UE Fluids Problems 7+10 Solution S'07

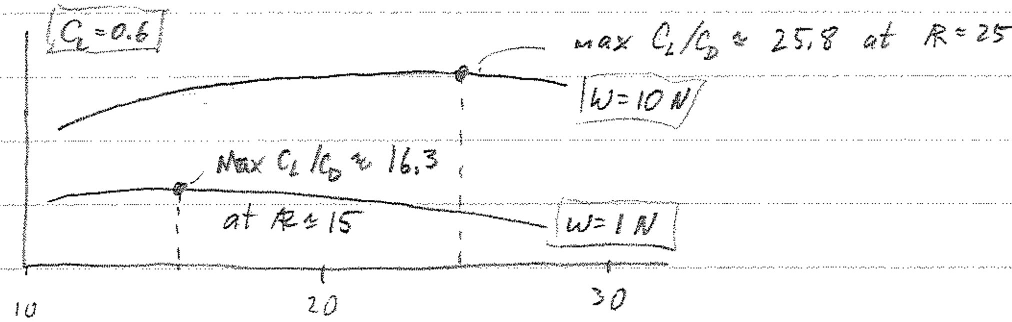
a) $b \cdot c_{avg} = S$, $b = \frac{S}{c_{avg}}$, $AR = \frac{b^2}{S} = \frac{(S/c_{avg})^2}{S} \rightarrow \boxed{c_{avg} = \sqrt{S/AR}}$

b) $W = L = \frac{1}{2}\rho V^2 S C_L \rightarrow V = \sqrt{\frac{2W}{S\rho C_L}}$, $\boxed{Re = \frac{V c_{avg}}{\nu} = \sqrt{\frac{2W}{S\rho C_L}}\cdot\sqrt{\frac{S}{AR}}\,\frac{1}{\nu} = \frac{1}{\nu}\sqrt{\frac{2W}{\rho C_L AR}}}$

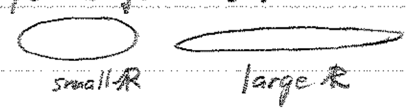
c) $C_D = C_d + C_{D_i} = 5.0\left[\frac{1}{\nu}\left(\frac{2W}{\rho C_L AR}\right)^{1/2}\right]^{-1/2} + \frac{C_L^2}{\pi AR}$

$$\boxed{C_D(C_L, AR) = 5.0\,\nu^{1/2}\left[\frac{\rho C_L AR}{2W}\right]^{1/4} + \frac{C_L^2}{\pi AR}}$$

d,e) $\frac{C_L}{C_D}$



d) The statement "performance improves with larger AR" addresses only C_{D_i} and assumes that C_d is constant. In reality, a larger AR implies a smaller c_{avg} and a smaller Re for a given S:



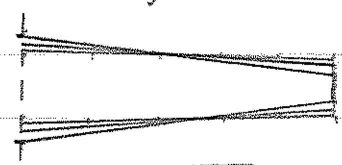
The smaller Re increases C_d, which eventually overcomes the decreasing C_{D_i}. This is basically a Re effect.

f) The heavier Albatross will have a larger optimum AR because of the Re effect. Survival & evolution favors maximum efficiency for minimum energy consumption in flight. So each bird is likely to be closest to its optimum, hence $AR_{Albatross} > AR_{Tern}$

QE Fluids Problem 8 Solution S'07

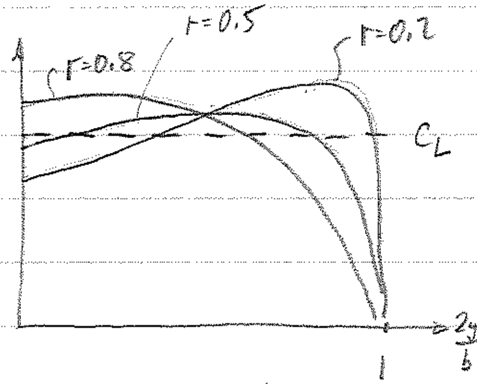
a) Linear interpolation between c_r and c_t:

$$c(y) = c_r + (c_t - c_r)\left|\frac{2y}{b}\right| = c_{avg}\cdot\frac{2}{1+r}\left[1+(r-1)\left|\frac{2y}{b}\right|\right]$$

