Unit M4.7
The Column and Buckling

Readings:

CDL 9.1 - 9.4

CDL 9.5, 9.6

16.003/004 -- "Unified Engineering"
Department of Aeronautics and Astronautics
Massachusetts Institute of Technology

LEARNING OBJECTIVES FOR UNIT **M4.7**

Through participation in the lectures, recitations, and work associated with Unit M4.7, it is intended that you will be able to………

- ….**explain** the concepts of stablity, instability, and bifurcation, and the issues associated with these
- ….**describe** the key aspects composing the model of a column and its potential buckling, and **identify** the associated limitations
- ….**apply** the basic equations of elasticity to **derive** the solution for the general case
- ….**identify** the parameters that characterize column behavior and **describe** their role

We are now going to consider the behavior of a rod under **compressive** loads. Such a structural member is called a ***column***. However, we must first become familiar with a particular phenomenon in structural behavior, the…..

Concept of Structural Stability/Instability

Key item is transition, with increasing load, from a **stable** mode of deformation (stable equilibrium for all possible [small] displacements/ deformations, a restoring force arises) to an **unstable** mode of deformation resulting in collapse (loss of load-carrying capability)

Thus far we have looked at structural systems in which the stiffness and loading are separate…..

System	Stiffness	Deflection		Load
Rod	EA	$\frac{du}{dx_1}$	=	P
Beam	EI	$\frac{d^2 w}{dx^2}$	=	M
Shaft	GJ	$\frac{d\square}{dx}$	=	T
General	k	x	=	F

There are, however, systems in which the *effective structural stiffness* depends on the loading

Define: *effective structural stiffness* (k) is a linear change in restoring force with deflection

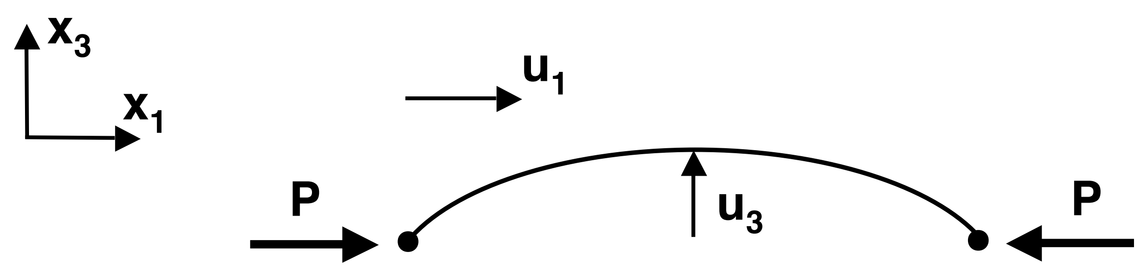
that is:
$$\boxed{\frac{dF}{dx} = k}$$

Examples

String (**stiffening**)

frequency changes with load and frequency is a function of stiffness

Ruler/pointer (**destiffening**)



easier to push in x_1, the more it deflects in u_3

--> From these concepts we can define a static (versus dynamic such as flutter -- window blinds) instability as:

"A system becomes unstable when a negative stiffness overcomes the natural stiffness of the structural system"

that is there is a

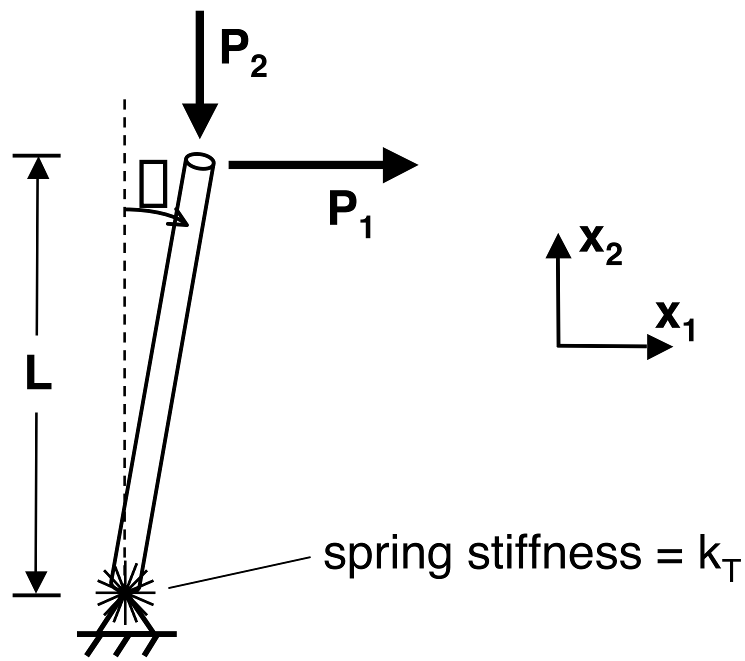
"loss of natural stiffness due to applied loads"

--> Physically, the more you push it, it gives even more and can build on itself!

Let's make a simple model to consider such phenomenon….

--> Consider a rigid rod with torsional spring with a load along the rod and perpendicular to the rod

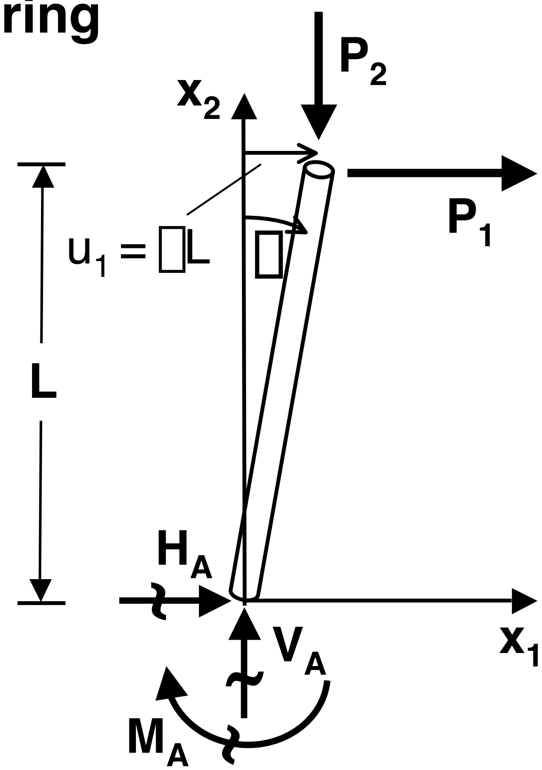
Figure M4.7-1 **Rigid rod attached to wall with torsional spring**



Restrict to small deflections (angles) such that $\sin\theta \approx \theta$

--> Draw Free Body Diagram

Figure M4.7-2 **Free Body Diagram of rigid rod attached to wall via torsional spring**



