Unit M4.6
Torsion of Rods/Shafts

Readings:
CDL 6.1-6.5

16.003/004 -- "Unified Engineering"
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LEARNING OBJECTIVES FOR UNIT **M4.6**

Through participation in the lectures, recitations, and work associated with Unit M4.6, it is intended that you will be able to………

- ….**describe** the key aspects composing the model of a (torsional) shaft and **identify** the associated limitations
- ….**apply** the basic equations of elasticity to **derive** the solution for the general case
- ….**identify** the parameters that characterize torsional behavior and **describe** their role

Thus far we've considered a long slender member under axial load (rod) and bending load (beam). Let's now look at a long slender member subjected to a torque. This is a shaft.

Let's begin with the…

Definition of a Shaft

*A **shaft** is a structural member that is long and slender and subjected to a torque moment about its long axis.*

Consider each of the three points that make up the definition and the true reality…

--> Modeling Assumptions

a) Geometry

(go back to indicial notation because it makes it easier to manipulate)

Figure M4.6-1 **General geometry of shaft**

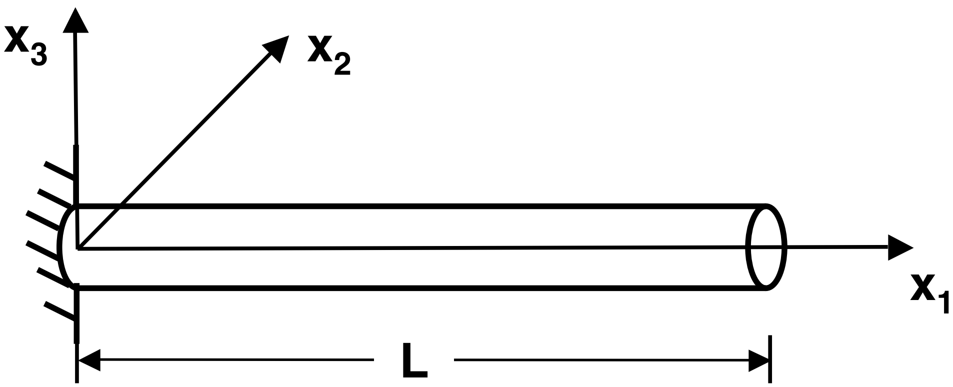
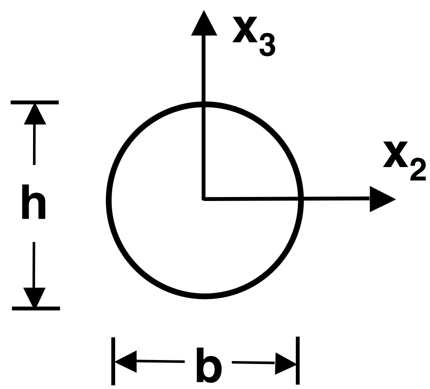


Figure M4.6-2 **Cross-section of shaft**



b, h = largest dimensions in x_2 and x_3

Assumption: "long" in x_1-direction

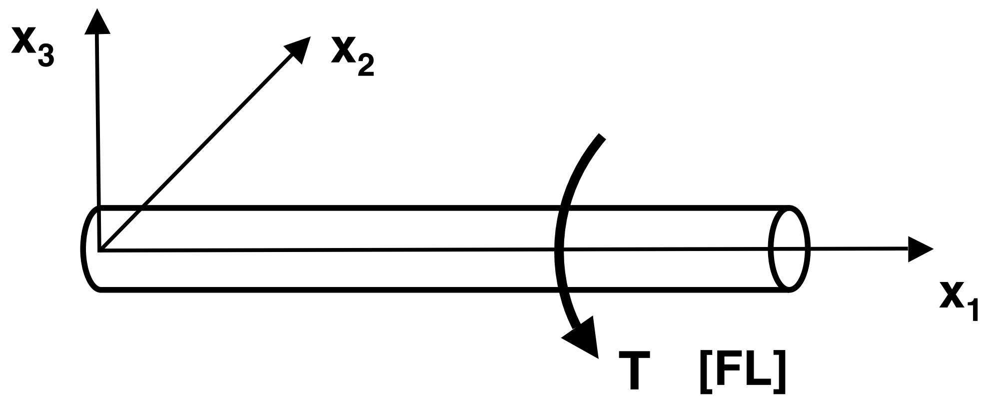
L > > b, h (slender member)

