Unit M4.3
Statics of Beams

Readings:

CDL 3.2 - 3.6

(CDL 3.8 -- extension to 3-D)

16.003/004 -- "Unified Engineering"
Department of Aeronautics and Astronautics
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LEARNING OBJECTIVES FOR UNIT **M4.3**

Through participation in the lectures, recitations, and work associated with Unit M4.3, it is intended that you will be able to………

- ….**describe** the aspects composing the model of a beam associated with geometry and loading and **identify** the associated limitations
- ….**apply** overall equilibrium to **calculate** the reactions and distributed internal forces (axial, shear, moment) for various beam configurations
- ….**use** equilibrium to **derive** the formal relationships between loading, shear, and moment (q, S, M) and **apply** these for various beam configurations

We now turn to looking at a slender member which can take bending loads. This is known as a beam. So let's first consider the…

Definition of a beam

"A beam is a structural member which is long and slender and is capable of carrying bending loads via deformation transverse to its long axis"

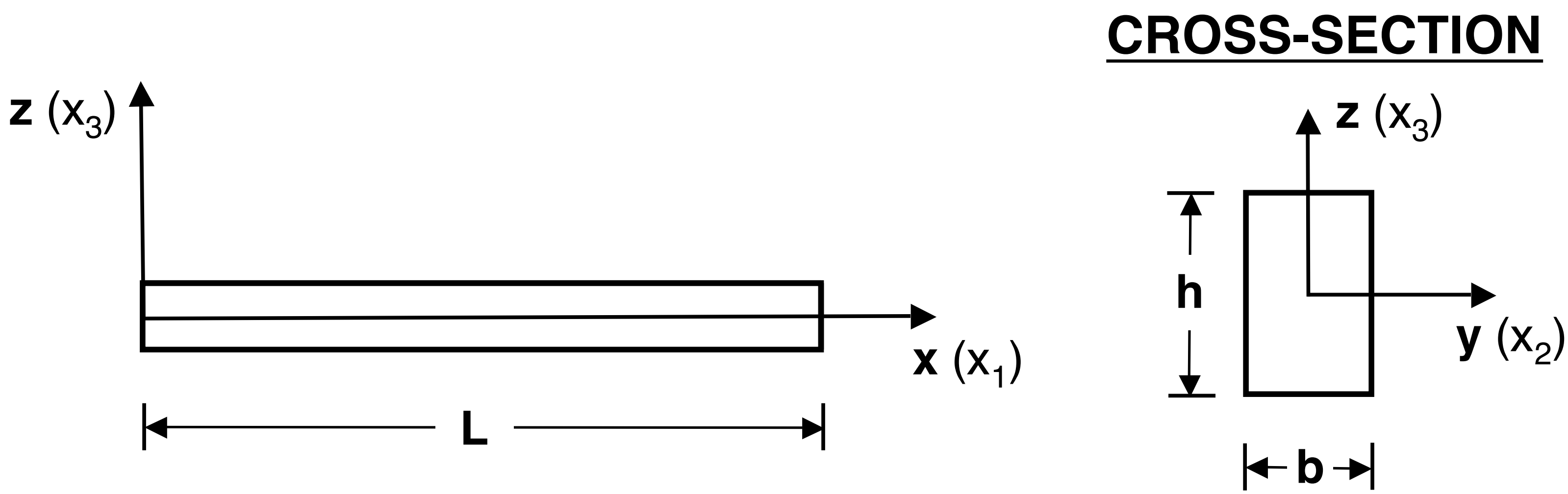
Note: bending loads are applied transverse to long axis

--> Look at specifics of Modeling Assumptions

a) Geometry

Note: (switch to $\underbrace{x, y, z}_{\text{engineering notation}}$ from $\underbrace{x_1, x_2, x_3}_{\text{indicial/tensorial notation}}$)

Figure M4.3-1 **Geometry of a "beam"**



Normal Convention: *(same as rod!)*

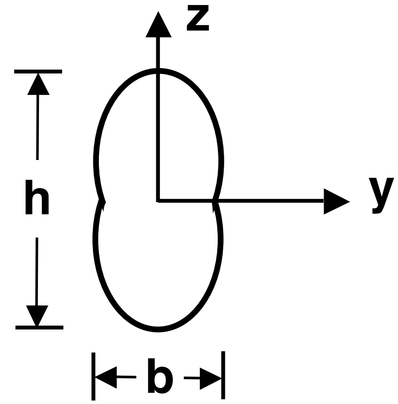
- L = length (x - dimension)
- b = width (y - dimension)
- h = thickness (z - dimension)

Assumption: "long" in x - direction

 L >> b, h (**slender** member)

--> has some arbitrary cross-section that is (*for now*) symmetric about y and z

Figure M4.3-2 **Geometry of arbitrary cross-section**



b) Loading

Assumption: Bending load transverse to x - direction

For one-dimensional case, load is in z-direction

(Note: *Higher order model* -- load in x-direction is "rod-load" making it a beam-bar/beam-rod for general case)

--> will look at implications on stresses later

c) Deformation

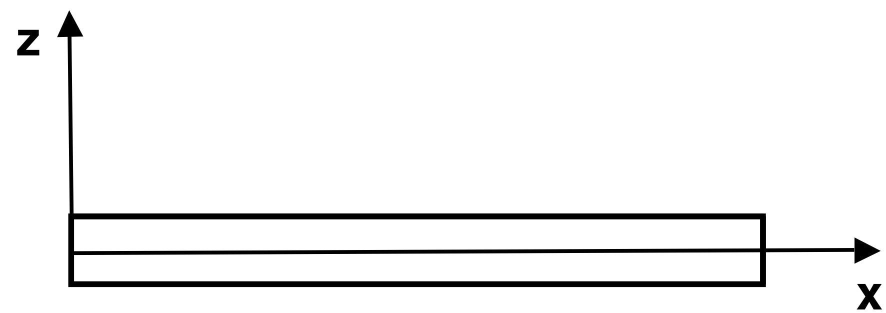
--> will look at later (there are different assumptions which can be made resulting in different models or "beams")

Let's next look at how a beam can be constrained and used….

Beam Types, Uses and Boundary Conditions

Q: How many Boundary Conditions (reactions) are needed to make a "1-D" beam statically determinate?

Figure M4.3-3 **Representation of beam with no boundary conditions**

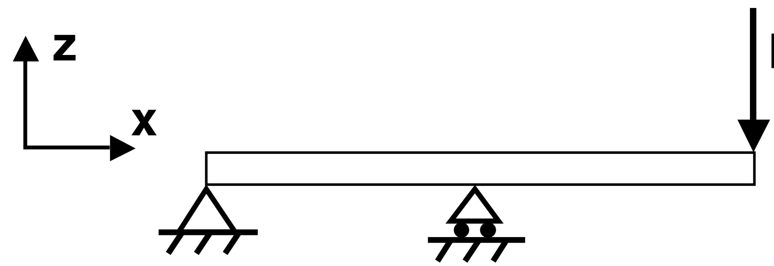


Rigid body of degrees of freedom: lateral in x, lateral in z, rotation about y } 3 d.o.f.'s

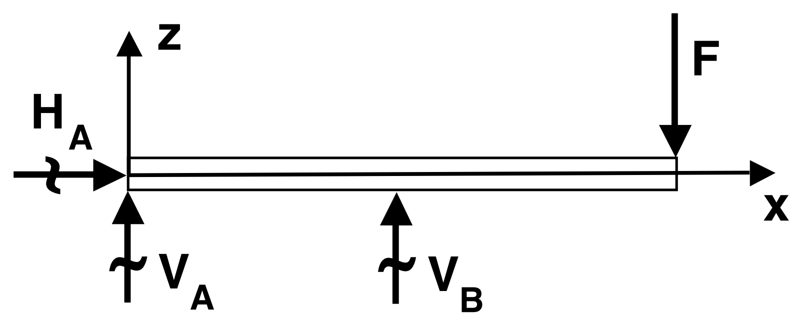
=> Need 3 boundary conditions (B.C.'s) for static determinance

--> Pinned beam (e.g., diving board)