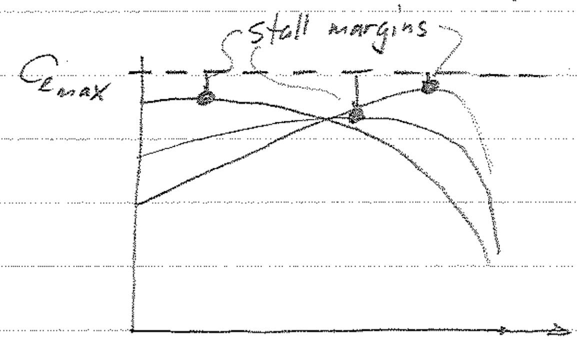


b) $\Gamma(y) = \frac{1}{2} c V c_\ell$, $c_\ell(y) = \dfrac{2\Gamma(y)}{c(y)V_\infty} = \dfrac{\Gamma_o}{c_{avg}V_\infty}(1+r)\dfrac{\sqrt{1-(2y/b)^2}}{1+(r-1)\left|\frac{2y}{b}\right|}$

$L = \frac{\pi}{4}\rho V_\infty \Gamma_o b$, $C_L = \dfrac{L}{\frac{1}{2}\rho V_\infty^2 b\, c_{avg}} = \dfrac{\pi}{2}\dfrac{\Gamma_o}{c_{avg}V_\infty}$



c) Local stall will occur first at the peak $c_\ell(y)$ location. Because the r=0.5 wing has the lowest c_ℓ value at its peak, it will have the greatest stall margin.



UE Fluids   Problem F9   Solution   S '07

a) $\frac{2y}{b} = \cos\theta \rightarrow \Gamma = \left[0.05 - 0.02\cos\theta\right]\sqrt{1-\cos^2\theta}$

$$= 0.05\sin\theta - 0.02\cos\theta\sin\theta$$

$$\boxed{\Gamma(\theta) = 0.05\sin\theta - 0.01\sin 2\theta}$$

b) $\Gamma = 2bV_\infty \sum A_n \sin n\theta$ ← same (*)

Could also use Fourier Analysis: Evaluate $\int_0^\pi (*)\sin\theta\,d\theta$, $\int_0^\pi (*)\sin 2\theta\,d\theta$, ⋮

By inspection, we must have $2bV_\infty A_1 = 0.05$

$2bV_\infty A_2 = -0.01$

Setting $V_\infty = 1$, $b = 2$:

$$\boxed{A_1 = 0.0125 \quad , \quad A_2 = -0.0025}$$

c) $L = \frac{\pi}{4}\rho V_\infty b A_1 = \frac{\pi}{4}\cdot 1\cdot 1\cdot 2\cdot 0.0125 = 0.01963$

$D_i = \pi b^2 \frac{1}{2}\rho V_\infty^2\left[A_1^2 + 2A_2^2\right] = \pi\cdot 4\cdot\frac{1}{2}\cdot\left[0.0125^2 + 2\cdot 0.0025^2\right] = 0.00106$

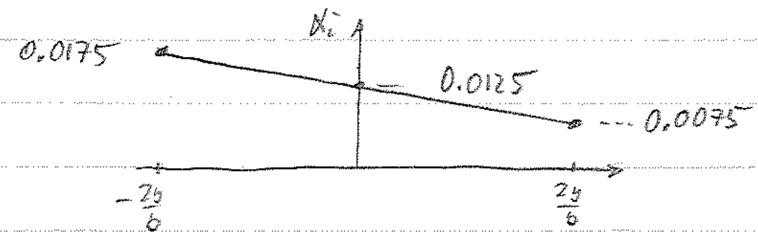
$e = \left[1 + 2(A_2/A_1)^2\right]^{-1} = \left[1 + 2\cdot(0.0025/0.0125)^2\right]^{-1} = 0.9259$

d) $M_{roll} = \int_{-b/2}^{b/2}\rho V_\infty \Gamma y\,dy = \int_0^\pi \rho V_\infty \underbrace{2bV_\infty\left[A_1\sin\theta + A_2\sin 2\theta\right]}_{\Gamma}\underbrace{\left(\frac{b}{2}\cos\theta\right)}_{y}\underbrace{\left(\frac{b}{2}\sin\theta\,d\theta\right)}_{-dy}$

$$M_{roll} = \frac{1}{2}\rho V_\infty^2 b^3 \int_0^\pi \left[A_1\sin\theta + A_2\sin 2\theta\right]\frac{1}{2}\sin 2\theta\,d\theta$$

$$\boxed{M_{roll} = \frac{1}{2}\rho V_\infty^2 b^3\left[A_1 0 + A_2\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{1}{2}\rho V_\infty^2 b^3\cdot A_2\cdot\frac{\pi}{4} = -0.00785}$$

e) $\alpha_i = A_1\frac{\sin\theta}{\sin\theta} + A_2\frac{\sin 2\theta}{\sin\theta} = A_1 + A_1 2\cos\theta = A_1 + A_2 2\left(\frac{2y}{b}\right) = 0.0125 - 0.005\left(\frac{2y}{b}\right)$



Problem S2 (Signals and Systems) SOLUTION

A system has step response given by

$$g_s(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & t \geq 0 \end{cases}$$

Find and plot the response of the system to the input

$$u(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-2t}, & t \geq 0 \end{cases}$$

using Duhamel's integral.

