Unified Engineering II **Spring 2007**

Problem S8 Solution (Signals and Systems)

1. $g(t) = \begin{cases} te^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

   Therefore,

$$G(s) = \int_0^\infty te^{-at}e^{-st}\,dt$$

   Integrate by parts to obtain

$$G(s) = -\frac{t}{s+a}e^{-(a+s)t}\Big|_{t=0}^{\infty} + \frac{1}{s+a}\int_0^\infty e^{-at}e^{-st}\,dt$$

   If Re[s] $> -a$, then the first term evaulates to 0; otherwise, it is undefined. The integral is just the LT of $e^{-at}\sigma(t)$. Therefore,

$$\begin{aligned} G(s) &= \frac{1}{s+a}\int_0^\infty e^{-at}e^{-st}\,dt \\ &= \frac{1}{(s+a)^2}, \qquad \text{Re}[s] > -a \end{aligned}$$

2. $g(t) = \begin{cases} t^2e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

   Integrate by parts twice to obtain

$$G(s) = \frac{2}{(s+a)^3}, \qquad \text{Re}[s] > -a$$

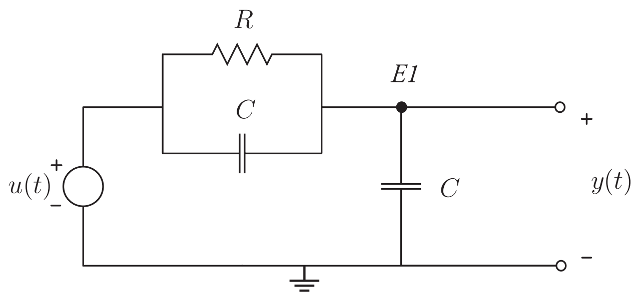
3. $g(t) = \begin{cases} t^ne^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, where n is a positive integer.

   In general,

$$G(s) = \frac{n!}{(s+a)^{n+1}}, \qquad \text{Re}[s] > -a$$

Problem S7 Solution (Signals and Systems)

1.



We can use impedance methods to solve for $Y(s)$ in terms of $U(s)$. Label ground and E_1 as shown. Then KCL at E_1 yields

$$Cs(E_1 - 0) + \left(Cs + \frac{1}{R}\right)(E_1 - U) = 0$$

Simplifying, we have

$$\left(2Cs + \frac{1}{R}\right) E_1 = \left(Cs + \frac{1}{R}\right) U$$

Since we are finding the step response,

$$U(s) = \frac{1}{s}, \qquad \text{Re}[s] > 0$$

Plugging in numbers, we have

$$(0.2s + 0.5)E_1(s) = (0.1s + 0.5)\,\frac{1}{s}$$

Solving for E_1, we have

$$E_1(s) = \frac{0.1s + 0.5}{(0.2s + 0.5)s} = \frac{0.5s + 2.5}{s(s + 2.5)}$$

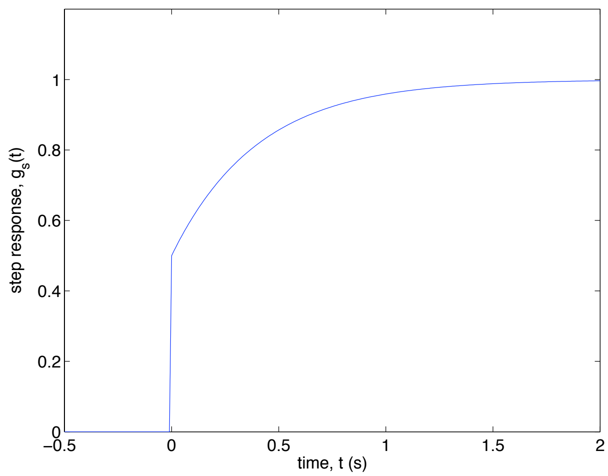
The region of convergence must be $\text{Re}[s] > 0$, since the step response is causal, and the pole at $s = 0$ is the rightmost pole. Using partial fraction expansions,

