Unit M4.4
Simple Beam Theory

Readings:

CDL 7.1 - 7.5, 8.1, 8.2

16.003/004 -- "Unified Engineering"
Department of Aeronautics and Astronautics
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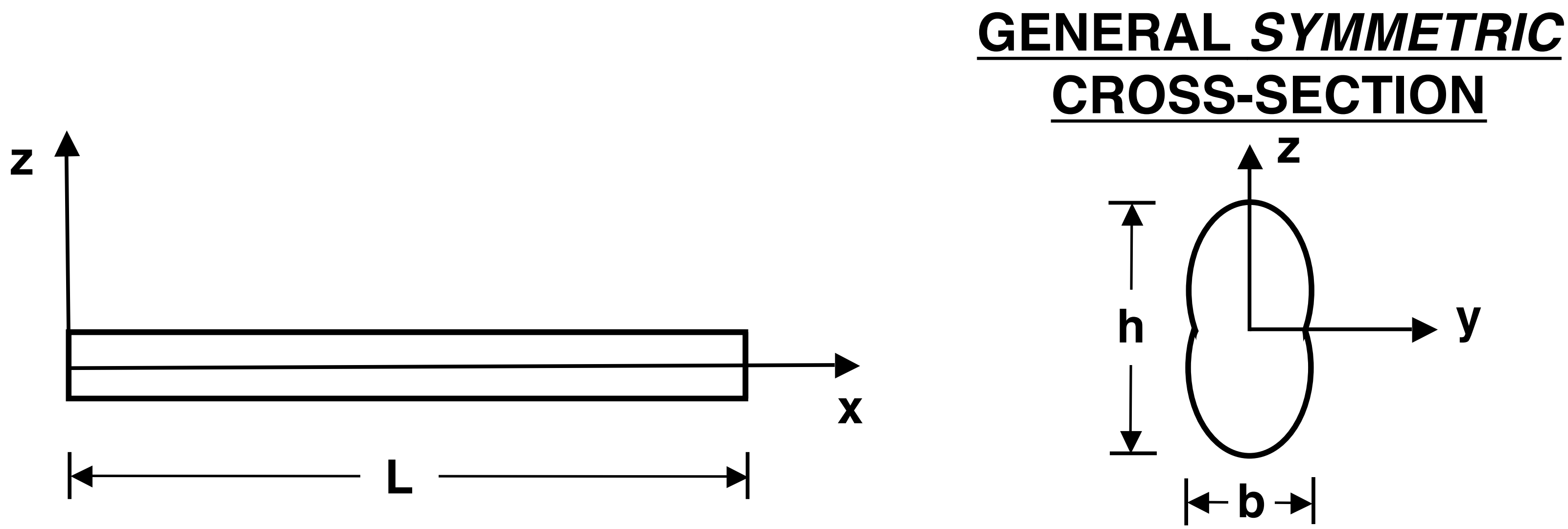
LEARNING OBJECTIVES FOR UNIT **M4.4**

Through participation in the lectures, recitations, and work associated with Unit M4.4, it is intended that you will be able to………

- ….**describe** the aspects composing the model of a beam associated with deformations/displacements and stresses (i.e. Simple Beam Theory) and **identify** the associated limitations
- ….**apply** the basic equations of elasticity to **derive** the solution for the general case
- ….**identify** the beam parameters that characterize beam behavior and **describe** their role

We have looked at the statics of a beam, but want to go further and look at internal stress and strain and the displacement/ deformation. This requires a particular model with additional assumptions besides those on geometry of "long and slender."

Figure M4.4-1 **Geometry of a beam**



h and b are "encompassing/extreme" dimensions

still have: L >> h, b

Now also consider

Assumptions on Stresses

We have said that loading is in the plane x-z and is transverse to the long axis (the x-axis)

The first resulting assumption from this is:

All loads in y - direction are zero

$\Rightarrow$ all stresses in y-direction are zero:

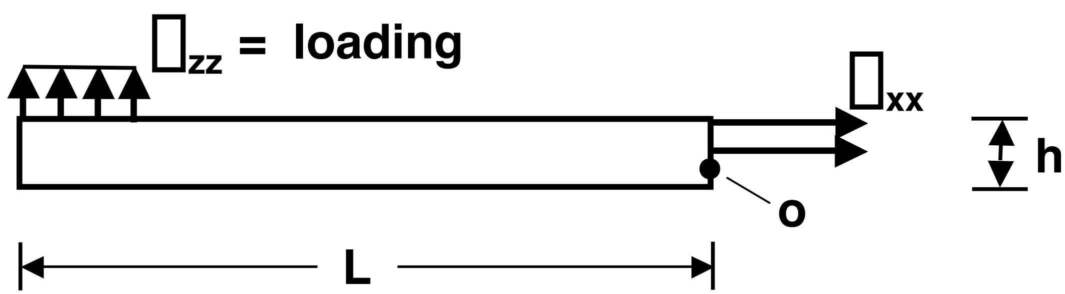
$$\sigma_{yy} = \sigma_{xy} = \sigma_{yz} = 0$$

--> Next, we "assume" that the only significant stresses are in the x-direction.

$$\Rightarrow \quad \sigma_{xx}, \sigma_{xz} >> \sigma_{zz}$$

--> Why (valid)? Look at isolated element and moment equilibrium

Figure M4.4-2 **Illustration of moment equilibrium of "isolated element" of beam**



$$\Sigma M_0 \text{ (magnitude)} \Rightarrow \sigma_{zz} \text{ (moment arm)} + \sigma_{xx} \text{ (moment arm)} = 0$$

$$\text{moment arm for } \sigma_{zz} \sim L; \quad \text{moment arm for } \sigma_{xx} \sim h$$

$$\text{but, } h << L \Rightarrow \sigma_{xx} >> \sigma_{zz}$$

can make same argument for σ_{xz}

--> thus, assumption is only non zero stresses are σ_{xz} and σ_{xx}

$$\Rightarrow \sigma_{zz} \approx 0$$

To complete our model, we need….