Solution: Duhamel's integral is

$$y(t) = u(0)g_s(t) + \int_0^\infty g_s(t-\tau)u'(\tau)\, d\tau$$

For $\tau \geq 0$, we can write

$$u'(\tau) = 2e^{-2\tau}$$

Also, note that $u(0) = 0$. Therefore, for $t \geq 0$,

$$\begin{aligned} y(t) &= \int_0^t \left(1 - e^{-(t-\tau)}\right) 2e^{-2\tau}\, d\tau \\ &= 2\int_0^t \left(e^{-2\tau} - e^{-t-\tau}\right)\, d\tau \\ &= 2\int_0^t e^{-2\tau}\, d\tau - 2e^{-t}\int_0^t e^{-\tau}\, d\tau \end{aligned}$$

Note that the upper limit of the integral is t, since g_s is causal, and therefore $g_s(t-\tau) = 0$ for $\tau > t$. Evaluating the integrals, we have

$$\begin{aligned} y(t) &= (1 - e^{-2t}) - 2e^{-t}(1 - e^{-t}) \\ &= 1 - e^{-2t} - 2e^{-t} + 2e^{-2t} \\ &= 1 - 2e^{-t} + e^{-2t} \end{aligned}$$

For $t < 0$, $y(t) = 0$, since there is no overlap between $g_s(t-\tau)$ and $u(\tau)$. Therefore,

$$y(t) = \left(1 - 2e^{-t} + e^{-2t}\right)\sigma(t)$$

Problem S3 (Signals and Systems) SOLUTION

SOLUTION: The step response of the airfoil (to unit change in angle of attack) is

$$g_s(t) = 2\pi\psi(\bar{t})$$

where $\bar{t} = 2Ut/c$ is the dimensionless time, and$\psi(\bar{t})$ is the Küssner function. The Küssner function can be approximated as

$$\psi(\bar{t}) = \begin{cases} 0, & \bar{t} < 0 \\ 1 - \frac{1}{2}e^{-0.13\bar{t}} - \frac{1}{2}e^{-\bar{t}}, & \bar{t} \geq 0 \end{cases}$$

The problem statement gives the following conditions:

$$\begin{aligned} c &= 1 \text{ m} \\ U &= 1 \text{ m/s} \end{aligned}$$

Therefore, $\bar{t} = 2t$, and the step response is then

$$g_s(t) = 2\pi \left(1 - \frac{1}{2}e^{-0.26t} - \frac{1}{2}e^{-2t}\right)\sigma(t)$$

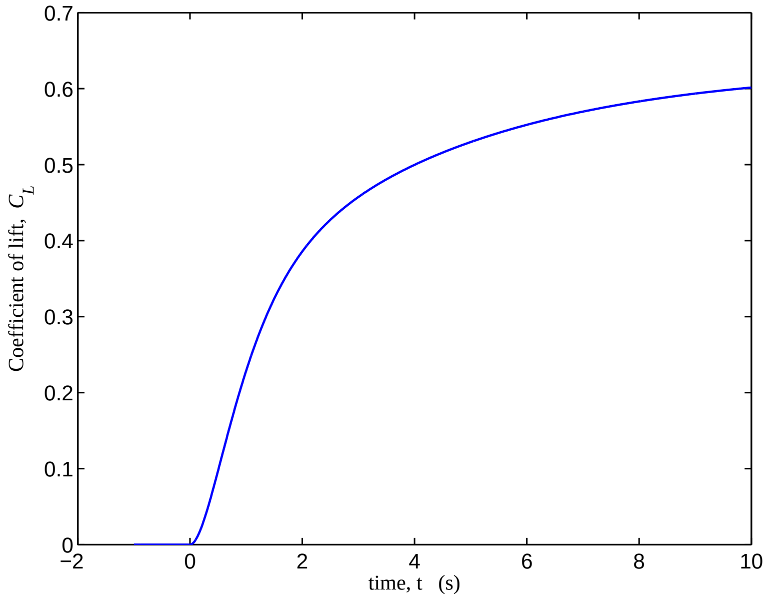
The angle of attack is

$$\begin{aligned} \alpha(t) &= w(t)/U \\ &= 0.1 \cdot \left(1 - e^{-2t}\right)\sigma(t) \\ &= u(t) \end{aligned}$$