Note: same as before

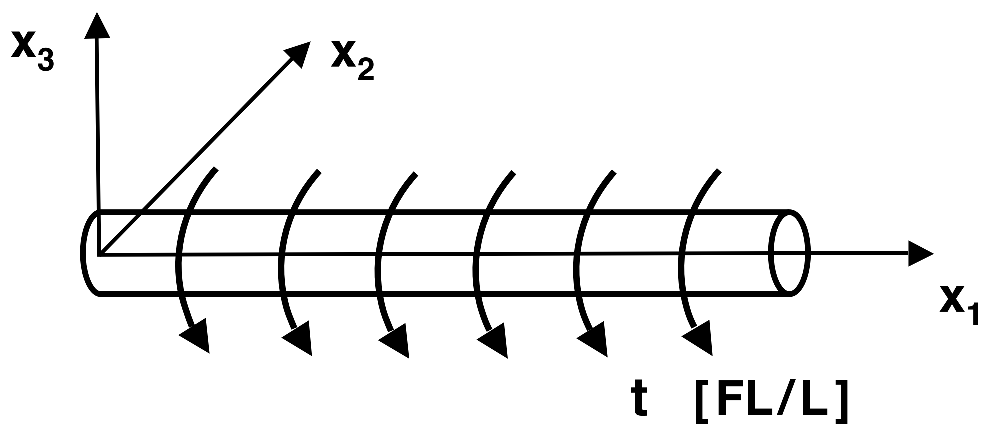
b) Loading

Assumption: Torque Moment about x_1-direction

- concentrated T



- distributed



No axial loads

$\Rightarrow$ at boundaries: $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$

finally look at:

c) <u>Deformation</u>

<u>Assumptions</u>:

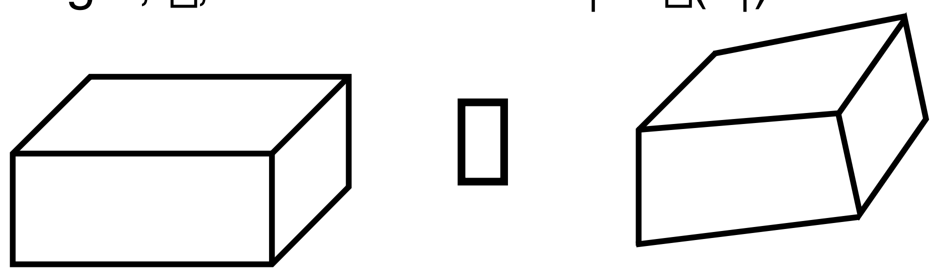
- At any location x_1, the cross-section rotates as a rigid body (□ no distortion of cross-section)

  <u>Note</u>: can also say "plane sections remain plane and perpendicular to midline"

- No deformation of cross-section in x_1-direction (no bending or extension)

□ Only deformation is rotation of cross-section through a twist angle.