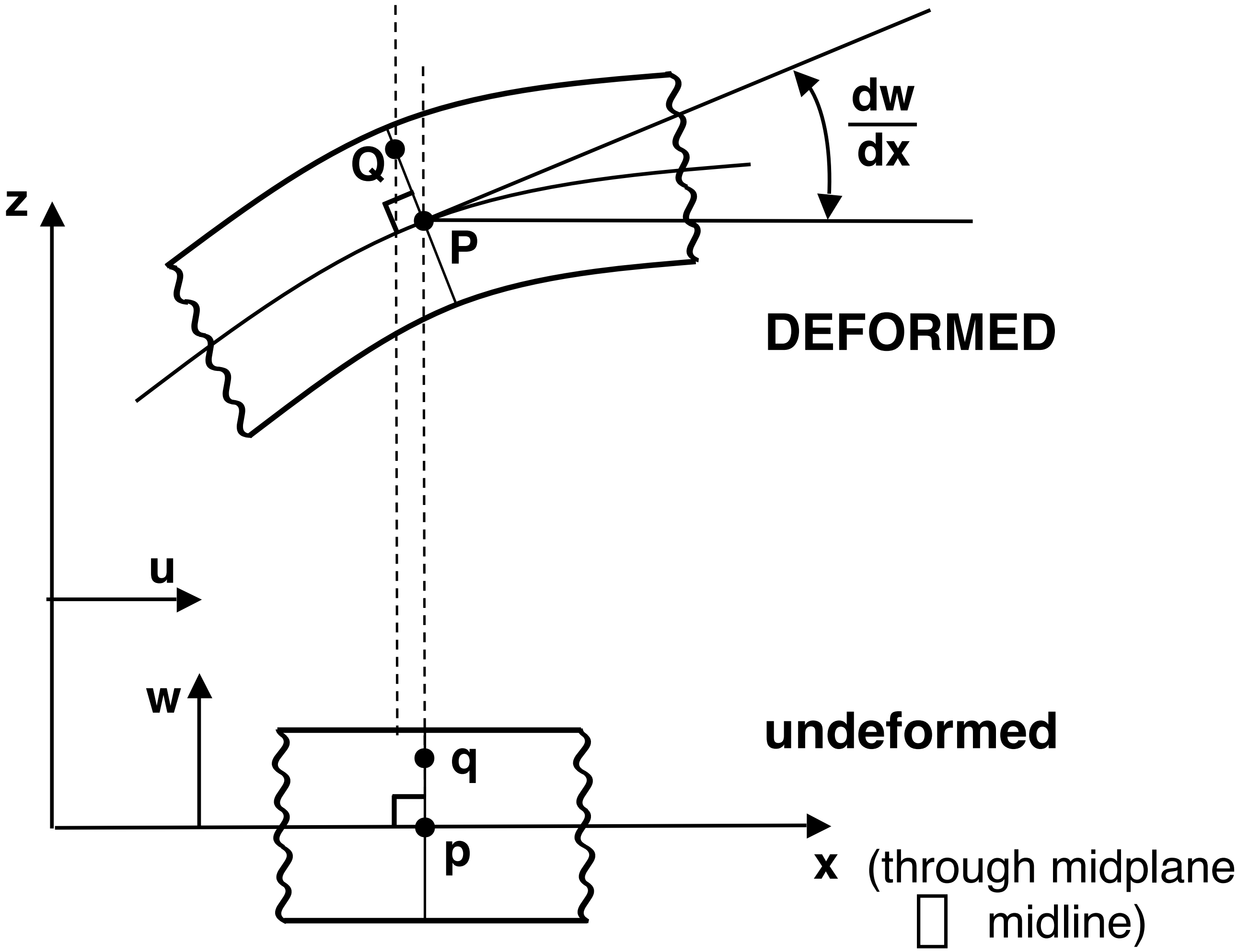
Assumptions on Deformations

The key here is the "Bernouilli-Euler Hypothesis" (~1750):

"Plane sections remain plane and perpendicular to the midplane after deformation"

--> To see what implications this has, consider an infinitessimal element that undergoes bending (transverse) deformation:

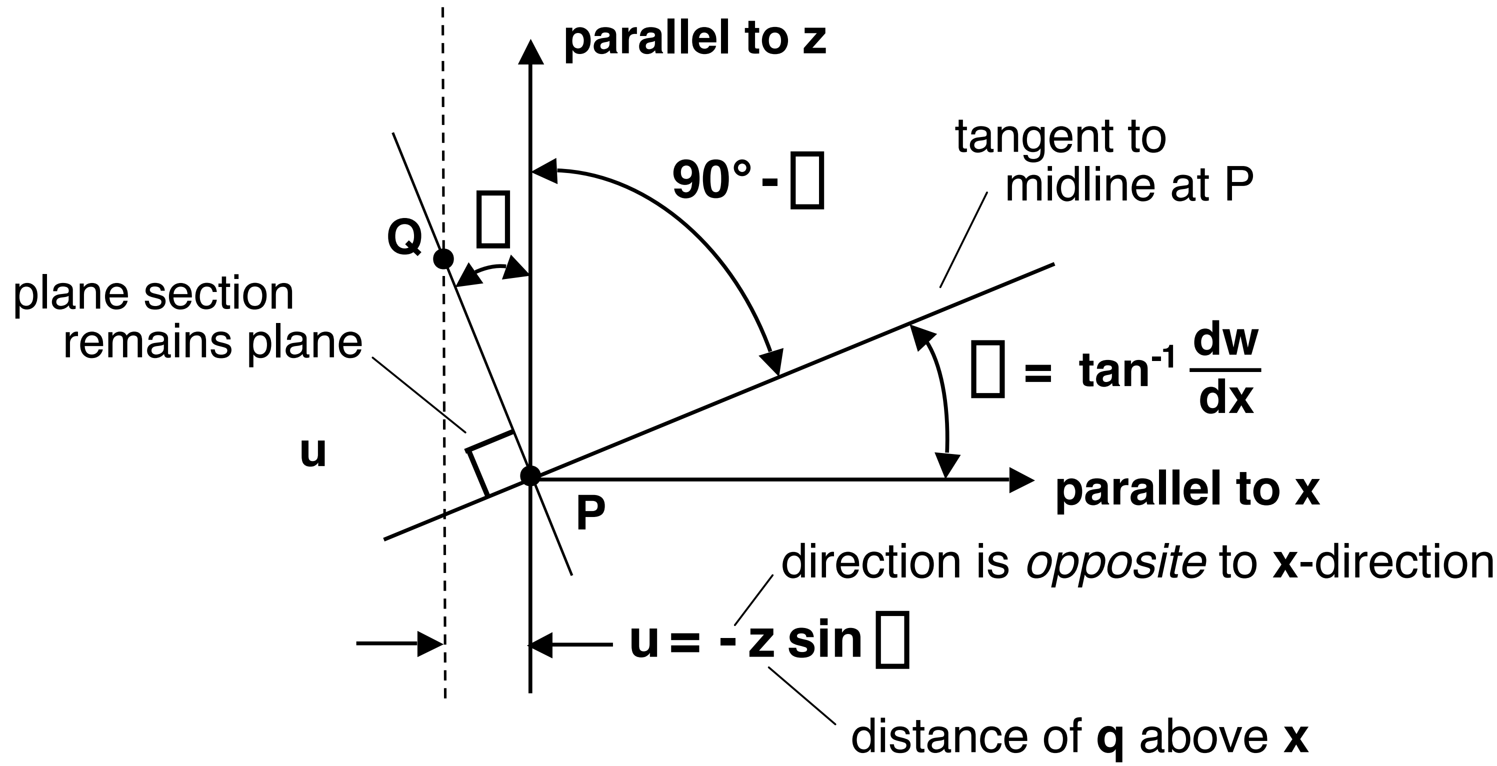
Figure M4.4-3 Basic deformation of infinitessimal element to beam according to “plane sections remains plane” (Bernouilli-Euler Hypothesis)