$$E_1(s) = \frac{1}{s} - \frac{0.5}{s + 2.5}$$

Therefore, $g_s(t) = y(t) = e_1(t)$ is the inverse transform of $E_1(t)$, so

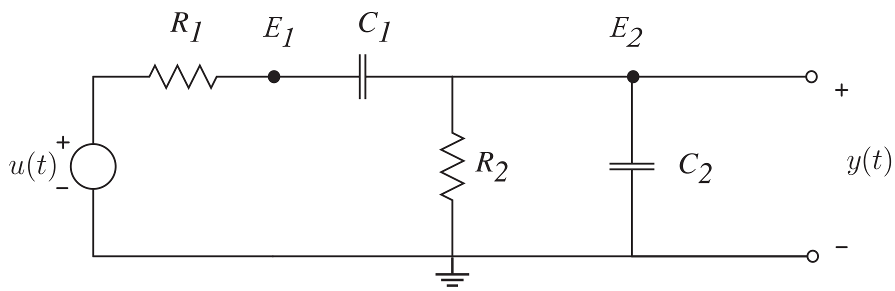
$$y(t) = \left(1 - 0.5e^{-2.5t}\right)\sigma(t)$$

The step response is plotted below:



Normal differential equation methods are difficult to apply, because we cannot apply the normal initial condition that $e_1(0) = 0$. This is because the chain of capacitors running from the voltage source to ground causes there to be an impulse of current at time $t = 0$, and the voltages across the capacitors change instantaneously at $t = 0$. It is possible to use differential equation methods, we just have to be more careful about the initial conditions. However, Laplace methods are easier.

2.



Again, use impedance methods, using the node labelling above. Then the node equations are

$$\begin{array}{rcrl} (C_1 s + G_1)E_1 & - & C_1 s E_2 & = G_1 U \\ -C_1 s E_1 & + & [(C_1 + C_2)s + G_2]E_2 & = 0 \end{array}$$

where $G = 1/R$. We can use Cramer's rule to solve for E_2:

$$\begin{aligned} E_2(s) &= \frac{\begin{vmatrix} C_1 s + G_1 & G_1 U(s) \\ -C_1 s & 0 \end{vmatrix}}{\begin{vmatrix} C_1 s + G_1 & -C_1 s \\ -C_1 s & (C_1 + C_2)s + G_2 \end{vmatrix}} \\ &= \frac{G_1 C_1 s}{C_1 C_2 s^2 + (G_1 C_1 + G_1 C_2 + G_2 C_1)s + G_1 G_2} U(s) \end{aligned}$$

Since we are finding the step response,

$$U(s) = \frac{1}{s}, \qquad \mathrm{Re}[s] > 0$$

Plugging in numbers, we have

$$Y(s) = E_2(s) = \frac{0.1s}{0.06s^2 + 0.35s + 0.25}\frac{1}{s} = \frac{5/3}{s^2 + 5.833\bar{3}s + 4.166\bar{6}}$$

In order to find $y(t)$, we must expand $Y(s)$ in a partial fraction expansion. To do so, we must factor the denominator, using either numerical techniques or the quadratic formula. The result is

$$s^2 + 5.833\bar{3}s + 4.166\bar{6} = (s + 5)(s + 0.833\bar{3})$$

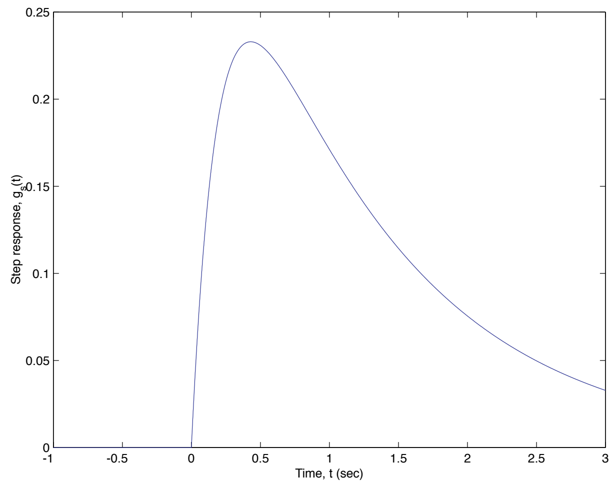
We can use the coverup method to factor $Y(s)$, so that

$$Y(s) = \frac{5/3}{(s + 5)(s + 0.833\bar{3})} = \frac{-0.4}{s + 5} + \frac{0.4}{s + 0.833\bar{3}}$$