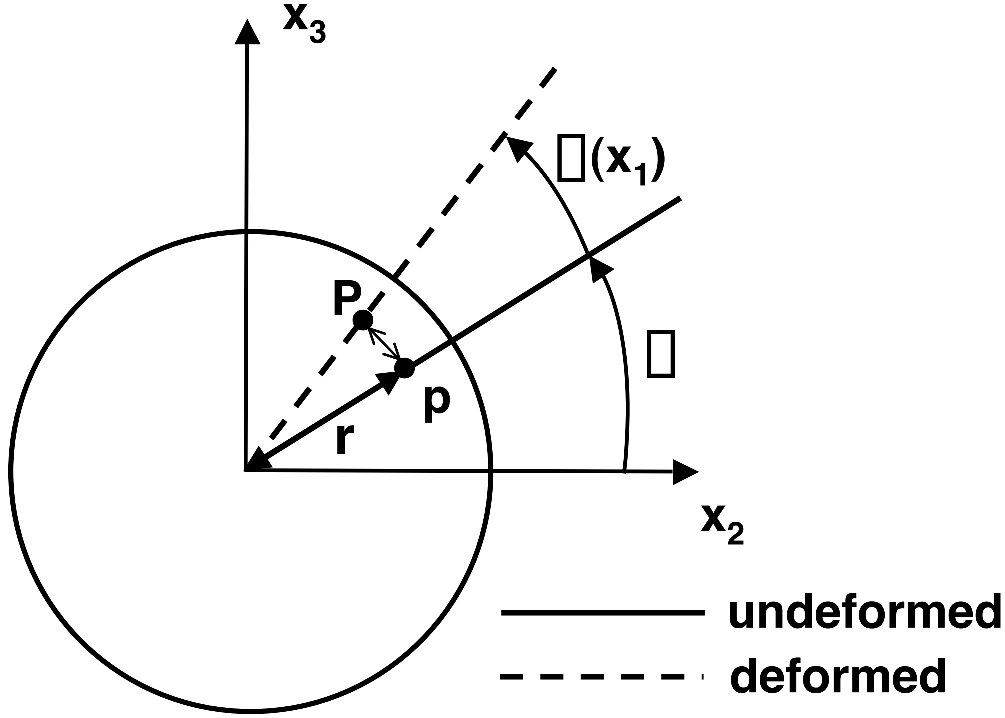
Define twist angle, □, as function of x_1 = □(x_1). Think of deck of cards:



We can, by geometry, relate the deformations, u_i, to the twist/rotation angle □(x_1).

Consider a cross-section at location x_1, and a point in a circular cross-section at angle □ from the reference axes.

Figure M4.6-2 **Illustration of deformation of shaft cross-section**

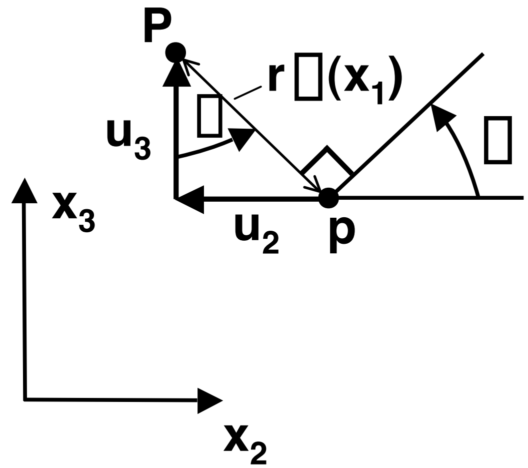


- Distance point p rotates (p to P) = r sin □(x_1)
- For small angles (<u>*assumed*</u> *here*)

  □ distance = r □(x_1)

Resolve into components along x_2 and x_3

Figure M4.6-3 **Resolution of deformation into components along x_2 and x_3**

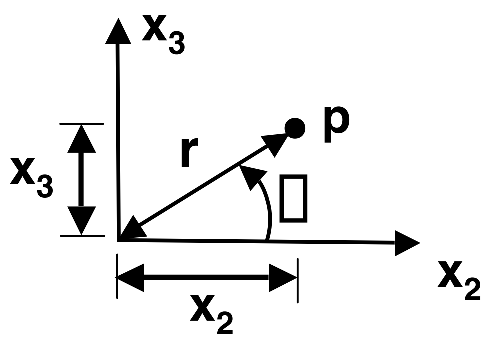


□ $u_2 = -r\,□(x_1)\sin □$

note direction

$u_3 = +r□(x_1)\cos □$

And □ is defined by the x_2 and x_3 location of p



$$r = \sqrt{x_2^2 + x_3^2}$$

$$\sin \square = \frac{x_3}{r}$$

$$\cos \square = \frac{x_2}{r}$$

So:

$$u_2 = \square \sqrt{x_2^2 + x_3^2} \; \square(x_1) \frac{x_3}{\sqrt{x_2^2 + x_3^2}} = \square\square(x_1) \, x_3$$

$$u_3 = \sqrt{x_2^2 + x_3^2} \; \square(x_1) \frac{x_2}{\sqrt{x_2^2 + x_3^2}} = \square(x_1) \, x_2$$

And we have, by assumption, no axial displacement. So the assumed displacement state is:

$$u_1 = 0 \qquad (1)$$

$$u_2 = \square\square(x_1) x_3 \qquad (2)$$

$$u_3 = \square(x_1) x_2 \qquad (3)$$

Let's now use the definitions in the equations of elasticity to get the…

Governing Equations

--> Strain-Displacement

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

consistent with assumption that cross-section does not deform in x

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$$

consistent with assumption that cross-section does not distort

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -\frac{1}{2}\, x_3 \frac{d\phi}{dx_1} \qquad (4)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\, x_2 \frac{d\phi}{dx_1} \qquad (5)$$

<u>Note</u>: $\frac{\partial}{\partial x_1} \Rightarrow \frac{d}{dx_1}$ since ϕ is a function of x_1 <u>only</u>

(partial) (total)

<u>Finally</u>: $$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(-\phi(x_1) + \phi(x_1)\right) = 0$$

$\Rightarrow$ also is consistent with assumption that cross-section does not distort

Next go to…

--> <u>Stress</u>-<u>Strain</u> <u>Equations</u>

(do for <u>isotropic</u> $\Rightarrow$ only one shear modulus)

Since ε_{11}, ε_{22}, and $\varepsilon_{33} = 0$ $\Rightarrow$ σ_{11}, σ_{22}, $\sigma_{33} = 0$

(**<u>consistent</u>**: no axial stresses)

$$\varepsilon_{23} = \frac{\sigma_{23}}{2G} = 0 \Rightarrow \sigma_{23} = 0$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} \qquad (6)$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{2G} \qquad (7)$$

$\Rightarrow$ only σ_{12} and σ_{13} are present

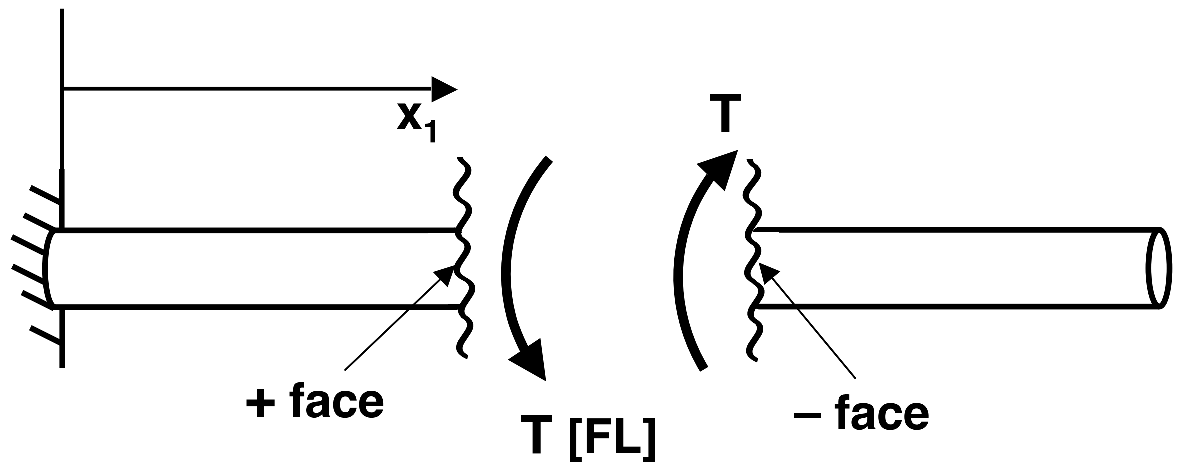
Finally we look at…

--> Equilibrium Equations

First we again define an internal stress resultant for the structural configuration. In this case, it will be the torque moment at any point.