Figure M4.3-4 **Geometry and free body diagram of pinned beam**

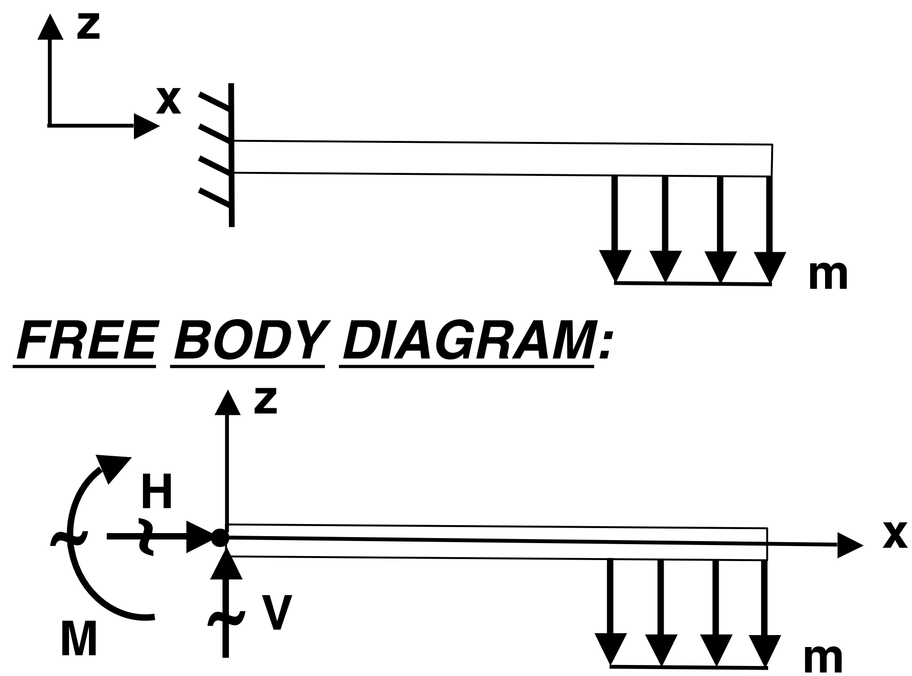


FREE BODY DIAGRAM:



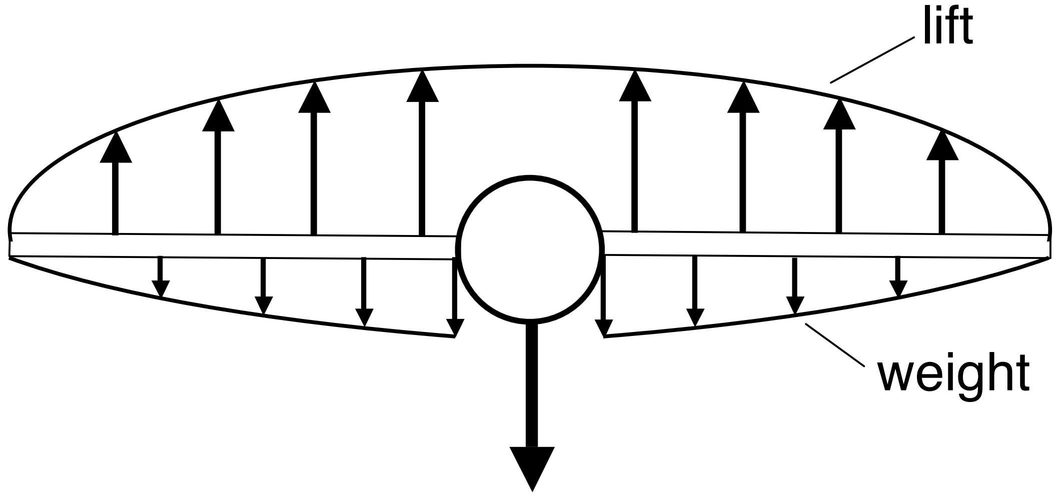
--> Cantilevered beam (e.g., flag pole)

Figure M4.3-5 **Geometry and free body diagram of cantilevered beam**



or an airplane wing

Figure M4.3-6 **Geometry of airplane wings as beam**

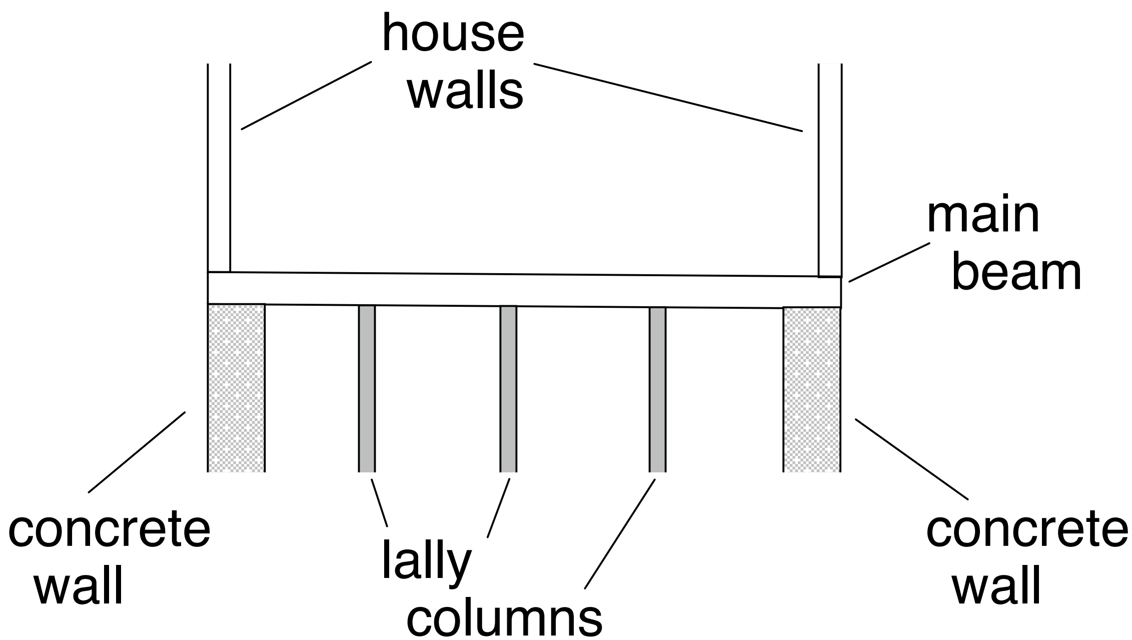


--> <u>Indeterminate</u> <u>beam</u>

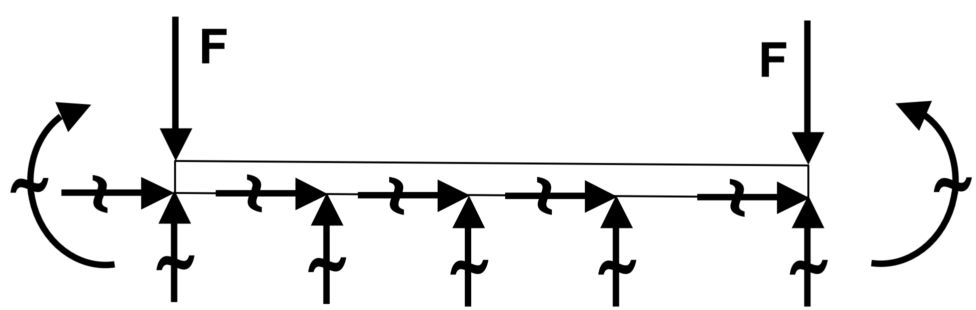
More reactions than d.o.f.'s:

(e.g., main house beam)

Figure M4.3-7 Geometry and free body diagram of indeterminate beam



FREE BODY DIAGRAM:



--> We will save looking at the statically indeterminate case for a later unit. Let's start off by considering….

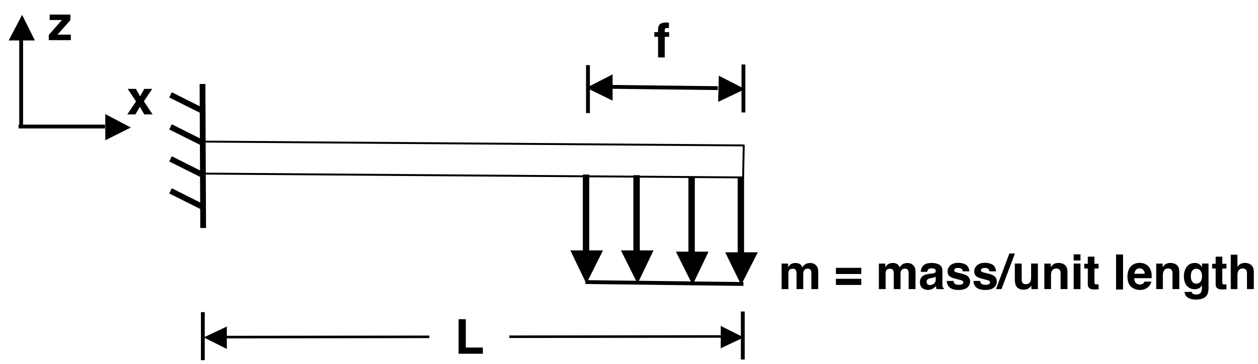
Static Determinance: The Reactions

There is no difference from what we've done before. Two steps:

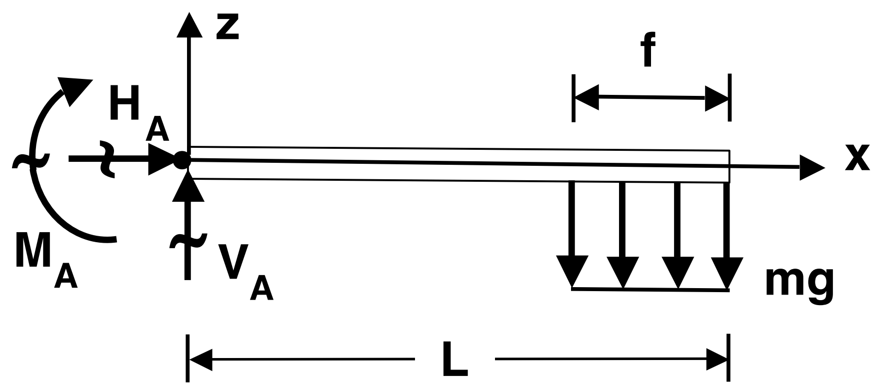
1. Draw Free Body Diagram
2. Apply Equilibrium

Example: Cantilevered Flag

Figure M4.3-8 **Geometry and free body diagram of cantilevered flag**



FREE BODY DIAGRAM:



$$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad H_a = 0$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow \quad V_a - mgf = 0 \quad \Rightarrow \quad V_a = mgf$$

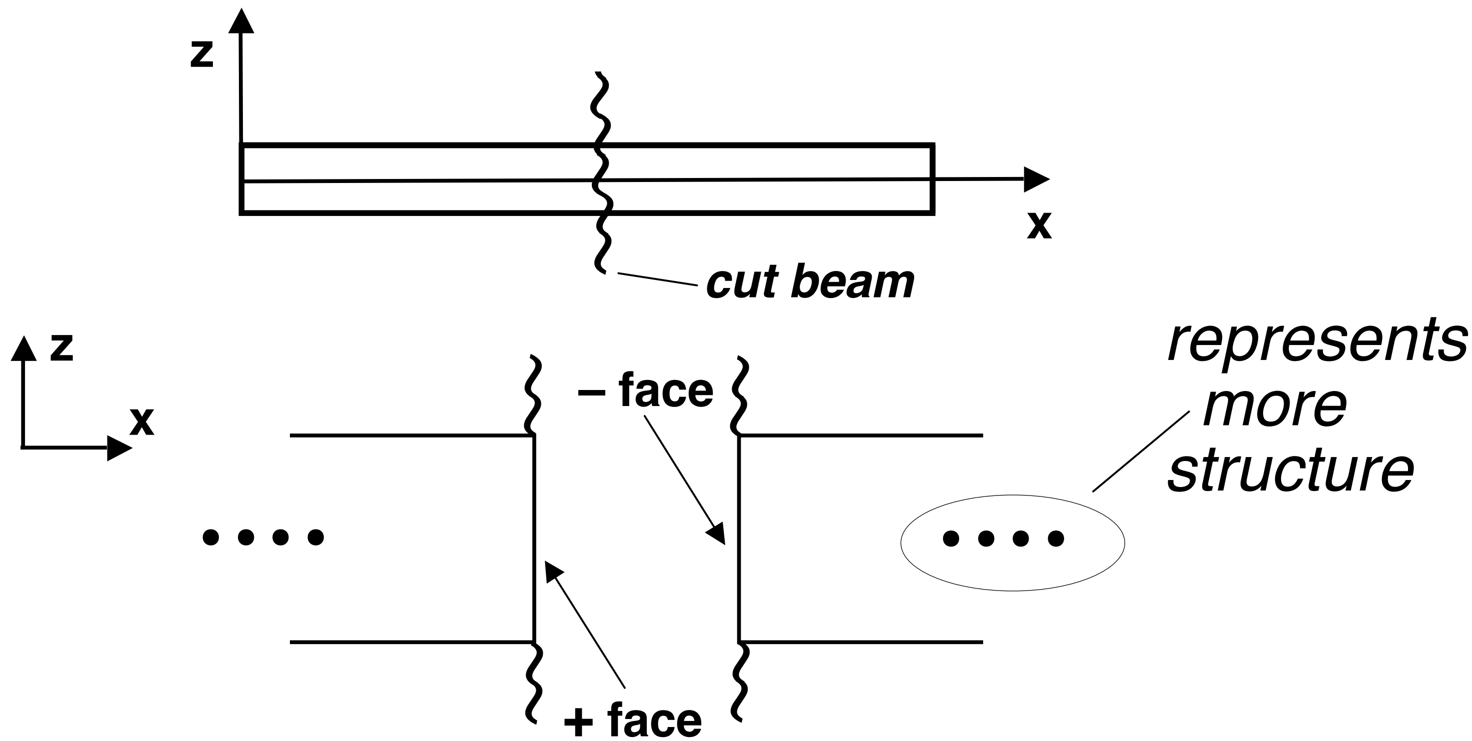
$$\sum M_y = 0 \quad \circlearrowleft + \quad \Rightarrow \quad M + \int_{L-f}^{L} mgx\,dx = 0 \quad \Rightarrow \quad M = -mg\left[\frac{x^2}{2}\right]_{L-f}^{L}$$

The next step is to determine what we can about the internal stress and strain state. Our first step here is to look at the….

Internal Forces

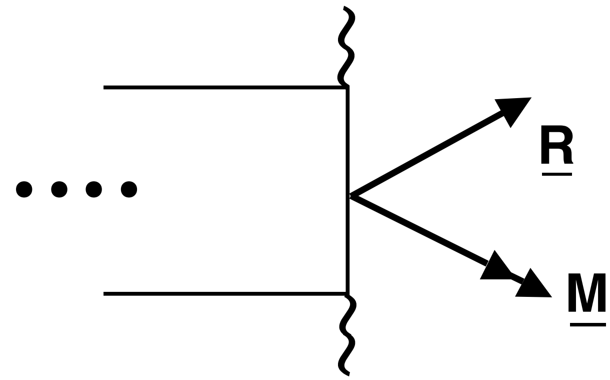
As we've done in the past (recall trusses, rods), we "cut" the structural member (the beam) at some point and consider the equilibrium of the cut face

Figure M4.3-9 **Illustration of generic cut beam and resulting cut faces**



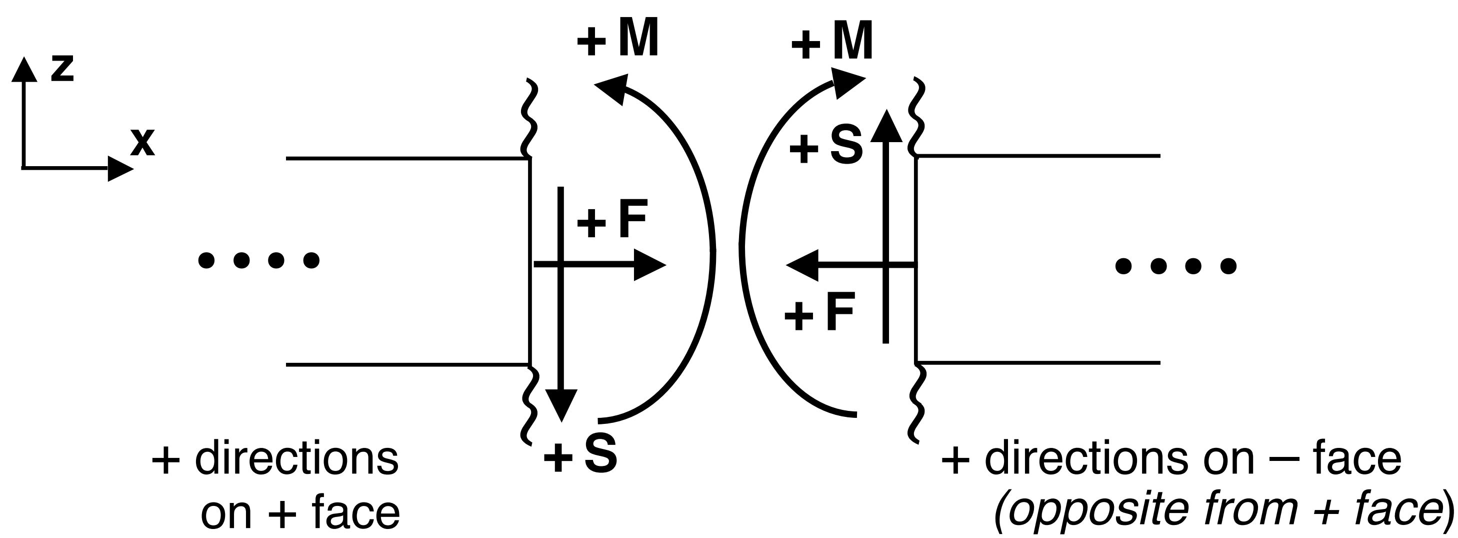
In 1-D case, can replace all forces on each face by overall resultant forces (moment and resultant)