The lift coeeficient can then be found from Duhamel's integral. Since $u(0) = 0$, we have

$$\begin{aligned} y(t) = C_L(t) &= \int_0^t g_s(t-\tau)u'(t)\,d\tau \\ &= \int_0^t 2\pi \left(1 - \frac{1}{2}e^{-0.26(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}\right) 0.2e^{-2\tau}\,d\tau \\ &= 0.4\pi \int_0^t \left(e^{-2\tau} - \frac{1}{2}e^{-0.26t-1.74\tau} - \frac{1}{2}e^{-2t}\right) d\tau \\ &= 0.4\pi \left[\frac{1}{2}\left(1 - e^{-2t}\right) - \frac{1}{2 \cdot 1.74}e^{-0.26t}\left(1 - e^{-1.74t}\right) - \frac{1}{2}te^{-2t}\right] \\ &= \pi\left(0.2 - 0.08506e^{-2t} - 0.1149e^{-0.26t} - 0.2te^{-2t}\right) \\ &= 0.62832 - 0.26722e^{-2t} - 0.3611e^{-0.26t} - 0.62832te^{-2t} \end{aligned}$$

A plot of the response is shown below:


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Coefficient of lift, C_L
−2
0
2
4
6
8
10
time, t (s)

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Problem S4 Solution (Signals and Systems)

1. Find the step response of the system

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = u(t)$$

Solution. First, find the homogeneous solution, that is, the solution to

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 0$$

If we assume an exponential solution of the form $y(t) = e^{st}$, then the characteristic equation becomes

$$s^2 + 3s + 2 = 0$$

The roots are $s_1 = -1$, $s_2 = -2$. The general homogeneous solution is then

$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

Next, we find the particular solution, for the input $u(t) = 1$, $t > 0$. Guess that the solution is a constant, $y(t) = c$. Plugging into the d.e. yields

$$2c = 1$$

and therefore $c = 1/2$. Therefore, the total solution is

$$y(t) = y_h(t) + y_p(t) = \frac{1}{2} + c_1 e^{-t} + c_2 e^{-2t}$$

The inital conditions are $y(0) = y^{'}(0) = 0$. Solving for the constants, $c_1 = -1$, $c_2 = 1/2$. Therefore,

$$g_s(t) = \left(\frac{1}{2} - e^{-t} + \frac{1}{2}\,e^{-2t}\right)\sigma(t)$$

2. Take the derivative of the step response to find the impulse response.

Solution. Take the derivative using the chain rule for multiplication

$$\begin{aligned} g(t) &= \frac{d}{dt}\left[\left(\frac{1}{2} - e^{-t} + \frac{1}{2}\,e^{-2t}\right)\sigma(t)\right] \\ &= \left(e^{-t} - e^{-2t}\right)\sigma(t) + \left(\frac{1}{2} - e^{-t} + \frac{1}{2}\,e^{-2t}\right)\delta(t) \end{aligned}$$

Since $\frac{1}{2} - e^{-t} + \frac{1}{2}\,e^{-2t} = 0$ at $t = 0$,

$$g(t) = \left(e^{-t} - e^{-2t}\right)\sigma(t)$$

3. Now assume that the input is given by

$$u(t) = e^{-2t}\sigma(t)$$

Before doing part (4), try to find the particular solution by the usual method, that is, by intelligent guessing. Be careful — it may not be what you expect!

Solution. The obvious thing to do is to guess that

$$y_p(t) = c\,e^{-2t}$$

But we already know that e^{-2t} is a solution to the homogeneouos equations. It can't be a particular solution! So instead, guess that

$$y_p(t) = c\,t\,e^{-2t}$$

Plugging into the differential equation gives

$$\begin{aligned} \frac{d^2}{dt^2}\, y_p(t) + 3\frac{d}{dt}\, y_p(t) + 2y_p(t) &= -c\,e^{-2t} \\ &= u(t) = e^{-2t} \end{aligned}$$

Therefore, $c = -1$, and

$$y_p(t) = -t\,e^{-2t}$$

4. Now find $y(t)$ using the superposition integral. Is the particular solution what you expected?

   Solution. Perform the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\, d\tau$$

   The result is

$$\begin{aligned} y(t) &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) e^{-2\tau}\, d\tau \\ &= \left(e^{-t} - e^{-2t} - te^{-2t}\right)\sigma(t) \end{aligned}$$

   Note that the correct particular solution is in the final result.