Cutting the shaft……

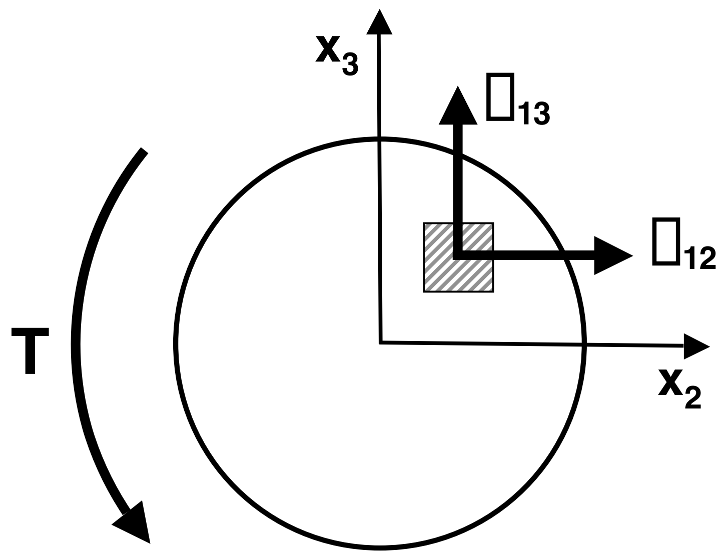
Figure M4.6-4 **Illustration of cutting shaft through cross-section and considering internal torque**



+ ***right hand rule*** -- gives equal and opposite

Express **T** in terms of the stress:

Figure M4.6-5 **Illustration of equipollence consideration for shaft cross-section**



equivalence/equipollence:

- Torque: $T = \iint \left(x_2 \sigma_{13} - x_3 \sigma_{12} \right) dx_2 dx_3$ (8)

So using equations of equilibrium (considering only non zero stresses)

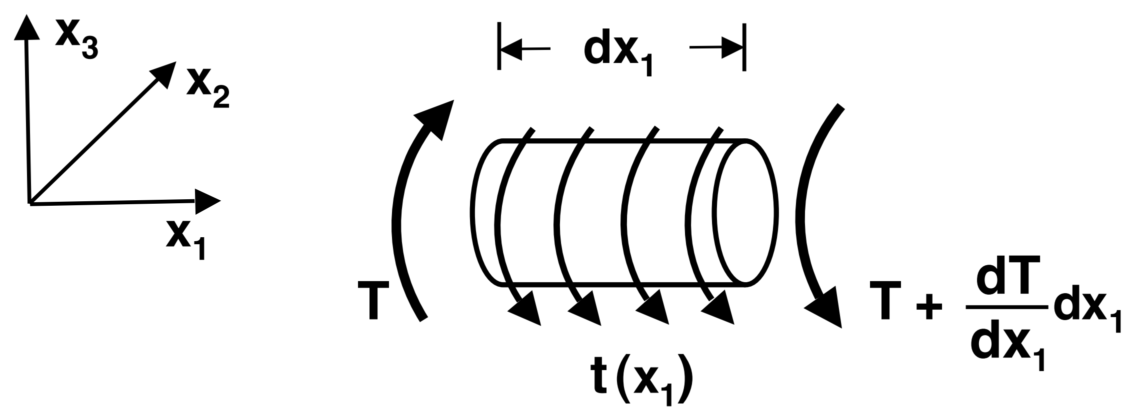
$$\frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \cancelto{0}{f_1} = 0 \qquad (9)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + f_2 = 0 \qquad (10)$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + f_3 = 0 \qquad (11)$$