Use moment equilibrium:

$$\sum M(origin) = 0 \quad \circlearrowleft + \quad \Rightarrow \quad - P_1 L - P_2 L \sin\theta + k_T \theta = 0$$

$$k_T \theta = M_A$$

get: $\left[\frac{k_T - P_2 L}{L} \right] \theta = P_1$

(underbrace) → effective torsional stiffness

i.e., $k_{eff} \theta = P$

Note: load affects stiffness: as P_2 increases, k_{eff} decreases

Important value: if $P_2 L = k_T$

$\Rightarrow \quad k_{eff} = 0$

Point of "static instability" or "buckling"

$$P_2 = \frac{k_T}{L}$$

Note terminology: ***eigenvalue*** = value of load for static instability

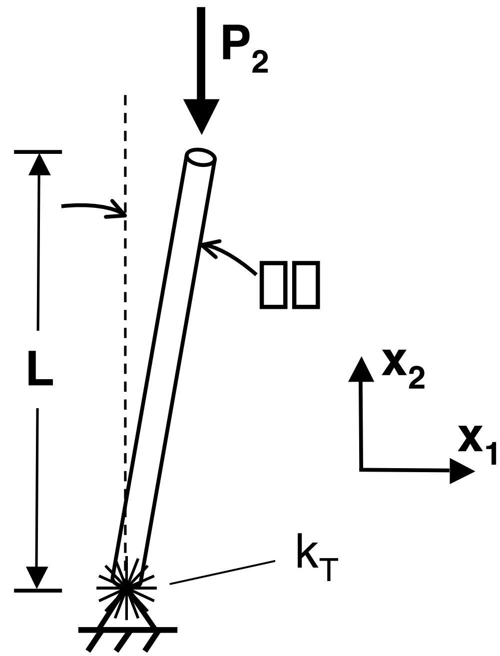
eigenvector = displacement shape/mode of structure *(we will revisit these terms)*

Also look at P_2 acting alone and "perturb" the system (give it a δ deflection; in this case $\delta\theta$)

stable: system returns to its condition

unstable: system moves away from condition

Figure M4.7-3 **Rod with torsional spring perturbed from stable point**



Sum moments to see direction of motion

$$\sum M \;(\circlearrowleft +) = -P_2 L \sin \Delta\theta + k_T \Delta\theta \sim \dot{\theta}$$

(proportional to change in θ)

$$\approx \left(k_T - P_2 L\right) \Delta\theta \sim \dot{\theta}$$

Note: + $\dot{\theta}$' is CCW (restoring)

− $\dot{\theta}$' is CW (unstable)

So: if $k_T > P_2 L$ □ ***stable*** and also get □ = 0

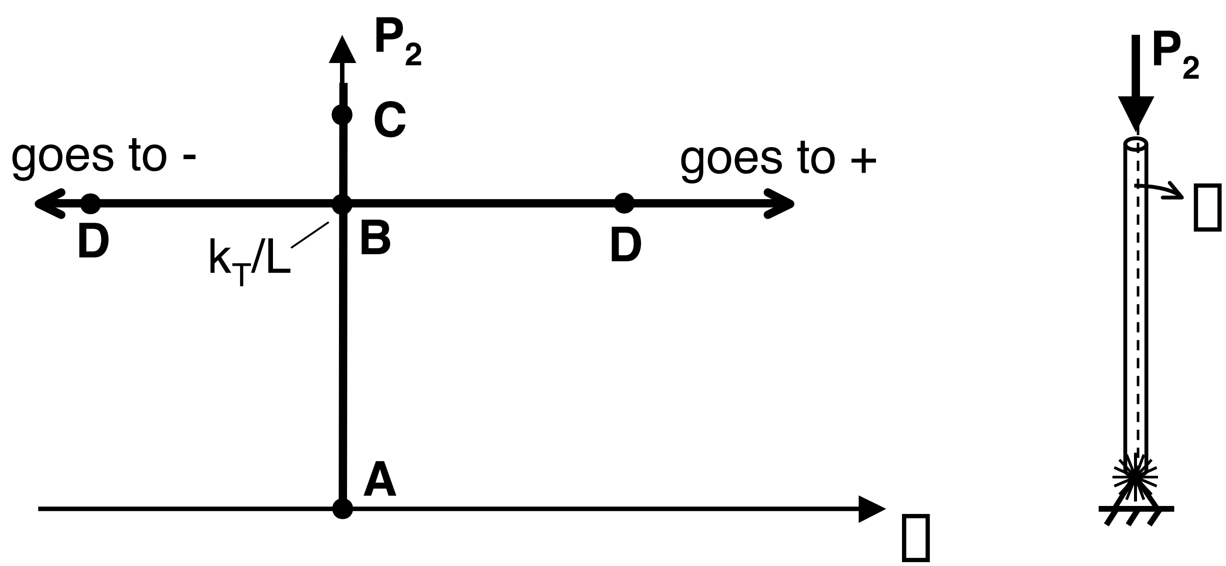
if $P_2 L \geq k_T$ □ ***unstable*** and also get □ =

critical point: $P_2 = \dfrac{k_T}{L}$

□ **spring cannot provide a sufficient restoring force**

--> so for P_2 acting alone:

Figure M4.7-4 **Response of rod with torsional spring to compressive load along rod**

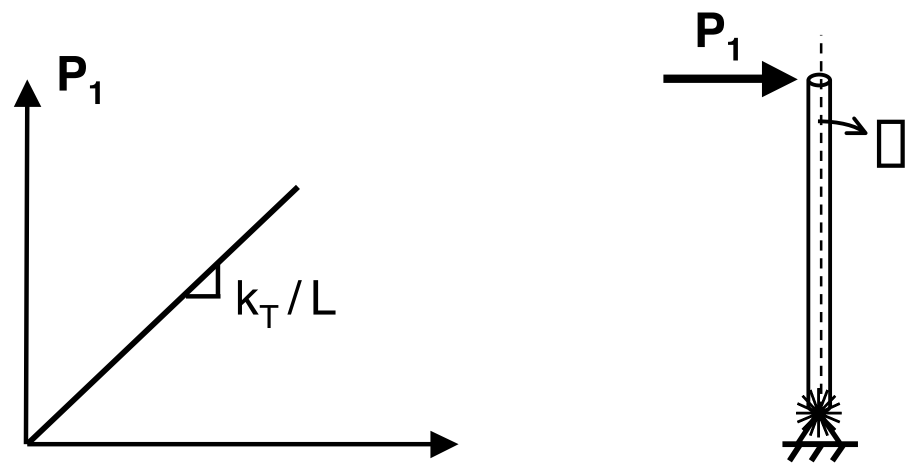


ABC - Equilibrium path, but not stable
ABD - Equilibrium path, deflection grows unbounded ("bifurcation") (B is bifurcation point, for simple model, …2 possible equilibrium paths)

Note: If P_2 is negative (i.e., upward), stiffness increases

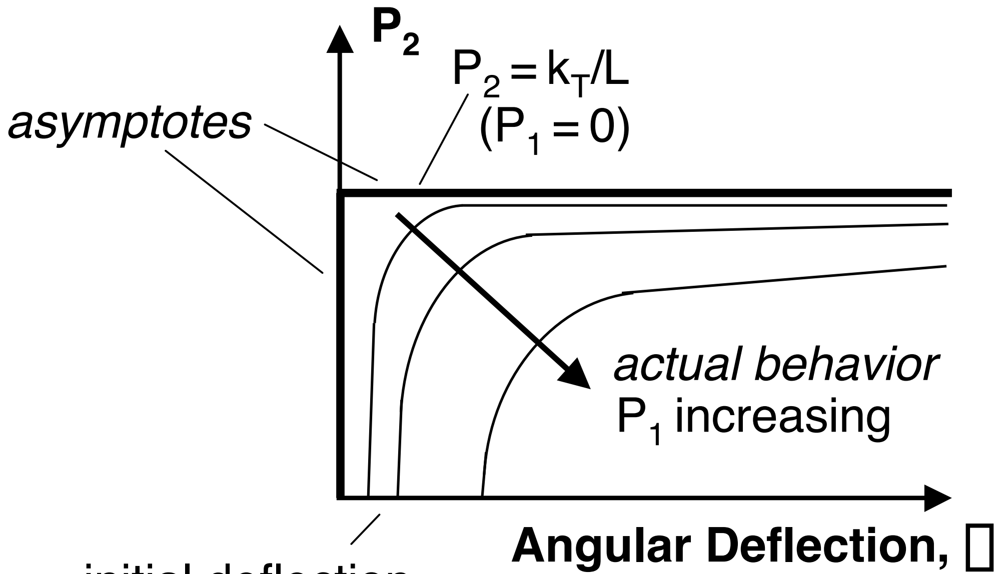
--> contrast to deflection for P_1 alone