Figure M4.3-10 **Overall resultant forces acting on cut face of beam**



Resolve this into forces aligned with axes:

Figure M4.3-11 **Overall aligned force system acting on cut faces of beam**



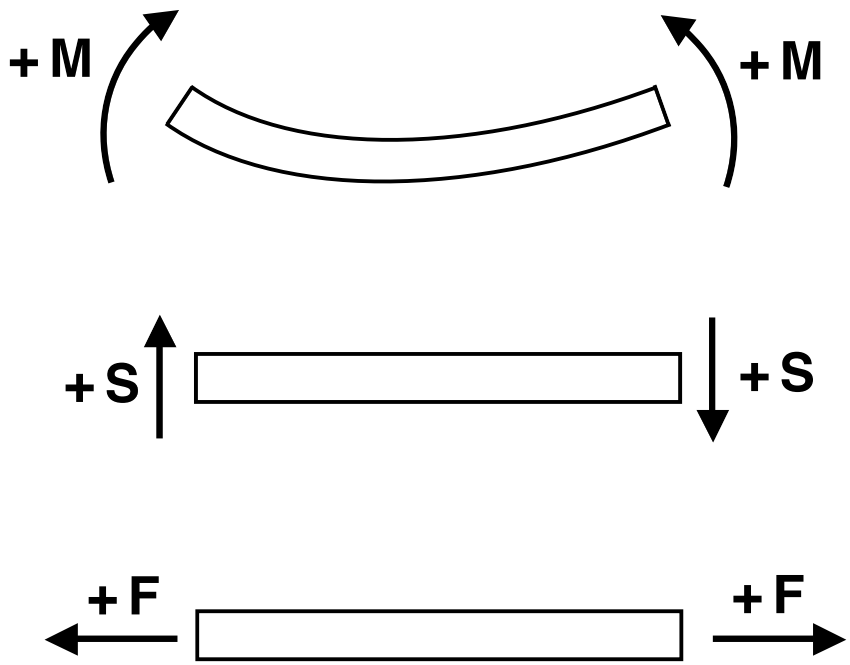
<u>directions</u> <u>by</u> <u>convention</u>

Note: CDL uses $V = -S$ (**always** look at definition being used)

where:

M = Bending Moment S = Shear Force	Pure Bending	"Beam-bar"
F = Axial Force	Bar	

Figure M4.3-12 **Illustration of positive conventions for beam internal forces**



+ M makes beam bend **up**

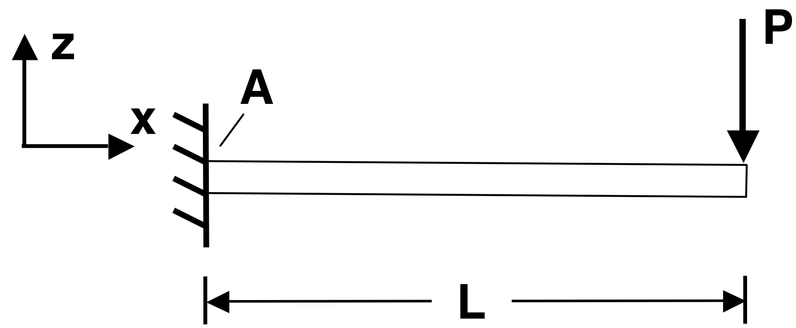
+ S makes beam shear **down** *(right hand side down relative to left)*

+ F makes beam **elongate** *(i.e., tension)*

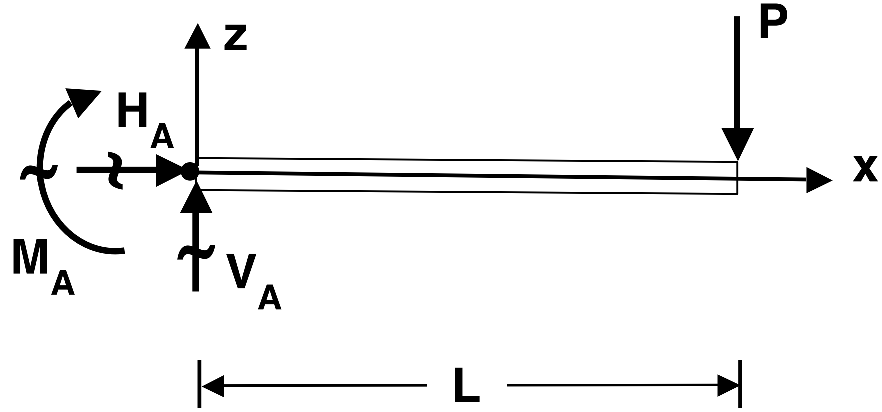
--> Determine values via equilibrium once reactions are found.

Example: tip loaded cantilevered beam

Figure M4.3-13 **Geometry and free body diagram of tip-loaded cantilevered beam**



FREE BODY DIAGRAM:



--> get reactions

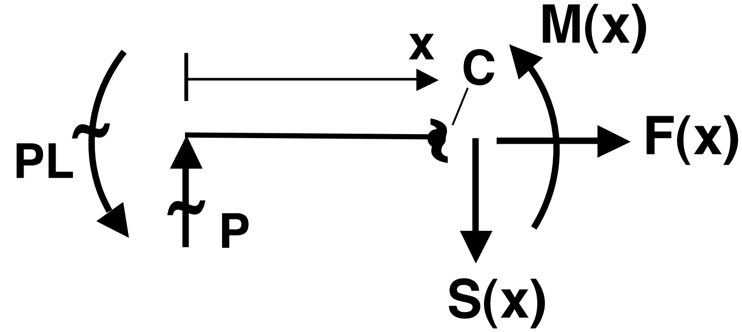
$$\sum F_H = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad H_A = 0$$

$$\sum F_V = 0 \quad \uparrow + \quad \Rightarrow \quad V_A - P = 0 \quad \Rightarrow \quad V_A = P$$

$$\sum M_y = 0 \quad \curvearrowright + \quad \Rightarrow \quad M_A + PL = 0 \quad \Rightarrow \quad M_A = -PL$$

--> "cut" beam at any point x (choose midpoint here) and replace cut with internal forces

Figure M4.3-14 **Free body diagram of beam with "cut" at point x = L/2**



--> apply equilibrium for cut section (here: x = L/2)

$$\sum F_H = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad F = 0$$

$$\sum F_V = 0 \quad \uparrow + \quad \Rightarrow \quad P - S = 0 \quad \Rightarrow \quad S = P$$

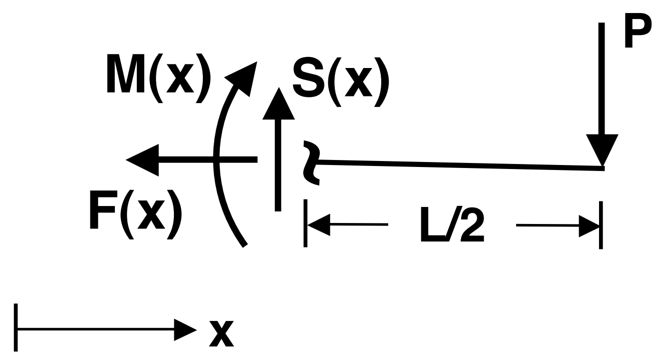
$$\sum M_c^{\ *} = 0 \quad \circlearrowright + \quad \Rightarrow \quad PL + M - \frac{PL}{2} = 0 \quad \Rightarrow \quad M = -\frac{PL}{2}$$

*<u>Note</u>: choose cut point to take moments about so S isn't involved!

can do this cut and equilibrium at any point

<u>Note</u>: could also do by considering outer section of cut….

Figure M4.3-15 **Free body diagram of outer section of beam with "cut" at point x = L/2**



--> again, use equilibrium:

$$\sum F_H = 0 \ \xrightarrow{+} \quad \Rightarrow \quad F = 0$$

$$\sum F_V = 0 \ \uparrow + \quad \Rightarrow \quad S - P = 0 \quad \Rightarrow \quad S = P$$

$$\sum M_c = 0 \ \circlearrowright + \quad \Rightarrow \quad - M - \frac{PL}{2} \quad \Rightarrow \quad M = -\frac{PL}{2}$$

Same results!!! (**better be!**)

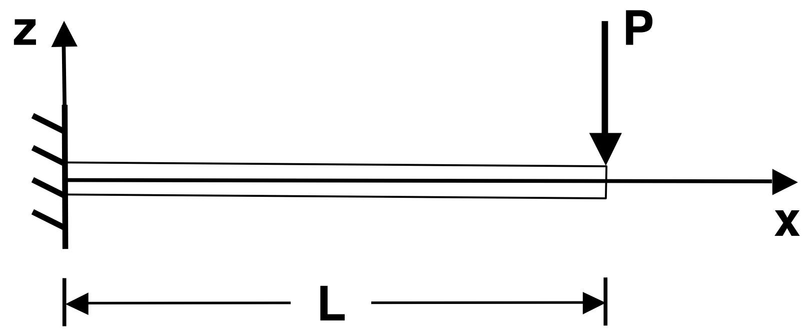
In general, we want to find the internal forces throughout the beam. So let's consider….