<u>Define</u>: w = deflection of midplane/midline (function of x only)

Use geometry to get deflection in x-direction, u, of point q (q to Q)

Figure M4.4-4 **Local geometry of deflection of any point of beam**



if deformations/angles are small: $\sin\theta \approx \theta$; $\theta \approx \dfrac{\partial w}{dx}$

$\Rightarrow \quad u \approx -z\theta$

Thus, implication of assumption on displacement is:

$$u\,(x,\ y,\ z) \ = \ -z\,\frac{dw}{dx} \qquad (1)$$

$$v\,(x,\ y,\ z) \ = \ 0$$

(nothing in y-direction)

$$w\,(x,\ y,\ z) \ = \ w(x) \qquad (2)$$

(cross-section deforms as a unit) $\Rightarrow$ *(plane sections remain plane)*

We have all the necessary assumptions as we have the structural member via assumptions on geometry, stress, and displacements/ deformations. We now use the Equations of Elasticity to get the….

Resulting Equations

--> First apply the Strain-Displacement Equations….

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{d^2 w}{dx^2} \qquad (3)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\frac{dw}{dx} + \frac{dw}{dx}\right) = 0 \text{ !!!}$$

⇒ This is consistent with assumption by B-E (no shearing gives plane sections remain plane and perpendicular)

--> Next use stress-strain.
We'll go to orthotropic as most general we can do

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} \qquad (4)$$

$$\varepsilon_{yy} = -\nu_{xy} \frac{\sigma_{xx}}{E_x}$$

$$\varepsilon_{zz} = -\nu_{xz} \frac{\sigma_{xx}}{E}$$

<u>Note</u>: "slight" inconsistency between assumed displacement state and those resulting strains, and the resulting strains from the stress-strain equations

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G_{xy}} = 0$$

$$\varepsilon_{yz} = \frac{\sigma_{yz}}{2G_{yz}} = 0$$

$$\varepsilon_{xz} = \frac{\sigma_{xz}}{2G_{xz}} \neq 0$$

<u>Note</u>: *again a "slight" inconsistency*

We "get around" these inconsistencies by saying that σ_{yy}, σ_{zz}, and ε_{xz} are very small but not quite zero. This is an approximation (part of model). Will check this later.

--> Finally use the Equilibrium Equations:

Assumption: no body forces ($f_i = 0$)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \qquad (5)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \quad \Rightarrow \quad 0 = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad (6)$$

--> So we have 5 unknowns: w, u, ε_{xx}, σ_{xx}, σ_{xz}

(Note: σ_{zz} is ignored)

--> And we have 5 equations: 1 from geometry: (1)

1 from strain-displacement: (3)
1 from stress-strain: (4)
2 from equilibrium: (5), (6)