The region of convergence must be $\mathrm{Re}[s] > -0.833\bar{3}$, since the step response is causal, and the r.o.c. is to the right of the right-most pole. Therefore, the step response is given by the inverse transform of $Y(s)$, so that

$$g_s(t) = \left(-0.4e^{-5t} + 0.4e^{-0.833\bar{3}t}\right)\sigma(t)$$

The step response is plotted below:


Step response, $g_s(t)$
Time, t (sec)
0.25
0.2
0.15
0.1
0.05
0
-1
-0.5
0
0.5
1
1.5
2
2.5
3

Problem S10 Solution (Signals and Systems)

1. Because the numerator is the same order as the denominator, the partial fraction expansion will have a constant term:

$$\begin{aligned} G(s) &= \frac{3s^2+3s-10}{s^2-4} \\ &= \frac{3s^2+3s-10}{(s-2)(s+2)} \\ &= a + \frac{b}{s-2} + \frac{c}{s+2} \end{aligned}$$

   To find a, b, and c, use coverup method:

$$\begin{aligned} a &= G(s)\big|_{s=\infty} = 3 \\ b &= \left.\frac{3s^2+3s-10}{s+2}\right|_{s=2} = 2 \\ c &= \left.\frac{3s^2+3s-10}{s-2}\right|_{s=-2} = 1 \end{aligned}$$

   So

$$G(s) = 3 + \frac{2}{s-2} + \frac{1}{s+2}, \qquad \text{Re}[s] > 2$$

   We can take the inverse LT by simple pattern matching. The result is that

$$g(t) = 3\delta(t) + \left(2e^{2t} + e^{-2t}\right)\sigma(t)$$

2.

$$\begin{aligned} G(s) &= \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} \\ &= \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3} \end{aligned}$$

   Using partial fraction expansions,

$$\begin{aligned} a &= \left.\frac{6s^2+26s+26}{(s+2)(s+3)}\right|_{s=-1} = 3 \\ b &= \left.\frac{6s^2+26s+26}{(s+1)(s+3)}\right|_{s=-2} = 2 \\ c &= \left.\frac{6s^2+26s+26}{(s+1)(s+2)}\right|_{s=-3} = 1 \end{aligned}$$

   So

$$G(s) = \frac{3}{s+1} + \frac{2}{s+2} + \frac{1}{s+3}, \qquad \text{Re}[s] > -1$$

   The inverse LT is given by

$$\left(3e^{-t} + 2e^{-2t} + e^{-3t}\right)\sigma(t)$$

3. This one is a little tricky — there is a second order pole at $s = -1$. So the partial fraction expansion is

$$G(s) = \frac{4s^2 + 11s + 9}{(s+1)^2(s+2)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{c}{s+2}$$

We can find b and c by the coverup method:

$$\begin{aligned} b &= \left.\frac{4s^2 + 11s + 9}{s+2}\right|_{s=-1} = 2 \\ c &= \left.\frac{4s^2 + 11s + 9}{(s+1)^2}\right|_{s=-2} = 3 \end{aligned}$$

So

$$G(s) = \frac{a}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2}$$

To find a, subtract the second and third terms from above, to obtain

$$\begin{aligned} \frac{a}{s+1} &= G(s) - \frac{2}{(s+1)^2} - \frac{3}{s+2} \\ &= \frac{4s^2 + 11s + 9}{(s+1)^2(s+2)} - \frac{2}{(s+1)^2} - \frac{3}{s+2} \\ &= \frac{4s^2 + 11s + 9 - 2(s+2) - 3(s+1)^2}{(s+1)^2(s+2)} \\ &= \frac{s^2 + 3s + 2}{(s+1)^2(s+2)} \\ &= \frac{1}{(s+1)} \end{aligned}$$

Therefore,

$$G(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2}, \qquad \mathrm{Re}[s] > -1$$

The inverse LT is then

$$g(t) = \left(e^{-t} + 2te^{-t} + 3e^{-2t}\right)\sigma(t)$$

4. This problem is similar to above. The partial fraction expansion is

$$G(s) = \frac{4s^3 + 11s^2 + 5s + 2}{s^2(s+1)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{(s+1)^2}$$

We can find b and d by the coverup method

$$\begin{aligned} b &= \left.\frac{4s^3 + 11s^2 + 5s + 2}{(s+1)^2}\right|_{s=0} = 2 \\ d &= \left.\frac{4s^3 + 11s^2 + 5s + 2}{s^2}\right|_{s=-1} = 4 \end{aligned}$$