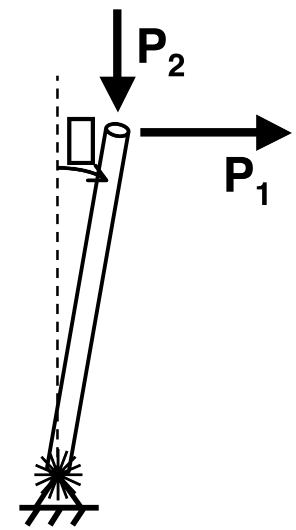
Figure M4.7-5 **Response of rod with torsional spring to load perpendicular to rod**



--> Now put on some given P_1 and then add P_2

Figure M4.7-6 **Response of rod with torsional spring to loads along and perpendicular to rod**





Note 1: If P_2 and P_1 removed prior to instability, spring brings bar back to original configuration (as structural stiffnesses do for various configurations)

Note 2: **Bifurcation** is a mathematical concept. The manifestations in actual systems are altered due to physical realities/imperfections. Sometimes these differences can be very important.

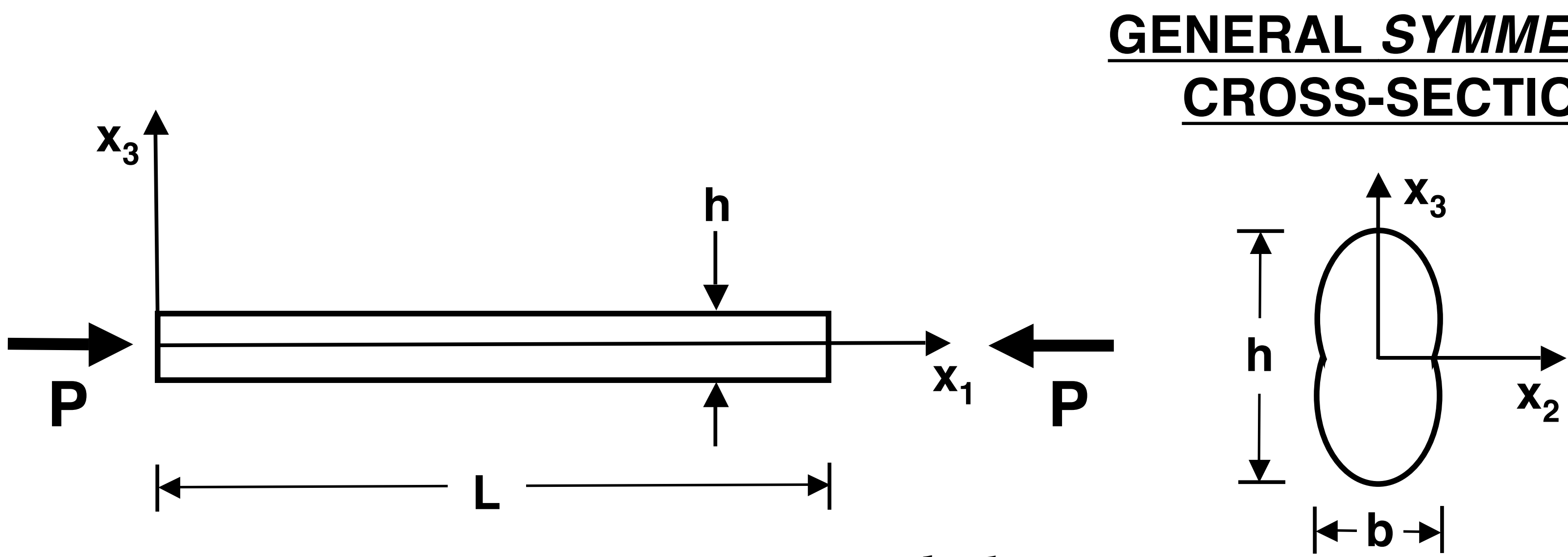
We'll touch on these later, but let's first develop the basic model and thus look at the….

Definition/Model of a Column

(Note: we include stiffness of continuous structure here. Will need to think about what is relevant structural stiffness here.)

a) Geometry - The basic geometry does not change from a rod/beam

Figure M4.7-7 **Basic geometry of column**



long and slender: $L >> b, h$

constant cross-section (assumption is EI = constant)

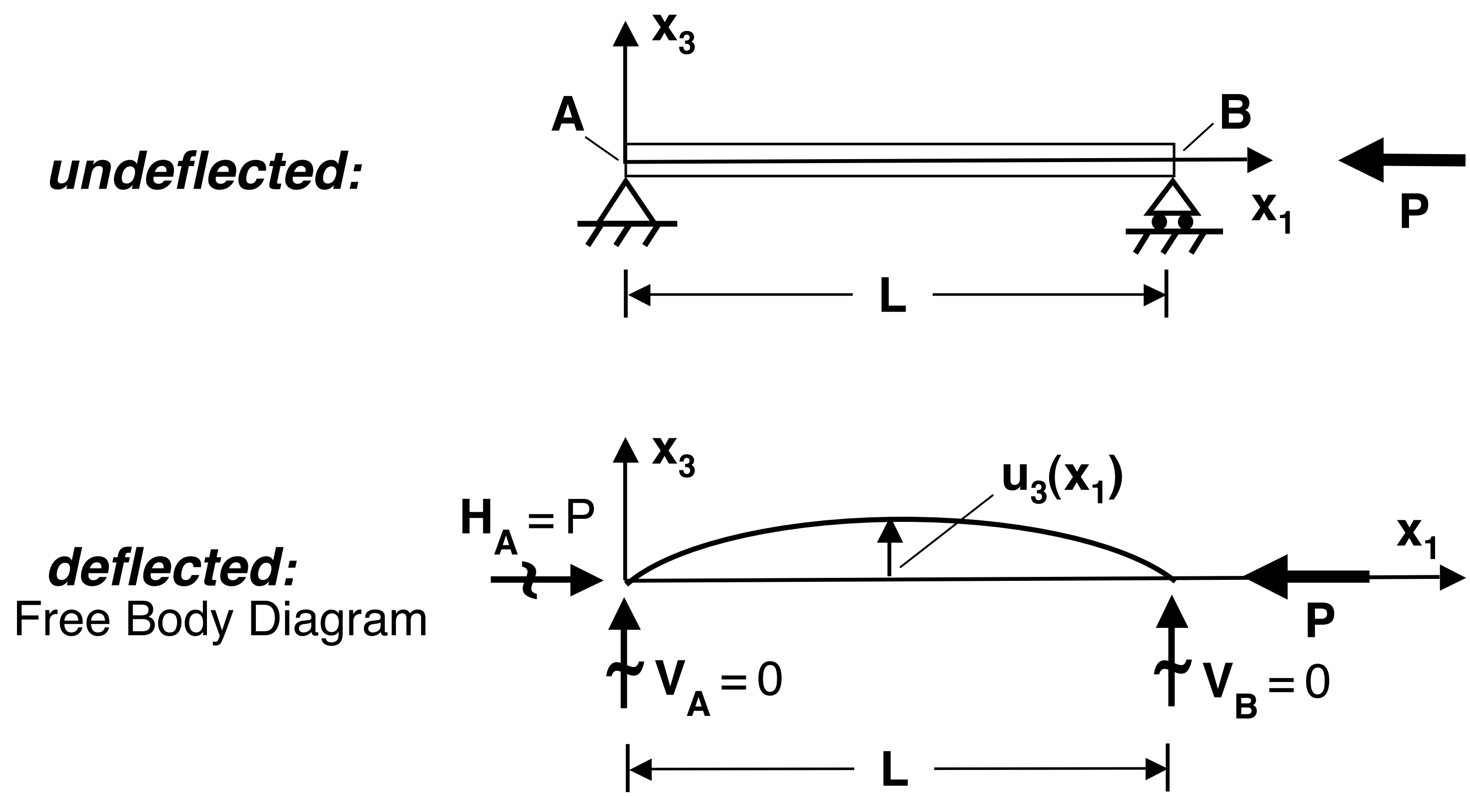
b) Loading - Unlike a rod where the load is tensile, or compressive here the load is only compressive but it is still along the long direction (x_1 - axis)