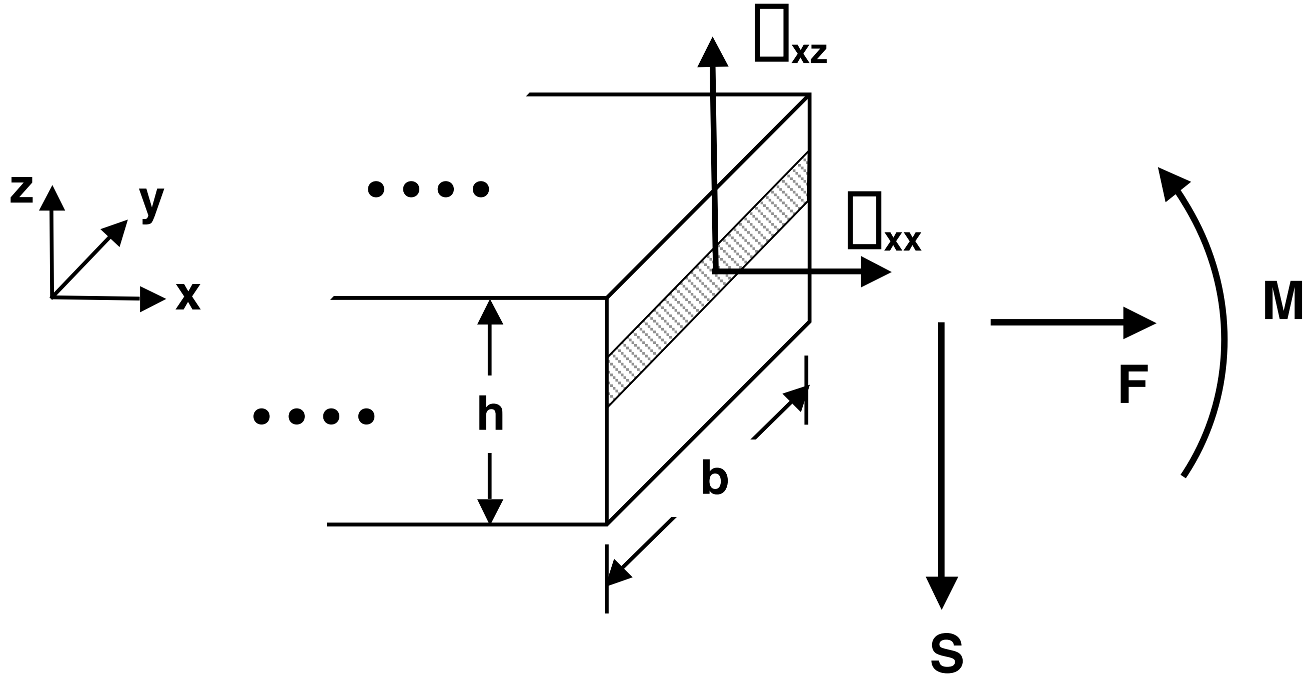
So then we have the right number of equations for the number of unknowns. So we consider the:

Solution: Stresses and Deflections

In doing this, it is first important to relate the point-by-point stresses to the average internal forces (F, S, M).

To do this, consider a cut face (do here for rectangular cross-section; will generalize later)

Figure M4.4-5 **Geometry of Equilibrium via stresses on cut face of beam**



Equilipollence (i.e., equally powerful) shows: (no variation in y)

$$F = \int_{-h/2}^{h/2} \sigma_{xx} \, b dz \qquad (7)$$

$$S = -\int_{-h/2}^{h/2} \tau_{xz} \, b dz \qquad (8)$$

$$M = -\int_{-h/2}^{h/2} \sigma_{xx} \, bz dz \qquad (9)$$

--> Now we begin substituting the various equations…

Put (1), (3) in (4) to get:

$$\sigma_{xx} = E_x \, \varepsilon_{xx} = -E_x \, z \, \frac{d^2 w}{dx^2} \qquad (10)$$

Now put this in (7):

$$F = -E_x \frac{d^2 w}{dx^2} \int_{-h/2}^{h/2} zb dz$$

$$= -E_x \frac{d^2 w}{dx^2} \left[\frac{z^2}{2} \, b \right]_{-h/2}^{h/2} = 0$$

since <u>no</u> <u>axial</u> <u>force</u> in <u>*pure beam*</u> case

(Note: something that carries axial and bending forces is known as a beam-column/rod)

--> we also place the result for σ_{xx} (10) in the equation for the internal moment (9):

$$M = E_x \frac{d^2 w}{dx^2} \int_{-h/2}^{h/2} z^2 b\,dz$$

we define:

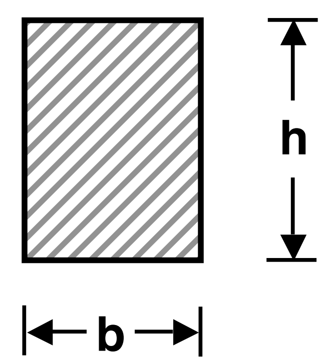
$$\boxed{I = \int_{-h/2}^{h/2} z^2 b\,dz} \quad (11)$$