F, S, and M Diagrams

We've started off by considering…

--> Point Load(s)

So let's continue that and our particular example of the tip-loaded cantilevered beam:

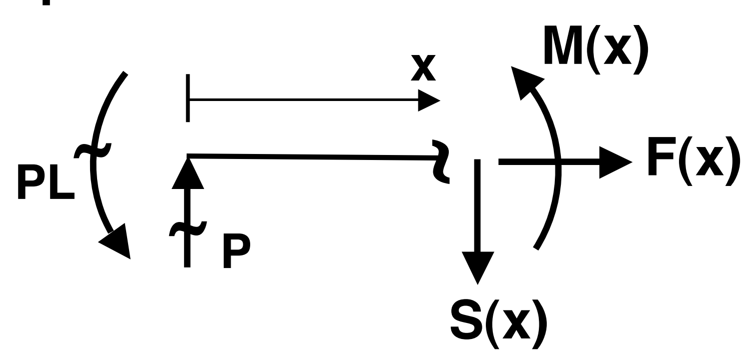


We could take a cut at several points, but let's do this "generically' by taking a cut at point "x":

$$0 < x < L$$

Let's use right cut side

Figure M4.3-16 **Free body diagram for cut at generic point of cantilevered beam with tip load**



Again, applying equilibrium

$$\sum F_H = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad F(x) = 0$$

$$\sum F_V = 0 \quad \uparrow + \quad \Rightarrow \quad P - S(x) = 0 \quad \Rightarrow \quad S(x) = P$$

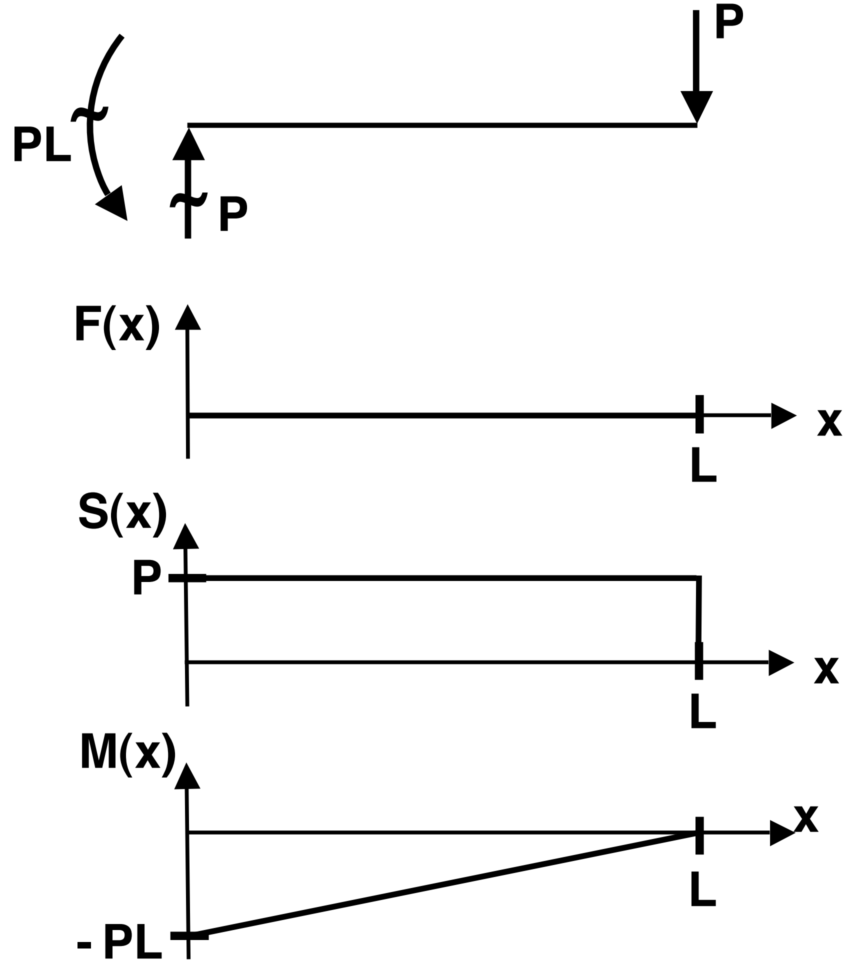
$$\sum M_c = 0 \quad \circlearrowleft + \quad \Rightarrow \quad PL + M(x) - Px = 0$$

$$\Rightarrow \quad M(x) = -P(L - x)$$

--> <u>Draw “sketches”</u> (diagrams)

Figure M4.3-17 Force diagrams for cantilevered beam with tip load

- Free Body Diagram
- Axial Force Diagram
- Shear Force Diagram
- Moment Diagram



Q: what happens at boundaries??

--> Values go to reaction values (include proper sense/direction)

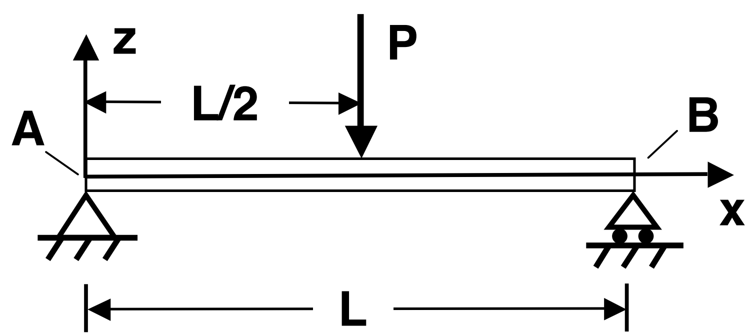
(as they should)

But, what if there is more than one point load or a point load is at the middle of the beam?

--> Need to make "generic" cuts on each side of such.

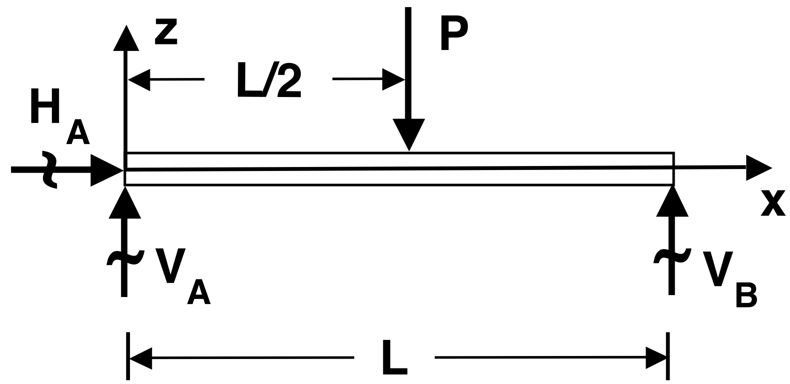
Example: simply-supported beam with mid-span load

Figure M4.3-18 **Geometry of simply-supported beam with mid-span load**



1. Get reactions

Figure M4.3-19 **Free body diagram of simply-supported beam with mid-span load**



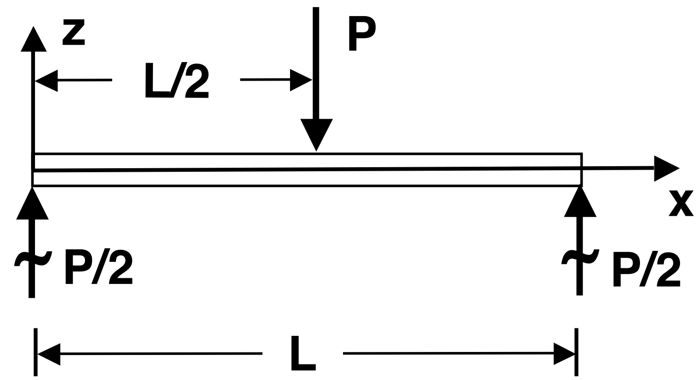
$$\sum F_H = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad H_A = 0$$

$$\sum F_V = 0 \quad \uparrow + \quad \Rightarrow \quad V_A + V_B - P = 0$$

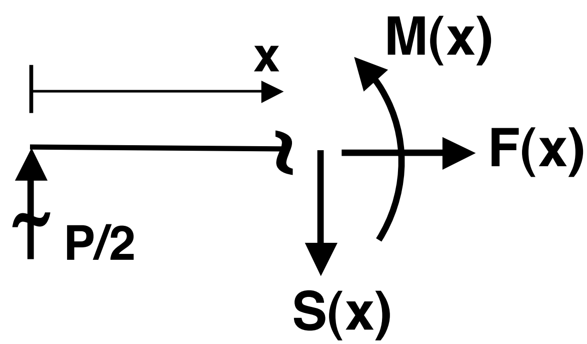
$$\sum M_A = 0 \quad \circlearrowright + \quad \Rightarrow \quad - \frac{PL}{2} + V_B L = 0 \quad \Rightarrow \quad V_B = \frac{P}{2}$$

gives: $V_A = \dfrac{P}{2}$

So:



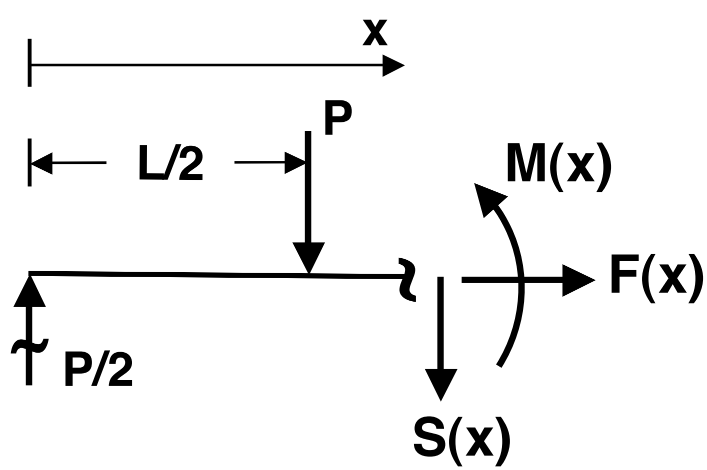
2. Take cut at $0 < x < L/2$



Note: No loads in x-direction so clearly F(x) = 0 everywhere in beam

$$\left.\begin{aligned} \sum F_V = 0 \ \uparrow + \quad &\Rightarrow \quad \frac{P}{2} - S(x) = 0 \\ &\Rightarrow \quad S(x) = \frac{P}{2} \\ \sum M_x = 0 \ \circlearrowleft + \quad &\Rightarrow \quad -\frac{P}{2}x + M(x) = 0 \\ &\Rightarrow \quad M(x) = \frac{Px}{2} \end{aligned}\right\} \quad 0 < x < \frac{L}{2}$$

3. Take cut at L/2 < x < L



showed F(x) = 0

$$\sum F_V = 0 \ \uparrow + \Rightarrow \frac{P}{2} - P - S(x) = 0$$

$$\Rightarrow S(x) = -\frac{P}{2}$$

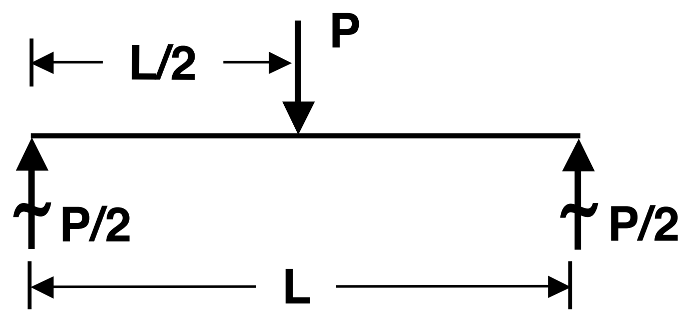
$$\sum M_x = 0 \ \circlearrowright + \Rightarrow -\frac{P}{2}x + P(x - L/2) + M(x) = 0$$

$$\Rightarrow M(x) = -\frac{P}{2}(x - L) = \frac{P}{2}(L - x)$$

$$\frac{L}{2} < x < L$$

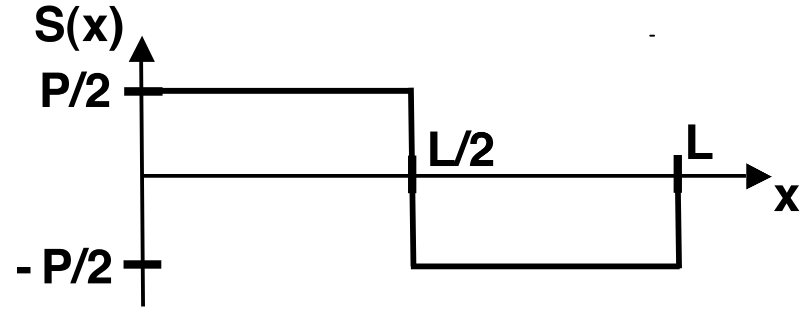
4. Draw diagrams

• Free Body Diagram

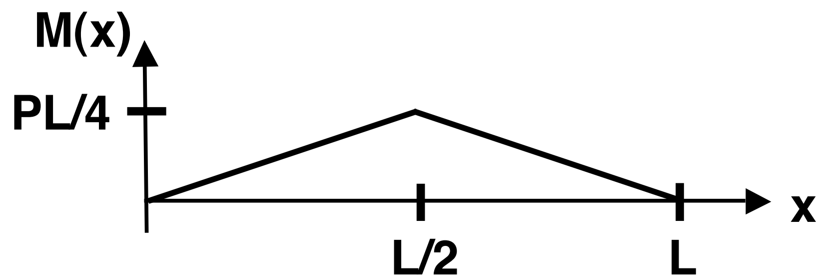


- Axial force: F(x) = 0 everywhere

- Shear Force Diagram



- Moment Diagram



Observations:

- Shear __________ between concentrated loads
- (Bending) moment______________between concentrated loads
- ______________________________________ at concentrated loads
- _______________ equals amount of concentrated load at point of application
- Values of S and M (and F)_______________________ at boundaries

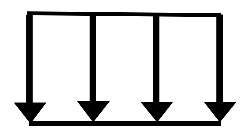
Beyond point loads we can also have….

--> Distributed Loads q (x)

units of [Force/Length] or can have intensity which varies with length

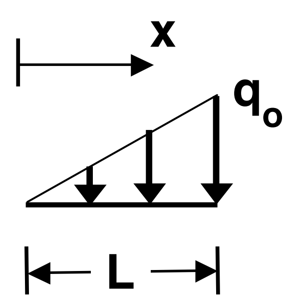
examples: gravity, pressure, inertial (D'Alembert)

Constant



force/length = q (x) = q

Linear



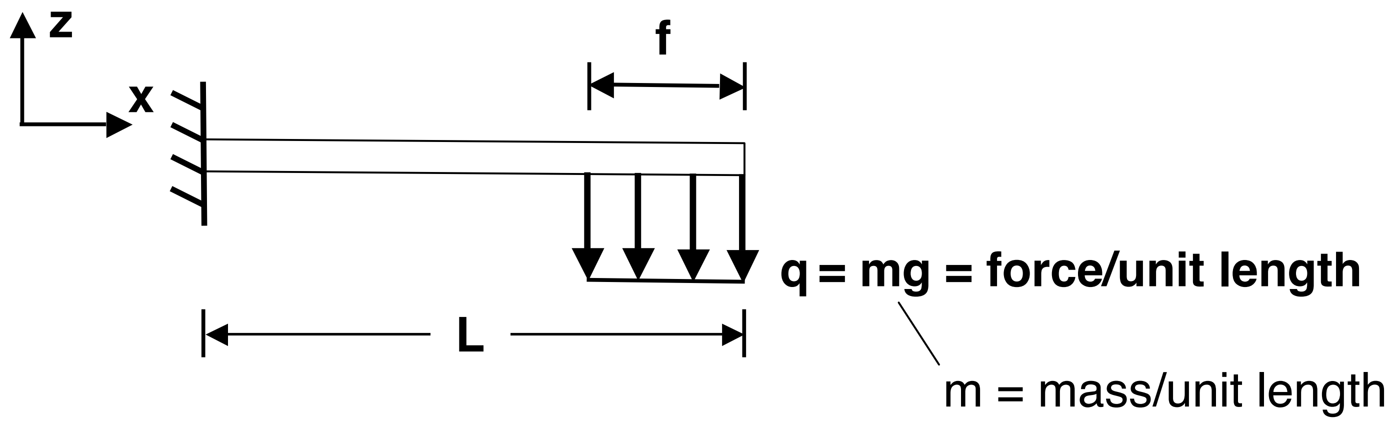
$q(x) = q_o\, x/L = (q_o/L)\, x$

$[q_o]$ = [force/length]

or other variation with position (e.g., function of x)

