 = 0$ is part of the solution, but we found that:

$$u_2 = -\frac{\nu_{12} P}{A E_1} x_2 \quad (\neq 0)$$

$$u_3 = -\frac{\nu_{13} P}{A E_1} x_3 \quad (\neq 0)$$

So the solution does not apply at the Boundary!

We invoke St. Venant's Principle:

"Remote from the boundary conditions, internal stresses and deformations will be insensitive to the exact form of the boundary condition."

--> St. Venant tells us that "far away" from the boundary the general solution holds. Thus, "ignore" the specifics of load introduction and boundary conditions.

- Generally have complicated stress states at such locations

- We replace details by a "statically equivalent" (equipollent condition)

Caution

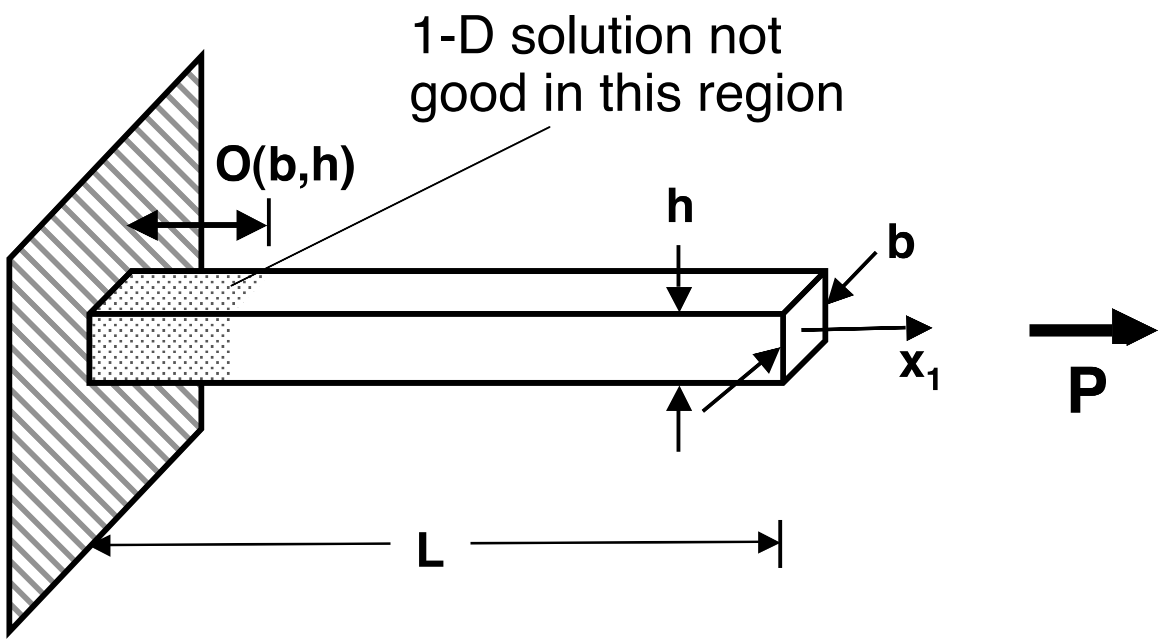
Failure often occurs at such locations (boundaries, points of load introduction)

Q: How far is "far away" (i.e., remote)?

This depends on the material, but a good rule is the larger of the dimensions associated with the boundary.

In this case: b or h

Figure M4.2-7 **Indication of where exact solution is not valid in the vicinity of boundary**



What can one do with this 1-D solution?

Besides looking at just the member (a single member in isolation), one can also consider deflection of a truss, …..

Use for Deflection of a Truss

Recall from Unit M1.5 that one can solve a general structural problem (indeterminate) by:

1. Applying equilibrium (get reactions, etc.)
2. Determining the constitutive relations
3. Enforcing compatibility (of displacements)
4. Solving the simultaneous equations

(…will consider in recitation)

Remarks on Static Determinance

Solution via static determinance is actually a model assuming no/small displacements.

As displacements are considered, geometry can change thus changing how load is carried (depending on the configuration)

--> Can normally ignore this

Always check once have solution to see that results are within acceptable limits of assumptions => **consistent**

Unit M4.2 (New) Nomenclature

b -- width of bar (dimension in x_2 - direction)
h -- thickness of bar (dimension in x_3 - direction)
L -- length of bar (dimension in x_1 - direction)
P -- applied load
A -- cross-sectional area of bar
u_1 -- deformation of bar in x_1 - direction
σ_{11} -- stress of bar in x_1 - direction
E_1 -- modulus of bar material in x_1 - direction