= Area (Second) Moment of Inertia of beam cross-section [about y-axis]

units of $[L^4]$

Note: For rectangular cross-section

$$I = \frac{bh^3}{12}$$



--> will look at this further in next unit

This results in the following:

$$\boxed{M = E_x \, I \, \frac{d^2 w}{dx^2}} \qquad (12)$$

Moment-Curvature relation for beam

Note: EI is controlling parameter - "*flexural rigidity*" or "*bending stiffness*". Has:
- geometrical contribution, I
- material contribution, E
- units: $\left[F \bullet L\right] = \left(\frac{F}{L^2}\right)\left[L^4\right]\left(\frac{L}{L^2}\right)$

--> Can also relate the internal shear, S, to these parameters. Use equation (5):

$$\frac{\partial \sigma_{zx}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} \qquad (5)$$

Multiply each side by b and integrate from z to h/2 to get:

$$\int_z^{h/2} b \, \frac{\partial \tau_{zx}}{\partial z} \, dz = -\int_z^{h/2} \frac{\partial \sigma_{xx}}{\partial x} \, b dz$$

First take (12) and put it in (10):

$$\sigma_{xx} = -E_x z \, \frac{d^2 w}{dx^2} = -E_x \, z \, \frac{M}{E_x I}$$

$$\Rightarrow \quad \boxed{\sigma_{xx} = -\frac{Mz}{I}} \qquad (13)$$

Units: $\left[\frac{F}{L^2}\right] = \frac{[FL][L]}{L^4}$

Now, work on integrating the pending equation:

$$- \; b \, \tau_{xz}(z)\Big]_z^{h/2} = -\int_z^{h/2} \left(-\frac{\partial M}{\partial x}\right) \frac{zb}{I} \, dz$$

Recall that: $\dfrac{dM}{dx} = S$ to get:

$$\Rightarrow \quad b\left[\tau_{xz}\left(\frac{h}{2}\right) - \tau_{xz}(z)\right] = +\int_{z}^{h/2} S \frac{zb}{I}\, dz$$

Note that the τ_{xz} at the top of surface is zero.

Also define:

$$\boxed{Q = \int_{z}^{h/2} zb\,dz} \quad = \text{(first) Moment of area about the center} \qquad (14)$$

So:

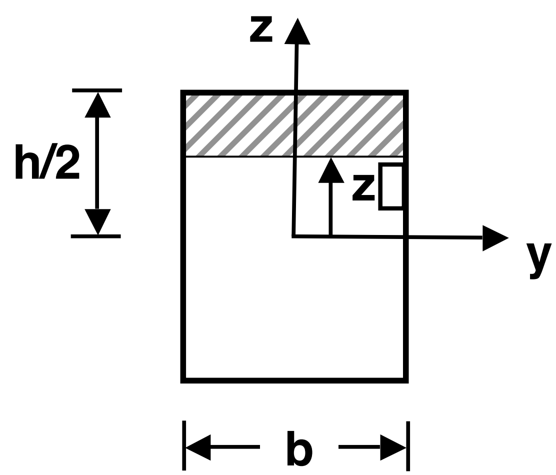
$$\boxed{\tau_{xz}(z) = -\frac{SQ}{Ib}} \qquad (15)$$

shear stress-Shear relation

Units: $\left[\frac{F}{L^2}\right] = \frac{[F]\left[L^3\right]}{\left[L^4\right][L]}$

For a rectangular section:

Figure M4.4-6 **Geometry for assessing (first) moment of area about centerline**



$$Q = \int_{z'}^{h/2} z\,b\,dz$$

$$= \left[\frac{z^2}{2} b\right]_{z'}^{h/2} = \frac{b}{2}\left[\frac{h^2}{4} - z'^2\right]$$

(maximum at $z' = 0$, *the centerline*)

--> Again, will look at this further and generalize in the next unit

The summary of how we can solve for the stress/strain/displacement states in a beam is presented in handout M-5

In the next section, we look at what this solution generally means and examine it for various situations.

Unit M4.4 (New) Nomenclature

EI -- *flexural rigidity* or *boundary stiffness* of beam cross-section
I -- Area (Second) Moment of Inertia of beam cross-section (about y-axis)
Q -- (First) Moment of area above the centerline
u -- deflection of point of beam in x-direction
v -- deflection of point of beam in y-direction
w -- deflection of (midpoint/midline of) beam in z-direction
ϕ -- slope of midplane of beam at any point x (= dw/dx)
d^2w/dx^2 -- curvature of beam (midplane/midline) at any point x of beam
σ_{xx} -- beam bending stress
τ_{xz} -- beam transverse shear stress