Now look at the equilibrium of a discrete segment (as we have in the past)

Figure M4.6-6 **Geometry for consideration of equilibrium of a discrete segment**



$$\sum M_{x_1} = 0 \quad \circlearrowleft + \quad \Rightarrow \quad -T + t(x_1)dx_1 + T + \frac{dT}{dx_1}dx_1 = 0$$

$$\Rightarrow \quad \boxed{\frac{dT}{dx_1} = -t(x_1)} \qquad (12)$$

"Torque-loading Relation"

$$\left(\text{like } \frac{dS}{dx} = q(x)\right)$$

Note: Can show average equilibrium relations [resulting in (12)] are consistent with pointwise relations [(9) - (11)].

We now have 6 unknowns (T, σ_{12}, σ_{13}, ε_{12}, ε_{13}, ϕ) and 6 equations { (4), (5), (6), (7), (8), (12) }. This allows us to solve the problem for this model.

So let's look at…..

Solution and Limitations of Model

Put equations (4) and (5) [Strain-Displacement] into the stress-strain equations (6) and (7):

$$\sigma_{12} = 2G\varepsilon_{12} = 2G\left(-\frac{1}{2}\, x_3\, \frac{d\phi}{dx_1}\right)$$

$$\Rightarrow \quad \sigma_{12} = -Gx_3\, \frac{d\phi}{dx_1} \qquad (13)$$

$$\sigma_{13} = 2G\varepsilon_{13} = 2G\left(\frac{1}{2}\, x_2\, \frac{d\phi}{dx_1}\right)$$

$$\Rightarrow \quad \sigma_{13} = Gx_2\, \frac{d\phi}{dx_1} \qquad (14)$$

Now place these results into the torque-stress equilibrium equation (8):

$$T = \iint \left(x_2 \sigma_{13} - x_3 \sigma_{12} \right) dx_2 dx_3$$

$$= \iint G \left[x_2^2 \frac{d\phi}{dx_1} + x_3^2 \frac{d\phi}{dx_1} \right] dx_2 dx_3$$

$$\Rightarrow \quad T = G \frac{d\phi}{dx_1} \iint \left(x_2^2 + x_3^2 \right) dA$$

<u>Define</u>:

$$\boxed{J \equiv \iint \left(x_2^2 + x_3^2 \right) dA} \qquad (15)$$

= <u>polar</u> (second) moment of inertia

$$= \frac{\pi R^4}{2} \quad \text{for circle}$$

So we write:

$$\boxed{T = GJ \frac{d\phi}{dx_1}} \qquad (16)$$

“Torque-Twist” relation

Note, again, overall “structural constitutive relation”

$$T = GJ \frac{d\phi}{dx_1}$$

(load) = (stiffness)(deformation)

Structural stiffness here is torsional stiffness = GJ

composed of two parts:

G - material contribution/parameter
J - geometrical contribution/parameter

similar to bending:

$$M = E_x I \frac{d^2 w}{dx^2}$$

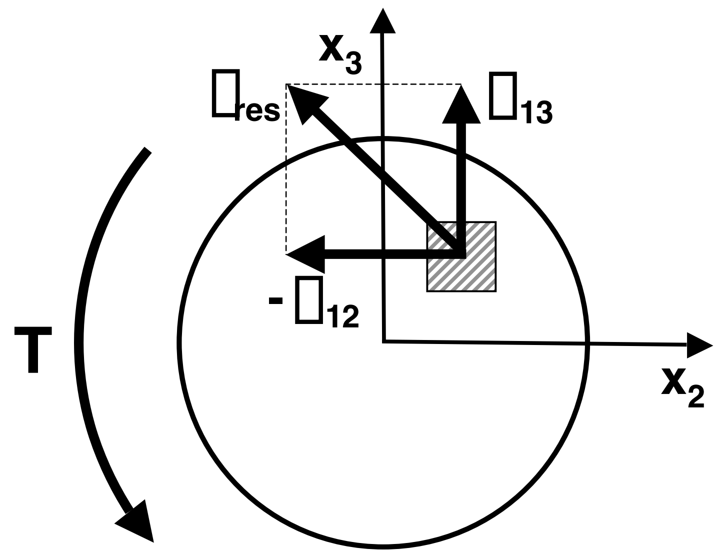
We can use the result $G \frac{d\phi}{dx_1} = \frac{T}{J}$ in equations (13) and (14) to relate stress to torque:

$$\sigma_{12} = -\frac{Tx_3}{J} \qquad (17)$$

$$\sigma_{13} = \frac{Tx_2}{J} \qquad (18)$$

Finally, can express the stress as a shear stress resultant:

Figure M4.6-7 **Illustration of shear stress resultant**



$$\sigma_{res} = \sqrt{\sigma_{12}^2 + \sigma_{13}^2}$$

$$= \frac{T}{J}\underbrace{\sqrt{x_3^2 + x_2^2}}_{= \, r}$$

$$\boxed{\tau_{res} = \frac{Tr}{J}} \qquad (19)$$

Note similarity to bending:

$$\sigma_{xx} = -\frac{Mz}{I}$$

form:

$$(\text{stress}) = \frac{\begin{bmatrix}\text{primary} \\ \text{loading}\end{bmatrix}\begin{bmatrix}\text{distance from} \\ \text{key point}\end{bmatrix}}{\begin{bmatrix}\text{geometrical} \\ \text{parameter}\end{bmatrix}}$$

Use of Model

Very similar to rod, beam…

1. Draw Free Body Diagram, determine reactions
2. Get internal stress resultant T(x_1)
3. Determine section property G J

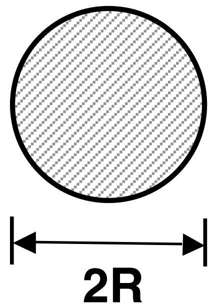
   (G: material constant; J: polar moment of inertia)
4. Use equation (16) to find rate of twist $\frac{d\phi}{dx_1}$
5. Use equations (17), (18), and (19) to determine stresses
6. Determine strains and displacements as needed

let's think about the…

--> <u>Limitations of the model</u>

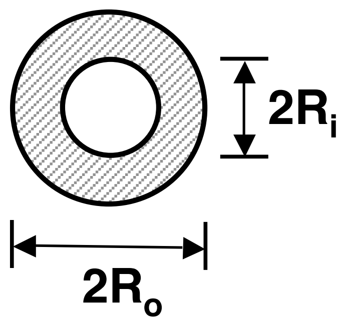
The assumptions give us an exact solution for circular closed cross-sections:

<u>Solid</u>



$$J = \frac{\pi R^4}{2}$$

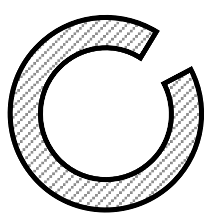
<u>Tube</u>



$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2}$$

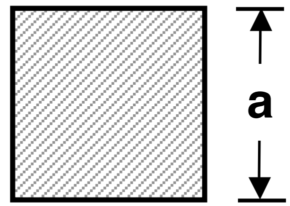
by <u>superposition</u>

<u>Not</u> good for an <u>open</u> section, for example…



Approximate for other closed sections:

e.g., **Square**



$$J = 0.141a^4$$

(assumption of no deformation of cross-section violated --> "warping")

[more in 16.20]

We'll next look at a rod under compression and look at an instability phenomenon known as "buckling". In this case, we call the structural member a **column**.

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Unit M4.6 (New) Nomenclature

G -- shear modulus (isotropic material)
GJ -- torsional stiffness
J -- polar (second) moment of inertia
R_o -- outer radius
R_i -- inner radius
T -- applied point torque load
t -- applied distributed torque load
□$_{res}$ -- shear stress resultant
□ -- twist angle