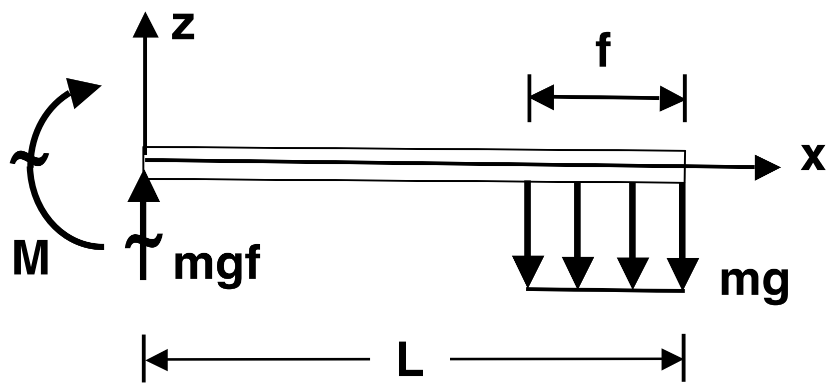
--> same basic procedure, but need to take a "generic cut" through each different section of distributed load and use integral within these sections

Example: Cantilevered Flag



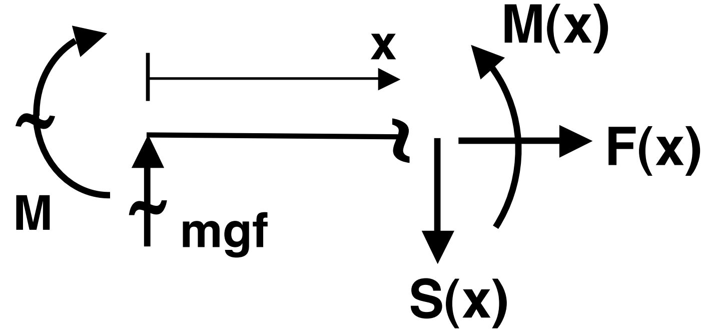
1. Get reactions

earlier showed FBD:



$$M = -mgLf + \frac{1}{2} mgf^2$$

2. Take cut at $0 < x < (L - f)$



Again note F(x) = 0 everywhere

$$\sum F_V = 0 \uparrow + \quad \Rightarrow \quad mgf - S(x) = 0 \quad \Rightarrow \quad S(x) = mgf$$

$$\sum M_x = 0 \circlearrowleft + \quad \Rightarrow \quad M(x) - M - mgfx = 0$$

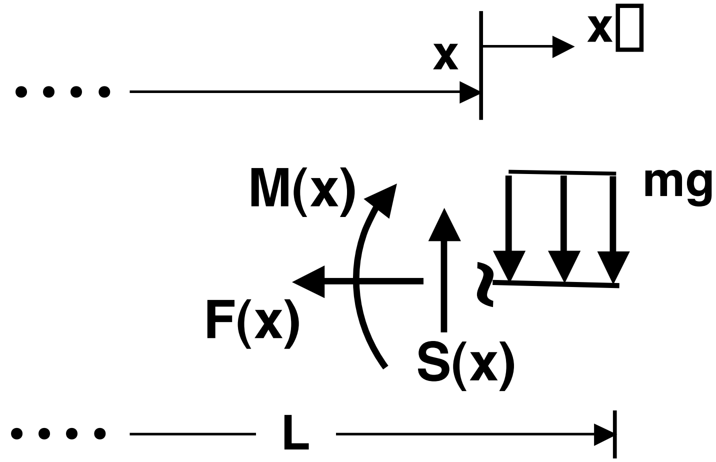
$$\Rightarrow \quad M(x) = -mgLf + \frac{1}{2} mgf^2 + mgfx$$

$$= \frac{1}{2} mgf^2 - mgf(L - x)$$

$$0 < x < (L - f)$$

3. Take cut at (L - f) < x < L

(use right side!)



F(x) = 0 everywhere

$$\sum F_V = 0 \uparrow + \Rightarrow -mg\,(L - x) + S(x) = 0$$

$$\Rightarrow S(x) = mg\,(L - x)$$

$$\sum M_x = 0 \circlearrowleft + \Rightarrow -M(x) - \int_0^{(L-x)} mgx'\,dx' = 0$$

$$\Rightarrow M(x) = -mg \left[\frac{x'^2}{2} \right]_0^{(L-x)}$$

$$= -\frac{1}{2}\, mg\,(L - x)^2$$

for $(L - f) < x < L$

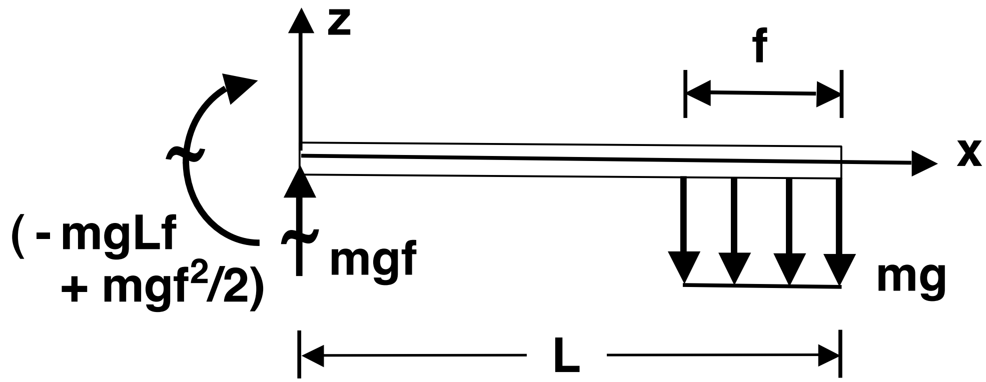
Calculate at “junction”: x = (L – f)

S (x) = mg (L – L + f) = mgf *same as before*

M (x) = – 1/2 mg (L – L + f)^2 = – 1/2 mgf^2 *same as before*

4. Draw diagrams

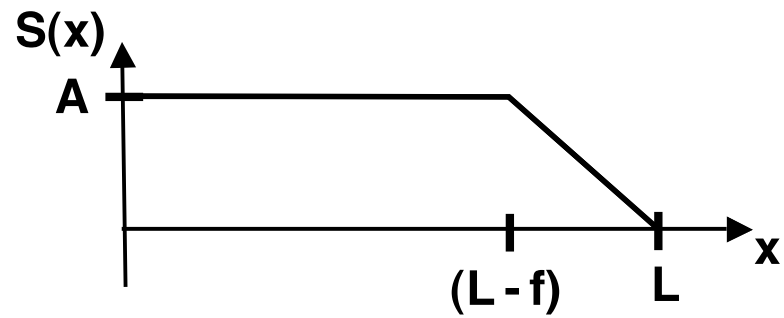
• Free Body Diagram



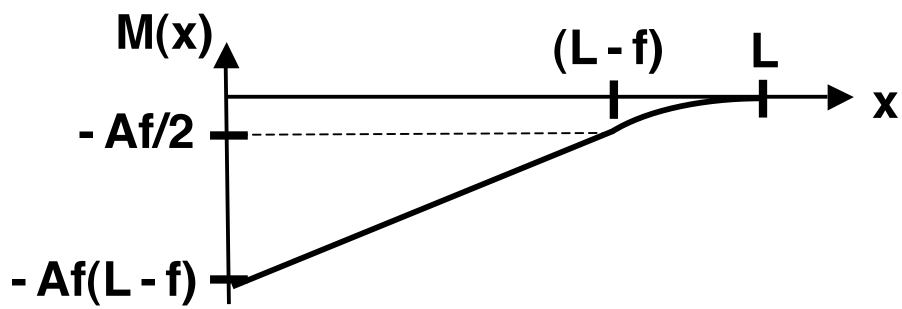
let A = mgf

• Axial Force: (recall F(x) = 0 everywhere)

• Shear Force Diagram



• Moment Diagram



(Additional) observation(s):

- Shear varies __________ over constant distributed load
- Moment varies _________ over constant shear; □ _______________ over linear shear region

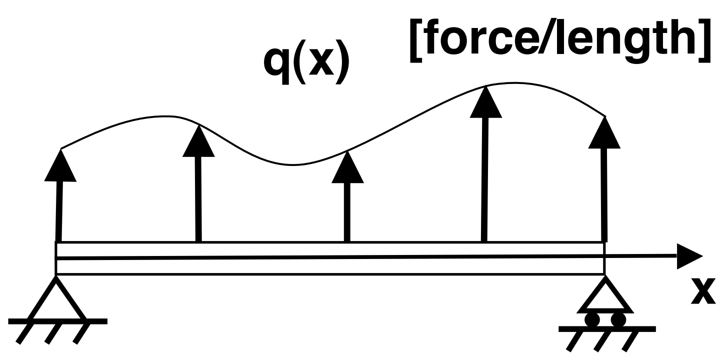
These observations and relationships between Loading, Shear and Moment indicate "there may be more there".