So

$$G(s) = \frac{4s^3 + 11s^2 + 5s + 2}{s^2(s+1)^2} = \frac{a}{s} + \frac{2}{s^2} + \frac{c}{s+1} + \frac{4}{(s+1)^2}$$

To find a and c, subtract both terms from both sides, so that

$$\begin{aligned}
\frac{a}{s} + \frac{c}{s+1} &= G(s) - \frac{2}{s^2} - \frac{4}{(s+1)^2} \\
&= \frac{4s^3 + 11s^2 + 5s + 2}{s^2(s+1)^2} - \frac{2}{s^2} - \frac{4}{(s+1)^2} \\
&= \frac{4s^3 + 11s^2 + 5s + 2 - 2(s+1)^2 - 4s^2}{s^2(s+1)^2} \\
&= \frac{4s^3 + 5s^2 + s}{s^2(s+1)^2} = \frac{s(4s+1)(s+1)}{s^2(s+1)^2} \\
&= \frac{4s+1}{s(s+1)} = \frac{1}{s} + \frac{3}{s+1}
\end{aligned}$$

So

$$G(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s+1} + \frac{4}{(s+1)^2}$$

and

$$g(t) = \left(1 + 2t + 3e^{-t} + 4te^{-t}\right)\sigma(t)$$

5. $G(s)$ can be expanded as

$$\begin{aligned}
G(s) &= \frac{s^3 + 3s^2 + 9s + 12}{\left(s^2+4\right)\left(s^2+9\right)} \\
&= \frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s+3j)(s-3j)} \\
&= \frac{a}{s+2j} + \frac{b}{s-2j} + \frac{c}{s+3j} + \frac{d}{s-3j}
\end{aligned}$$

The coefficients can be found by the coverup method:

$$\begin{aligned}
a &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s-2j)(s+3j)(s-3j)}\right|_{s=-2j} = 0.5 \\
b &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s+3j)(s-3j)}\right|_{s=+2j} = 0.5 \\
c &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s-3j)}\right|_{s=-3j} = 0.5j \\
d &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s+3j)}\right|_{s=+3j} = -0.5j
\end{aligned}$$

Therefore

$$G(s) = \frac{0.5}{s+2j} + \frac{0.5}{s-2j} + \frac{0.5j}{s+3j} + \frac{-0.5j}{s-3j}, \qquad \mathrm{Re}[s] > 0$$

and the inverse LT is

$$g(t) = 0.5\left(e^{-2jt} + e^{2jt} + je^{-3jt} - je^{3jt}\right)\sigma(t)$$

This can be expanded using Euler's formula, which states that

$$e^{ajt} = \cos at + j\sin at$$

Applying Euler's formula yields

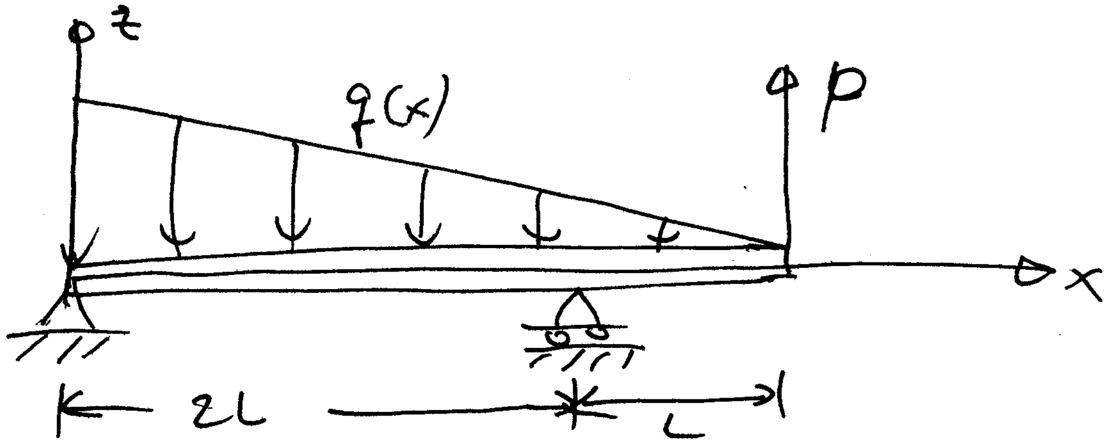
$$g(t) = (\cos 2t + \sin 2t)\,\sigma(t)$$