c) Deflection - Here there is a considerable difference. Initially, it is the same as a rod in that deflection occurs along x_1 (u_1 -- shortening for compressive loads)

But we consider whether buckling (instability) can occur. In this case, we also have deflection transverse to the long axis, u_3. This u_3 is governed by bending relations:

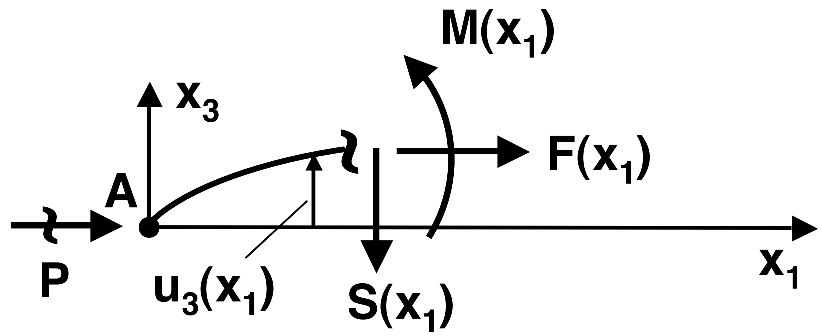
$$\frac{d^2 u_3}{d{x_1}^2} = \frac{M}{EI} \quad (u_3 = w)$$

Figure M4.7-8 **Representation of undeflected and deflected geometries of column**



We again take a "cut" in the structure and use stress resultants:

Figure M4.7-9 **Representation of "cut" column with resultant loads**



Now use equilibrium:

$$\Sigma F_1 = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad P + F(x_1) = 0$$

$$\Rightarrow \quad F(x_1) = -P$$

$$\Sigma F_3 = 0 \quad \uparrow + \quad \Rightarrow \quad S(x_1) = 0$$

$$\Sigma M_A = 0 \quad \circlearrowleft + \quad \Rightarrow \quad M(x_1) - F(x_1)\, u_3(x_1) = 0$$

$$\Rightarrow \quad M(x_1) + P u_3(x_1) = 0$$

Use the relationship between M and u_3 to get:

$$EI\frac{d^2u_3}{dx_1^{\ 2}} + Pu_3 = 0$$

governing differential equation for Euler buckling (2nd order differential equation)

always stabilizing (restoring)--basic beam: basic bending stiffness of structure resists deflection (pushes back)

destabilizing for compressive load ($u_3 > 0$ □ larger force to deflect); stabilizing for tensile load ($F = -P$) ($u_3 > 0$ □ restoring force to get $u_3 = 0$)

Note: + P is compressive

We now need to solve this equation and thus we look at the…..

(Solution for) Euler Buckling

First the

--> Basic Solution

(Note: may have seen similar governing for differential equation for harmonic notation:

$$\frac{d^2 w}{dx^2} + kw = 0)$$

From Differential Equations (18.03), can recognize this as an eigenvalue problem. Thus use:

$$u_3 = e^{\square x_1}$$

Write the governing equation as:

$$\frac{d^2 u_3}{dx_1{}^2} + \frac{P}{EI} u_3 = 0$$

<u>Note</u>: will often see form
(differentiate twice for general B.C.'s)

$$\frac{d^2}{dx_1^{\ 2}}\left[EI\,\frac{d^2u_3}{dx_1^{\ 2}}\right] + \frac{d^2}{dx_1^{\ 2}}\left(Pu_3\right) = 0$$

This is more general but reduces to our current form if EI and P do not vary in x_1

Returning to: $$\frac{d^2u_3}{dx_1^{\ 2}} + \frac{P}{EI}\,u_3 = 0$$

We end up with: $$\lambda^2 e^{\lambda x_1} + \frac{P}{EI}e^{\lambda x_1} = 0$$

$$\Rightarrow \quad \lambda^2 = -\frac{P}{EI}$$

$$\Rightarrow \quad \lambda = \pm\sqrt{\frac{P}{EI}}\;i$$

(also 0, 0 for 4th order Ordinary Differential Equation [O.D.E.])

where: $i = \sqrt{-1}$

We end up with the following <u>general</u> <u>homogeneous</u> <u>solution</u>:

$$u_3 = A \sin \sqrt{\frac{P}{EI}}\, x_1 + B \cos \sqrt{\frac{P}{EI}}\, x_1 + \underbrace{C + Dx_1}_{\text{comes from 4th order O.D.E. considerations}}$$

We get the constants A, B, C, D by using the ***<u>Boundary</u> <u>Conditions</u>***

(4 constants from the 4th under O.D.E.
$\Rightarrow$ need 2 B.C.'s at each end)

For the simply-supported case we are considering:

$$@\ x_1 = 0 \quad \begin{cases} u_3 = 0 \\ M = EI \dfrac{d^2 u_3}{dx_1^{\ 2}} = 0 \quad \Rightarrow \quad \dfrac{d^2 u_3}{dx_1^{\ 2}} = 0 \end{cases}$$

$$@\ x_1 = L \quad \begin{cases} u_3 = 0 \\ M = EI \dfrac{d^2 u_3}{dx_1^{\ 2}} = 0 \quad \Rightarrow \quad \dfrac{d^2 u_3}{dx_1^{\ 2}} = 0 \end{cases}$$

<u>Note</u>: $$\frac{d^2 u_3}{dx_1^{\ 2}} = \frac{P}{EI} A \sin \sqrt{\frac{P}{EI}}\, x_1 - \frac{P}{EI} B \cos \sqrt{\frac{P}{EI}}\, x_1$$

So using the B.C.'s:

$$\left.\begin{array}{ll} u_3 \left(x_1 = 0\right) = 0 & \Rightarrow \quad B + C = 0 \\ \dfrac{d^2 u_3}{dx_1^{\ 2}} \left(x_1 = 0\right) = 0 & \Rightarrow \quad B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} B = 0 \\ C = 0 \end{array}$$

$$\left.\begin{array}{ll} u_3 \left(x_1 = L\right) = 0 & \Rightarrow \quad A \sin \sqrt{\dfrac{P}{EI}}\, L + DL = 0 \\ \dfrac{d^2 u_3}{dx_1^{\ 2}} \left(x_1 = L\right) = 0 & \Rightarrow \quad -A \sin \sqrt{\dfrac{P}{EI}}\, L = 0 \end{array}\right\} \Rightarrow \quad D = 0$$