There is! So let's investigate…..

Formal Relations Between q, S, M

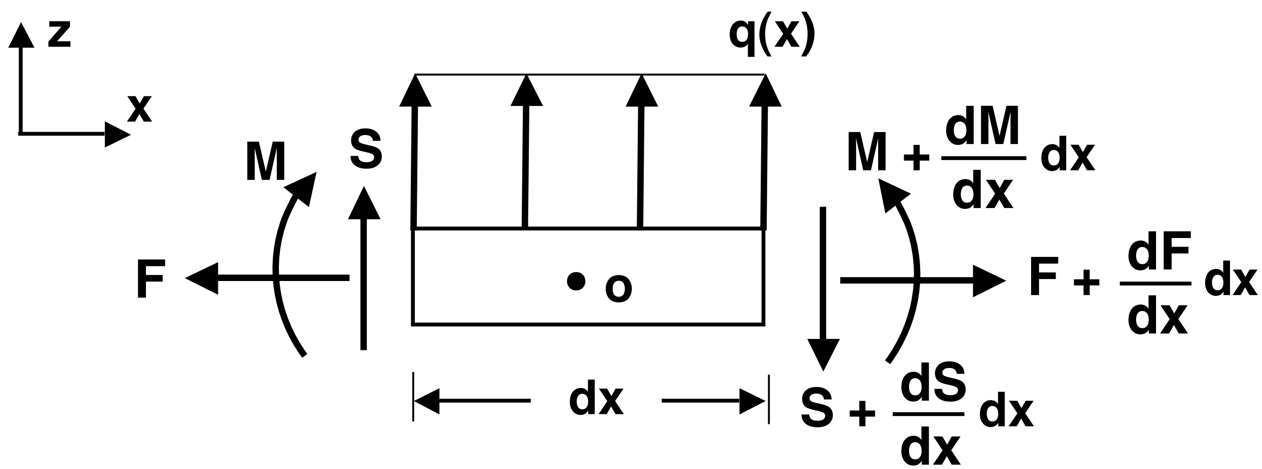
--> Consider a beam under some arbitrary loading q(x)

Figure M4.3-20 **Geometry of general beam loading**



--> As we did for a full body, let's consider an infinitesimal element of length dx

Figure M4.3-21 **Differential loading in infinitesimal element of generally loaded beam**



or

Equipollent load on element = qdx at midpoint (consider q approximately constant over infinitesimal length dx)

Now use Equilibrium…..

$$\sum F_x = 0 \quad \overset{+}{\longrightarrow} \quad \Rightarrow \quad -F + F + \frac{dF}{dx}dx = 0$$

$$\boxed{\Rightarrow \quad \frac{dF}{dx} = 0}$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow \quad S - S - \frac{dS}{dx}dx + q(x)dx = 0$$

$$\boxed{\Rightarrow \quad \frac{dS}{dx} = q(x)}$$

$$\sum M_o = 0 \quad \circlearrowright + \quad \Rightarrow \quad -\overset{①}{\cancel{M}} + \overset{①}{\cancel{M}} + \frac{dM}{dx}dx - S\frac{dx}{2}$$

$$- \left(S + \frac{dS}{dx}dx\right)\frac{dx}{2} = 0$$

(Note: q(x) has no net moment since it is symmetrical about point o)

$$\square \quad \frac{dM}{dx}dx \;\square\; Sdx \;\square\; \frac{1}{2}\,\frac{dS}{dx}\,(dx)^2 = 0$$

$$\square\; 0 \text{ (Higher Order Term } = dx^2\text{)}$$

$$\square \quad \boxed{\frac{dM}{dx} = S}$$

<u>Summarizing</u>:

$$\frac{dF}{dx} = 0 \qquad \text{(bar)}$$

$$\frac{dS}{dx} = q$$

$$\frac{dM}{dx} = S \;\square\; \frac{d^2M}{dx^2} = q$$

<u>Note</u>: can also use relations in integral form:

$$S = \int q\,dx$$

$$M = \int S\,dx$$

or in definite integral form:

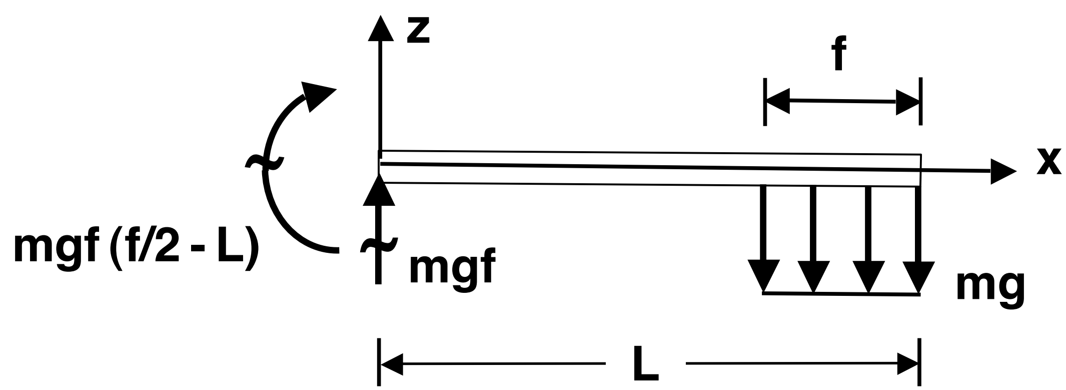
$$S_b - S_a = \int_{x_a}^{x_b} q\,dx$$

$$M_b - M_a = \int_{x_a}^{x_b} S\,dx$$

Example: redo cantilevered flagpole

Note: same procedure (basically)

1. Get reactions, show FBD



Must still do for each "differently" loaded point

2. Consider portion: $0 < x < (L - f)$

here: $q(x) = 0$

$$\Rightarrow \quad S(x) = \int q(x)dx = C$$

get value of constant by using reaction (i.e., boundary condition on Shear)

@ $x = 0$, $S = mgf$ $\Rightarrow$ $C = mgf$

$\Rightarrow$ $S(x) = mgf$ for $0 < x < (L - f)$

Now $M(x) = \int S(x)dx = \int mgfdx = mgfx + C$

Again, use reaction: @ $x = 0$, $M = mgf\ (f/2 - L)$

$\Rightarrow$ $C = mgf\ (f/2 - L)$

$\Rightarrow$ $M(x) = -mgf\ (L - f/2 - x)$ for $0 < x < (L - f)$

3. Consider portion: $(L - f) < x < L$

here: $q(x) = -mg$

$$\Rightarrow \quad S(x) = \int -mgdx = -mgx + C$$

"Boundary condition" on Shear and Moment this time is at edge of this section ($x = L - f$)

From before (part 2): @ x = (L – f), S = mgf

$\Rightarrow$ mgf = – mg(L – f) + C

$\Rightarrow$ C = mgL

$\Rightarrow$ S(x) = mg(L – x) (L – f) < x < L

and using $M(x) = \int S(x)dx = \int mg(L - x)dx$

$$\Rightarrow \quad M(x) = mg\left(Lx - \frac{x^2}{2}\right) + C$$

From before (part 2): @ x = (L – f), M = – mgf (f/2) = –mg $f^2/2$

$\Rightarrow$ – mg $f^2/2$ = mg(L^2 – Lf – $L^2/2$ + Lf – $f^2/2$) + C

$\Rightarrow$ C = – 1/2 mgL^2

So: M(x) = – mg ($L^2/2$ – Lx + $x^2/2$)

= – 1/2 mg $(L – x)^2$ for (L – f) < x < L

Same as using other method!

4. Draw diagrams

(will be the same as before)

Note: label FBD with

--> Can check work via:

- dM/dx = S; dS/dx = q in sections (look at slopes)
- taking cuts at specific points and comparing values calculated by each method

Notes on general solution procedure

- This is all *linear*, so can use *superposition*
  (add point and continuous techniques)
  (divide up and put together)

- Can make use of equipollent forces in calculating reactions, etc. (See CDL section 3.4)

- Can express point/concentrated forces and moments by mathematical expressions (recall dirac delta) (See CDL section 3.6)

Unit M4.3 (New) Nomenclature

b -- width of beam
F -- force
g -- gravity
h -- height of beam
H -- horizontal reaction
L -- length of beam
m -- mass per unit length
M -- moment (reaction or internal)
P -- applied load
$\underline{R}$ -- resultant force
V -- vertical reaction
x (x_1) -- coordinate along long direction (axis) of beam
y (x_2) -- coordinate along width direction of beam
z (x_3) -- coordinate along height direction of beam
M(x) -- internal beam moment at point x
S(x) -- internal beam shear force at point x
F(x) -- internal beam axial force of point x
F_V -- vertical force
F_H -- horizontal force
q(x) -- distributed applied load (at point x)