PAL
3/11/07

UNIFIED ENGINEERING
PROBLEM SET - WEEK 6 Spring, 2006

SOLUTIONS

M 6.1



First determine the relative magnitude of $q(x)$ is terms of P.

we are given:

$$\left| \int_0^{3L} q(x)\,dx \right| = |P|$$

Need a functional expression for $q(x)$. Give it a value of q_0 at $x = 0$. It linearly tapers to zero at $x = 3L$. So:

$$q(x) = \left(\frac{3L-x}{3L}\right) q_0$$

Check: @ $x=0$: $q(0) = q_0$

@ $x = 3L$: $q(3L) = 0$

$$\frac{\partial q(x)}{\partial x} = -\frac{q_0}{3L} \Rightarrow \text{decreases linearly}$$

✓ checks

Use this in the previous equation:

$$\left| \int_0^{3L} \frac{q_0}{3L}(3L-x)\,dx \right| = P$$

$$\Rightarrow \left| \frac{q_0}{3L}\left(3Lx - \frac{x^2}{2}\right)\Big]_0^{3L} \right| = P$$

$$\Rightarrow \left| \frac{q_0}{3L}\left(9L^2 - \frac{9L^2}{2}\right) \right| = P$$

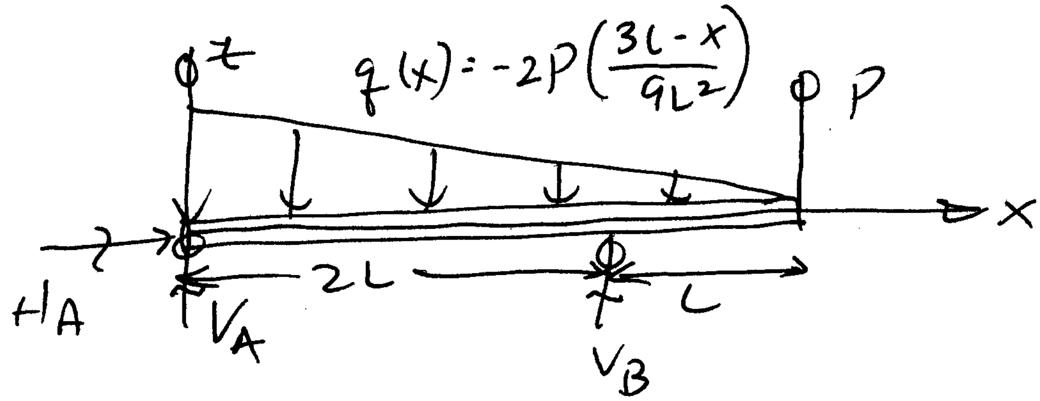
$$\Rightarrow q_0 \frac{3L^2}{2} = P \quad \Rightarrow q_0 = \frac{2P}{3L}$$

So: $\underset{\text{magnitude}}{\rightarrow} |q(x)| = \frac{2P}{3L}\left(\frac{3L-x}{3L}\right) = 2P\left(\frac{3L-x}{9L^2}\right)$

with direction, $q(x)$ is downward, so:

$$q(x) = -2P\left(\frac{3L-x}{9L^2}\right)$$

(a) Draw the free Body Diagram:



Use equilibrium:

$\sum F_x = 0 \xrightarrow{+} \Rightarrow H_A = 0$

$\sum F_z = 0 \uparrow + \Rightarrow V_A + V_B + P - \underbrace{\int_0^{3L} (2P)\left(\frac{3L-x}{9L^2}\right)dx}_{} = 0$

Recall this integrates to a magnitude of P with a negative direction. So:

$V_A + V_B + P - P = 0 \Rightarrow V_A + V_B = 0$

$\sum M_A = 0 \;\circlearrowleft + \Rightarrow V_B(2L) + P(3L) - \int_0^{3P} 2P\left(\frac{3L-x}{9L^2}\right)x\,dx = 0$

$\Rightarrow V_B(2L) + P(3L) - \frac{2P}{9L^2}\left(\frac{3Lx^2}{2} - \frac{x^3}{3}\right)\Big]_0^{3L} = 0$

$\Rightarrow V_B(2L) + P(3L) - \frac{2P}{9L^2}\left(\frac{27L^3}{2} - \frac{27L^3}{3}\right) = 0$