So we are left with:

$$A \sin \sqrt{\frac{P}{EI}}\, L = 0$$

This occurs if:

- A = 0 (trivial solution, $\Rightarrow$ $u_3 = 0$)
- $\sin \sqrt{\frac{P}{EI}}\, L = 0$

$$\Rightarrow \quad \sqrt{\frac{P}{EI}}\, L = n\pi$$

(n → integer)

Thus, buckling occurs in a simply-supported column if:

$$\boxed{P = \frac{n^2 \pi^2 EI}{L^2}}$$

eigenvalues

associated with each load (eigenvalue) is a shape (eigenmode)

$$\boxed{u_3 = A \sin \frac{n \pi x}{L}}$$

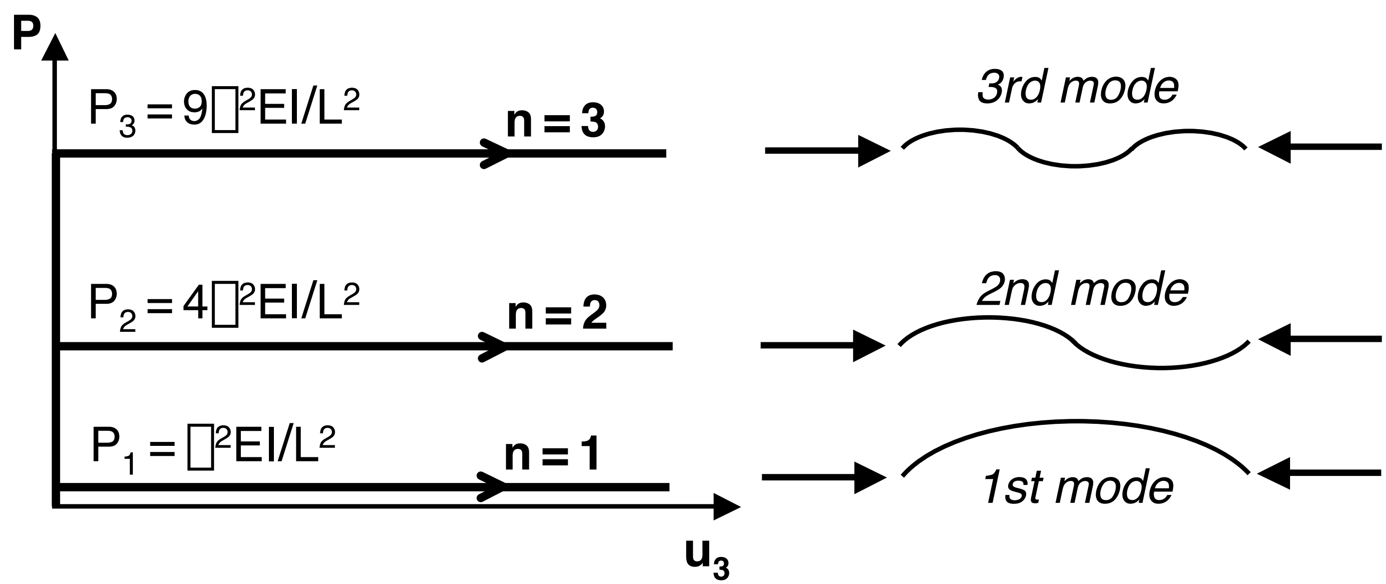
eigenmodes

Note: A is still undefined. This is an instability ($u_3 \rightarrow \infty$), so any value satisfies the equations.

[Recall, bifurcation is a mathematical concept]

Consider the buckling loads and associated mode shape (n possible)

Figure M4.7-10 **Potential buckling loads and modes for one-dimensional column**



The lowest value is the one where buckling occurs:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Euler (critical) buckling load (~1750)

for simply-supported column

(Note: The higher critical loads can be reached if the column is "artificially restrained" at lower bifurcation loads)

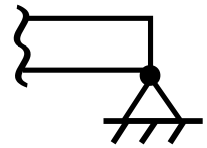
There are also other configurations, we need to consider….

--> Other Boundary Conditions

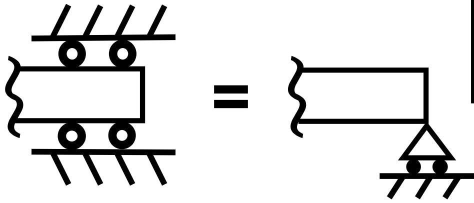
There are 3 (/4) allowable Boundary Conditions on u_3 (need two on each end) which are homogeneous (B.C.'s…. = 0)

--> simply-supported

(*pinned*)



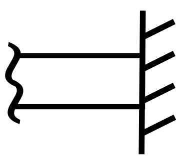
(*roller*)



$$\begin{cases} u_3 = 0 \\ M = EI\dfrac{d^2u_3}{dx_1^{\ 2}} = 0 \quad \Rightarrow \quad \dfrac{d^2u_3}{dx_1^{\ 2}} = 0 \end{cases}$$

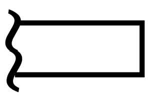
--> fixed end

(*clamped*)



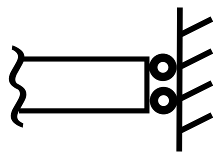
$$\begin{cases} u_3 = 0 \\ \dfrac{du_3}{dx_1} = 0 \end{cases}$$

--> free end



$$\begin{cases} M = EI\dfrac{d^2u_3}{dx_1^{\ 2}} = 0 \quad \Rightarrow \quad \dfrac{d^2u_3}{dx_1^{\ 2}} = 0 \\ S = 0 = \dfrac{dM}{dx_1} = \dfrac{d}{dx_1}\left(EI\dfrac{d^2u_3}{dx_1^{\ 2}}\right) \quad \Rightarrow \quad \dfrac{d^3u_3}{dx_1^{\ 3}} = 0 \end{cases}$$

--> sliding

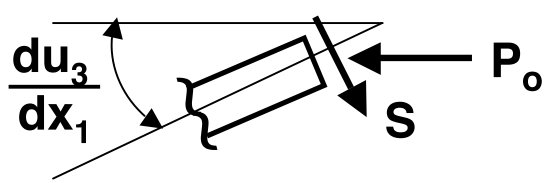


$$\left\{\begin{array}{l} \dfrac{du_3}{dx_1} = 0 \\[2ex] S = 0 \quad \Rightarrow \quad \dfrac{d^3 u_3}{dx_1{}^3} = 0 \end{array}\right.$$

There are combinations of these which are inhomogeneous Boundary Conditions.

Examples…

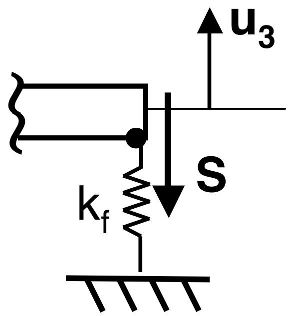
--> free end with an axial load



$$\left\{\begin{array}{l} M = 0 \\[2ex] S = -P_0 \dfrac{du_3}{dx_1} \end{array}\right.$$

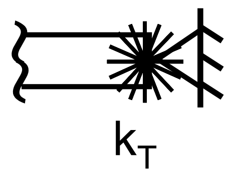
--> springs

(*vertical*)



$$\begin{cases} M = 0 \\ S = k_f \, u_3 \end{cases}$$

(*torsional*)



$$\begin{cases} u_3 = 0 \\ M = -k_T \dfrac{du_3}{dx_1} \end{cases}$$

Need a <u>general</u> <u>solution</u> <u>procedure</u> to find P_{cr}

Do the same as in the basic case.