$2V_B + 3P - 2P\left(\frac{1}{2}\right) = 0$

So: $2V_B = -2P \Rightarrow V_B = -P$

from before $V_A = -V_B = +P$

Summarizing, the reactions are:

$$\boxed{\begin{aligned} H_A &= 0 \\ V_A &= +P \\ V_B &= -P \end{aligned}}$$

b) This needs to be done in parts since there are point loads (reactions) along the beam

for $0 \leq x \leq 2L$

there is no loading in x so: $\boxed{f(x) = 0}$

$$q(x) = -2P\left(\frac{3L - x}{9L^2}\right)$$

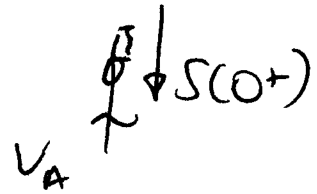
use $\frac{dS}{dx} = q(x)$

$$\Rightarrow S(x) = -\int 2P\left(\frac{3L-x}{9L^2}\right)dx$$

$$= -\frac{2P}{9L^2}\left(3Lx - \frac{x^2}{2}\right) + C_1$$

use a boundary condition to get the constant of integration.

Look at $x = 0+$



$\Sigma F_z = 0 \uparrow + \Rightarrow V_A = S(0+)$

so $S(0+) = V_A = +P = C_1 \Rightarrow C_1 = +P$

giving $S(x) = \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right) + P$

Now use $\frac{dM}{dx} = S$

$$\Rightarrow M(x) = \int \left\{ \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right) + P \right\} dx$$

$$= \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + Px + C_2$$

Again, use a boundary condition.

@ $x = 0$, $M = 0$

$\Rightarrow C_2 = 0$

Summarizing:

$$\boxed{\begin{array}{l} \text{for } 0 \leq x \leq 2L \\ f(x) = 0 \\ q(x) = -\frac{2P}{9L^2}(3L - x) \\ S(x) = \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right) + P \\ M(x) = \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + Px \end{array}}$$

→ Move on to $2L \leq x \leq 3L$

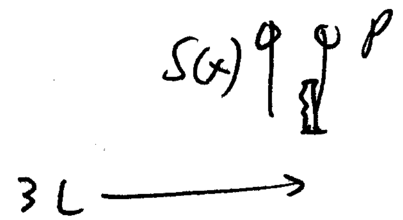
There is still no loading in x, so: $\boxed{f(x) = 0}$

distributed $q(x) = -2P\left(\frac{3L - x}{9L^2}\right)$

so using $\frac{dS}{dx} = q(x)$

we again get: $S(x) = \int q(x) = -\frac{2P}{9L^2}\left(3Lx - \frac{x^2}{2}\right) + C_3$

but there are different boundary conditions in this sector. Go to the tip ($x = 3L$) and take a cut giving a "negative" face:



$\sum F_z = 0 \uparrow + \Rightarrow S(3L) = -P$

using this in our expression:

$$S(3L) = -P = \frac{-2P}{9L^2}\left(9L^2 - \frac{9L^2}{2}\right) + C_3$$

$$\Rightarrow -P = -P + C_3 \quad \Rightarrow C_3 = 0$$

so: $S(x) = \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right)$

again use $\frac{dM}{dx} = S(x)$

$$\Rightarrow M(x) = \int S(x)\,dx$$

$$= \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + C_4$$

going to the tip ($x = 3L$) $M = 0$

using this:

$$0 = M(3L) = \frac{2P}{9L^2}\left(\frac{27L^3}{6} - \frac{27L^3}{2}\right) + C_4$$

$$\Rightarrow 0 = 2P(-L) + C_4 \quad \Rightarrow C_4 = 2PL$$

giving: $M(x) = \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + 2PL$

Summarizing: for $2L \le x \le 3L$

$$\boxed{\begin{aligned} F(x) &= 0 \\ q(x) &= -\frac{2P}{9L^2}(3L - x) \\ S(x) &= \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right) \\ M(x) &= \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + 2PL \end{aligned}}$$

There are no point moments applied, so the solutions for $M(x)$ for the two segments must be equal at $x = 2L$. CHECK:

$$\frac{2P}{9L^2}\left(\frac{8L^3}{6} - \frac{12L^3}{2}\right) + 2PL \stackrel{?}{=} \frac{2P}{9L^2}\left(\frac{8L^3}{6} - \frac{12L^3}{2}\right) + 2PL$$