- same assumed solution $u_3 = e^{\lambda x_1}$
- yields basic general homogeneous solution

$$u_3 = A \sin \sqrt{\frac{P}{EI}} \, x_1 + B \cos \sqrt{\frac{P}{EI}} \, x_1 + C + D x_1$$

- use B.C.'s (two at each end) to get four equations in four unknowns (A, B, C, D)
- solve this set of equations to find non-trivial value(s) of P

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0$$

4 x 4 matrix

homogeneous equation

- set determinant of matrix to zero ($\Delta = 0$) and find roots (solve resulting equation)

  roots = ***eigenvalues*** = buckling loads
  also get associated……
  eigenmodes = buckling shapes

--> will find that for homogeneous case, the critical buckling load has the generic form:

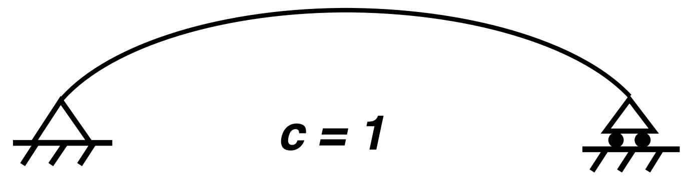
$$P_{cr} = \frac{c \pi^2 EI}{L^2}$$

where: c = coefficient of edge fixity
→ depends on B.C.'s

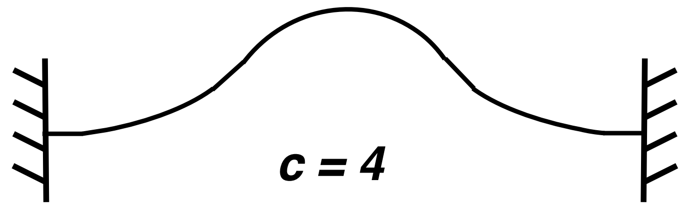
For aircraft and structures, often use c □ 2 for “fixed ends”.

Why?

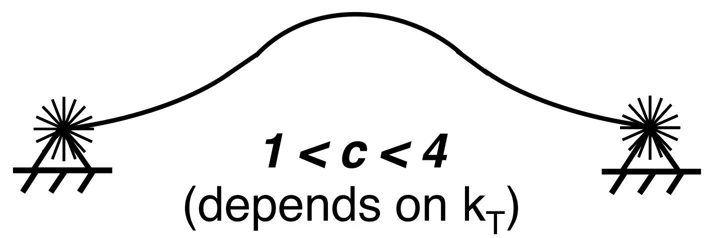
- simply-supported is too conservative



- cannot truly get clamped ends



- actual supports are basically “torsional springs”, empirically c = 2 works well and remains conservative



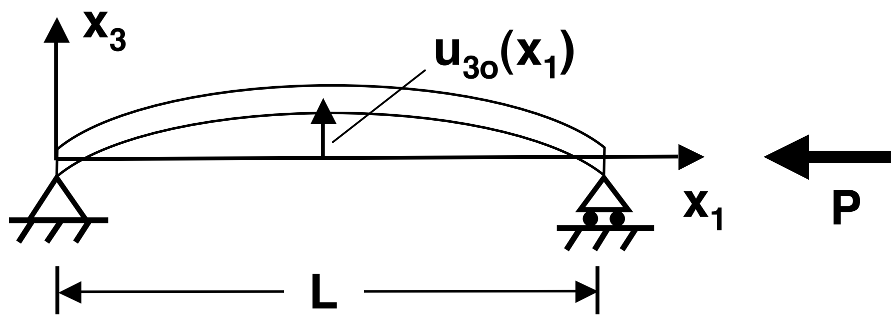
We've considered the "perfect" case of bifurcation where we get the instability in our mathematical model. Recall the opening example where that wasn't quite the case. Let's look at some realities here. First consider….

Effects of Initial Imperfections

We can think about two types…

Type 1 -- initial deflection in the column (due to manufacturing, etc.)

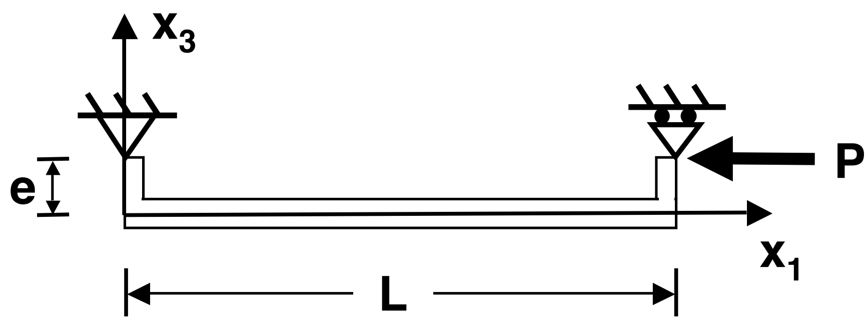
Figure M4.7-11 **Representation of initial imperfection in column**



Type 2 -- load not applied along centerline of column

Define: e = eccentricity (+↓ downwards)

Figure M4.7-12 **Representation of load applied off-line (eccentrically)**



(a beam-column)

moment PL plus axial load P

The two cases are basically handled the same way, but let's consider Type 2 to illustrate…

The governing equation is still the same:

$$\frac{d^2 u_3}{dx_1^{\ 2}} + \frac{P}{EI} u_3 = 0$$

Take a cut and equilibrium gives the same equations **<u>except</u>** there is an additional moment due to the eccentricity at the support: M = –Pe

Use the same basic solution:

$$u_3 = A \sin \sqrt{\frac{P}{EI}}\, x_1 + B \cos \sqrt{\frac{P}{EI}}\, x_1 + C + D x_1$$

and take care of this moment in the <u>Boundary</u> <u>Conditions</u>:

Here:

$$@\ x_1 = 0 \quad \left\{ \begin{array}{l} u_3 = 0 \quad \Rightarrow \quad B + C = 0 \\ M = EI \dfrac{d^2 u_3}{dx_1^{\ 2}} = -Pe \quad \Rightarrow \quad -PB = -Pe \end{array} \right\}$$

$$\Rightarrow \begin{array}{l} B = e \\ C = -e \end{array}$$

$$@ \; x_1 = L \; \begin{cases} u_3 = 0 \quad \Rightarrow \quad .... \\ M = EI \dfrac{d^2 u_3}{dx_1^{\;2}} = -Pe \quad \Rightarrow \quad .... \end{cases}$$

Doing the algebra find:

$$D = 0$$

$$A = \frac{e\left(1 - \cos\sqrt{\dfrac{P}{EI}}L\right)}{\sin\sqrt{\dfrac{P}{EI}}\,L}$$

actual value for A!

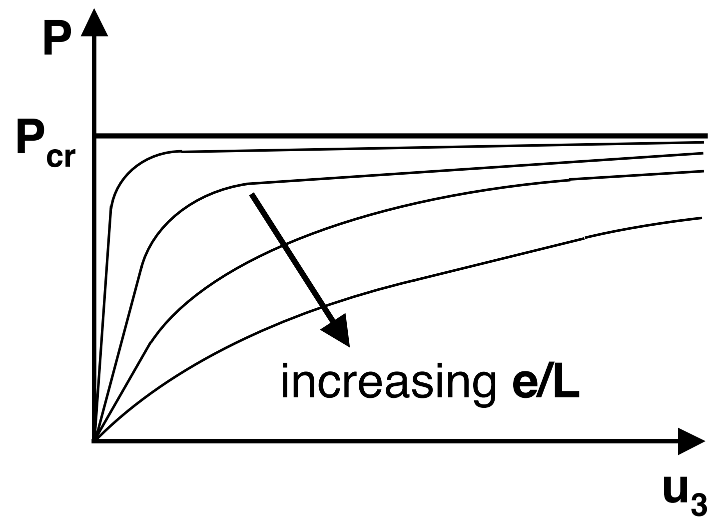
Putting this all together:

$$u_3 = e\left[\frac{\left(1 - \cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}\,L}\sin\sqrt{\frac{P}{EI}}\,x_1 + \cos\sqrt{\frac{P}{EI}}\,x_1 - 1\right]$$

<u>Notes</u>:

- Now get finite values of u_3 for values of P.
- As $P \rightarrow P_{cr} = \frac{\pi^2 EI}{L^2}$, still find u_3 becomes unbounded $\left(u_3 \rightarrow \infty\right)$

Figure M4.7-13 **Response of column to eccentric load**



Nondimensionalize by dividing through by L

- Bifurcation is asymptote
- u_3 approaches bifurcation as P --> P_{cr}
- As e/L (imperfection) increases, behavior is less like perfect case (bifurcation)

The other "deviation" from the model deals with looking at the general….

Failure of Columns

Clearly, in the "perfect" case, a column will fail if it buckles

$u_3 \rightarrow \infty$ (not very useful)

$u_3 \uparrow \Rightarrow M \uparrow \Rightarrow \sigma \uparrow \Rightarrow$ material fails!

Let's consider what else could happen depending on geometry

--> For long, slender case

$$P_{cr} = \frac{c\pi^2 EI}{L^2}$$

with:

$$\sigma_{11} = \frac{P}{A}$$

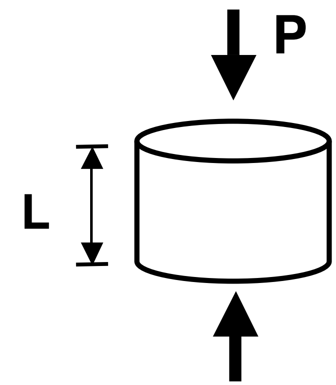
$$\Rightarrow \boxed{\sigma_{cr} = \frac{c\pi^2 EI}{L^2 A}}$$ for buckling failure

--> For <u>short</u> columns

if no buckling occurs, column fails when stress reaches material ultimate

(σ_{cu} = ultimate compressive stress)

$$\sigma = \frac{P}{A} = \sigma_{cu}$$



failure by "squashing"

--> Behavior of columns of various geometries characterized via:

effective length: $L' = \frac{L}{\sqrt{c}}$ (depends on Boundary Conditions)

radius of gyration: $\rho = \sqrt{\frac{I}{A}}$ (ratio of moment of inertia to area)

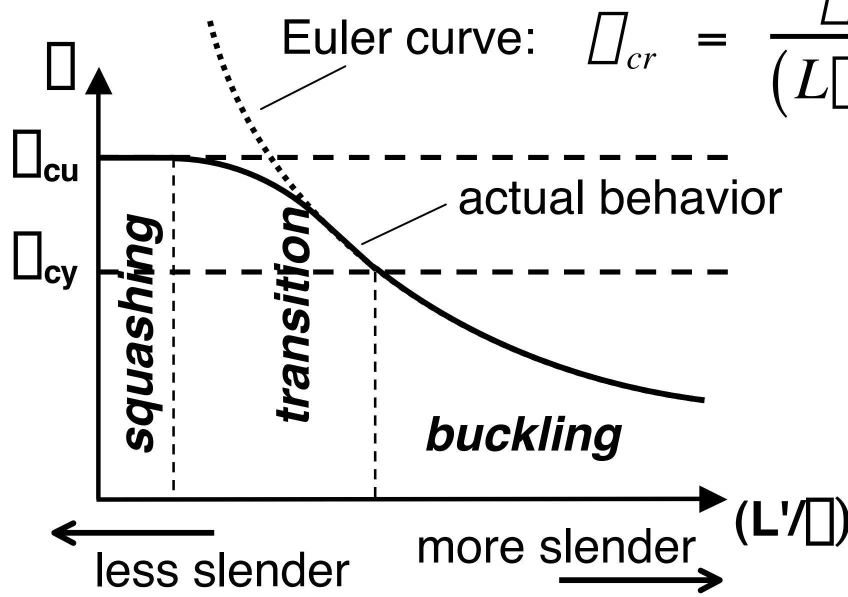
Look at equation for σ_{cr}, can write as:

$$\sigma_{cr} = \frac{\pi^2 E}{(L'/\rho)^2}$$

Can capture behavior of columns of various geometries on one plot

using: $\left(\frac{L'}{\rho}\right)$ = "slenderness ratio"

Figure M4.7-14 **Representation of general behavior for columns of various slenderness ratios**



where:
σ_{cy} = compressive yield stress

Notes:

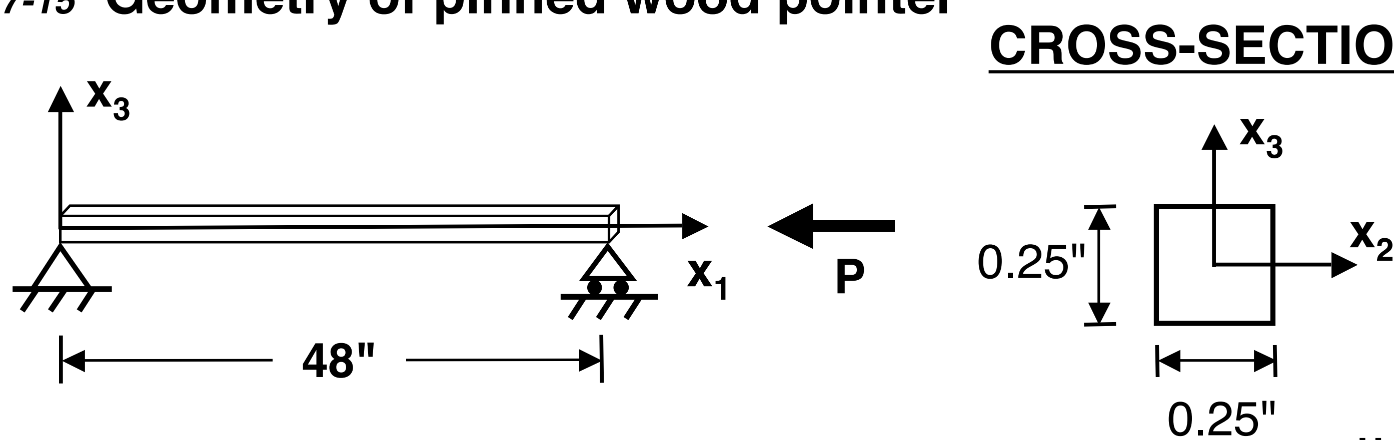
- for $\left(\frac{L}{\rho}\right)$ "large", column fails by buckling
- for $\left(\frac{L}{\rho}\right)$ "small", column squashes
- in transition region, plastic deformation (yielding) is taking place

$$\sigma_{cy} < \sigma < \sigma_{cu}$$

Let's look at all this via an…

Example: a wood pointer-- assume it is pinned and about 4 feet long

Figure M4.7-15 **Geometry of pinned wood pointer**



Material properties:
(Basswood)

$E = 1.4 \times 10^6$ psi

$\sigma_{cu} \cong 4800$ psi

--> Find maximum load P

Step 1: Find pertinent cross-section properties:

$A = b \times h = (0.25 \text{ in}) \times (0.25 \text{ in}) = 0.0625 \text{ in}^2$

$I = bh^3/12 = (0.25 \text{ in})(0.25 \text{ in})^3/12 = 3.25 \times 10^{-4} \text{ in}^4$

Step 2: Check for buckling

use:

$$P_{cr} = \frac{c\pi^2 EI}{L^2}$$

simply-supported $\Rightarrow$ c = 1

So: $P_{cr} = \dfrac{\pi^2 \left(1.4 \times 10^6 \ lbs / in^2\right)\left(3.26 \times 10^{-4} \ in^{-4}\right)}{\left(48 \ in\right)^2}$

$$\Rightarrow \quad \boxed{P_{cr} = 1.96 \ lbs}$$

Step 3: Check to see if it buckles or squashes

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1.96 \ lbs}{0.0625 \ in^2} = 31.4 \ psi$$

So: $\sigma_{cr} < \sigma_{cu} \Rightarrow$ BUCKLING!

--> Variations

1. What is "transition" length?

Determine where "squashing" becomes a concern (approximately)

$$\Rightarrow \quad \sigma_{cr} = \sigma_{cu}$$

--> work <u>backwards</u>

$$\sigma_{cr} = \frac{P_{cr}}{A} = 4800 \ psi$$

$$\Rightarrow \quad P_{cr} = \left(4800 \ \ lbs/in^2\right)\left(0.0625 \ in^2\right)$$

$$\Rightarrow \quad P_{cr} = 300 \ lbs$$

--> Next use:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

where L is the variable, gives:

$$L^2 = \frac{\pi^2 EI}{P_{cr}}$$

$$\Rightarrow \quad L = \sqrt{\frac{\pi^2\left(1.4 \ \times \ 10^6 \ \ lbs/in^2\right)\left(3.26 \ \times \ 10^{-4} \ in^4\right)}{300 \ lbs}}$$

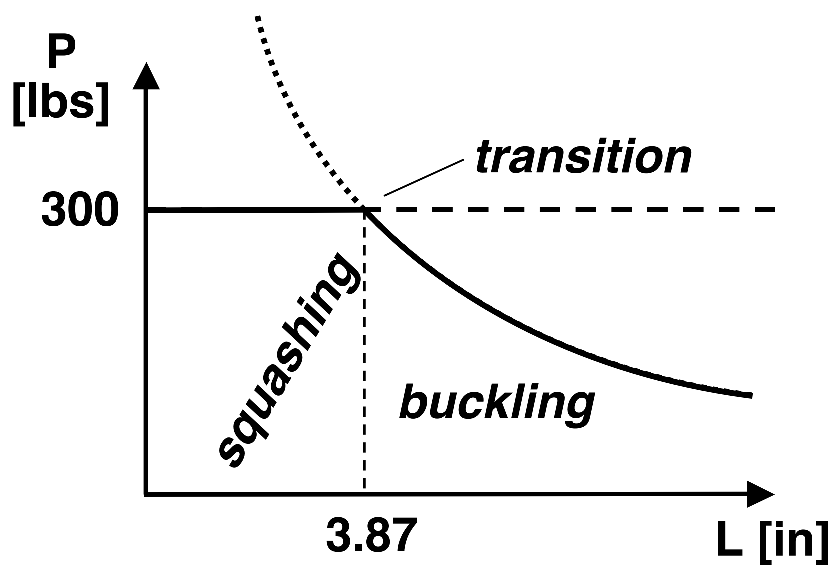
$$\Rightarrow \quad L = \sqrt{15.01 \ in^2} \Rightarrow \quad \boxed{L = 3.87 \ in}$$

Finally….

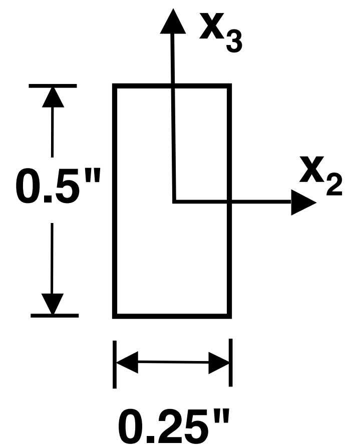
If L > 3.87 in buckling
If L < 3.87 in squashing

Note transition "around" 3.87 in due to yielding (basswood relatively brittle)

Figure M4.7-16 **Behavior of 4-foot basswood pointer subjected to compressive load**



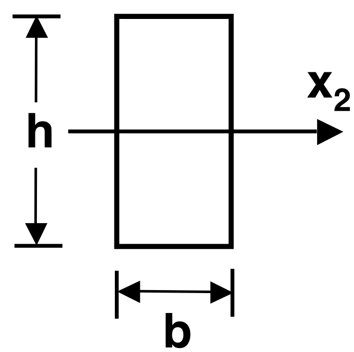
2. What if rectangular cross-section?



Does it still buckle in x_3 - direction?

Consider I about x_2 - axis and x_3 - axis

--> x_2 - axis □

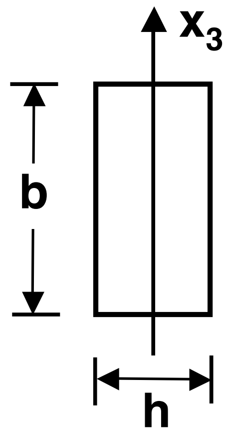


□ h = 0.5 in, b = 0.25 in

$$\Rightarrow \quad I_2 = \frac{bh^3}{12} = \frac{(0.25\ in)(0.50\ in)^3}{12}$$

$= 0.0026\ in^4$
$= 2.60 \times 10^{-3}\ in^4$

--> x_3 - axis $\Rightarrow$



$\Rightarrow$ h = 0.25 in, b = 0.5 in

$$I_3 = \frac{bh^3}{12} = \frac{(0.50\ in)(0.25\ in)^3}{12}$$

$= 0.00065\ in^4$
$= 0.65 \times 10^{-3}\ in^4$

then use:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

and find:

$$I_3 < I_2$$

$\Rightarrow$ P_{cr} smaller for buckling about x_3 - axis.

Important For buckling, h is the <u>shorter/smaller</u> cross-section dimension since buckling occurs about axis with smallest I !

--> Final note on <u>buckling</u>

…possibility of occurrence in any structure where there is a <u>compressive</u> load (thinner structures most susceptible)

Unit M4.7 (New) Nomenclature

c -- coeffcient of edge fixity
e -- eccentricity (due to loading off line or initial imperfection)
I_2 -- moment of inertia about x_2 - axis
I_3 -- moment of inertia about x_3 - axis
k_{eff} -- effective stiffness
k_f -- axial stiffness
k_T -- torsional stiffness
L -- effective length (in buckling considerations)
L'/ρ -- slenderness ratio
P -- compressive load along column
P_{cr} -- critical (buckling) load (for instability)
ρ -- radius of gyration (square root of ratios of moment of inertia to area)
σ_{cr} -- critical buckling stress
σ_{cu} -- compressive ultimate stress
σ_{cy} -- compressive yield stress