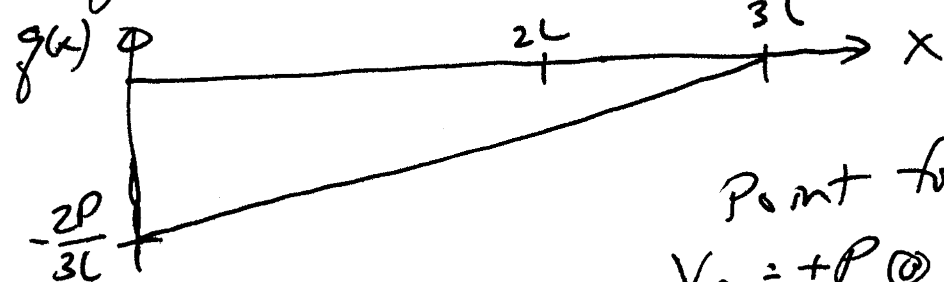
$$\left(\begin{aligned} &= 2P\left(\frac{8L}{54} - \frac{36L}{54}\right) + 2PL \\ &= +\frac{26}{27}PL \end{aligned} \right)$$

YES!

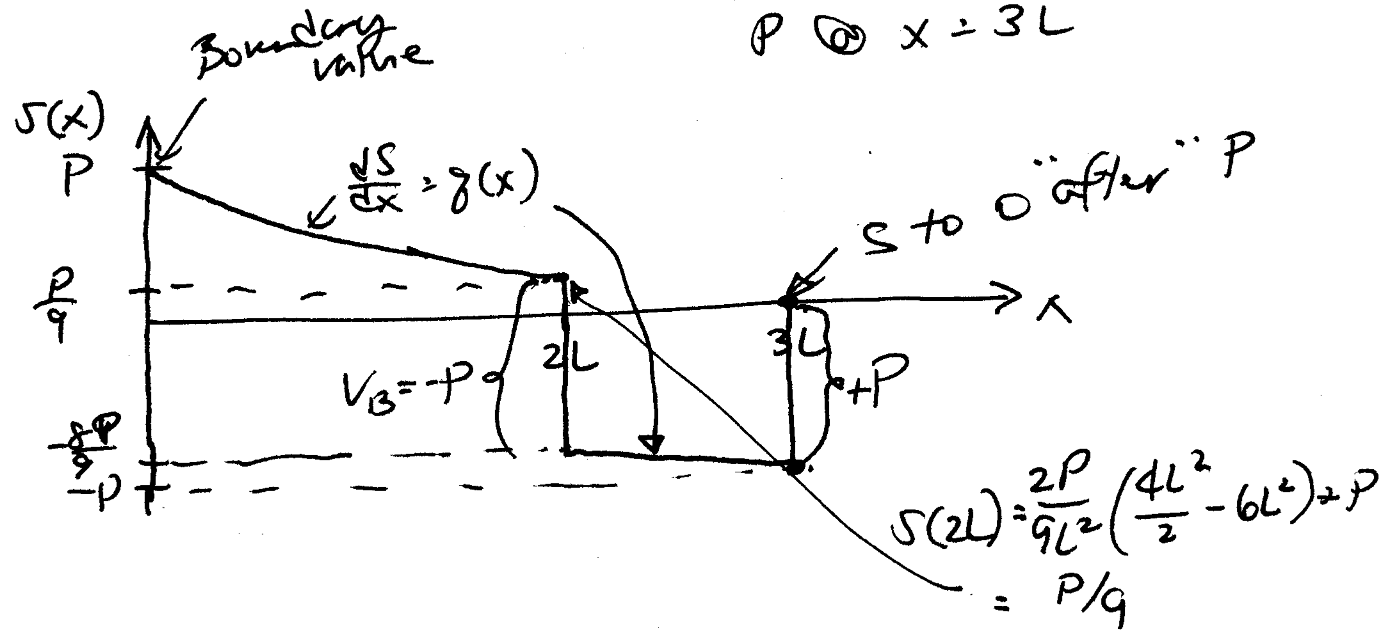
Now draw these. In sketching, use the relations of the derivatives to get a shape. Calculate end point values to begin. And recall that point loads cause equal jumps in shear (account for proper direction and sign).

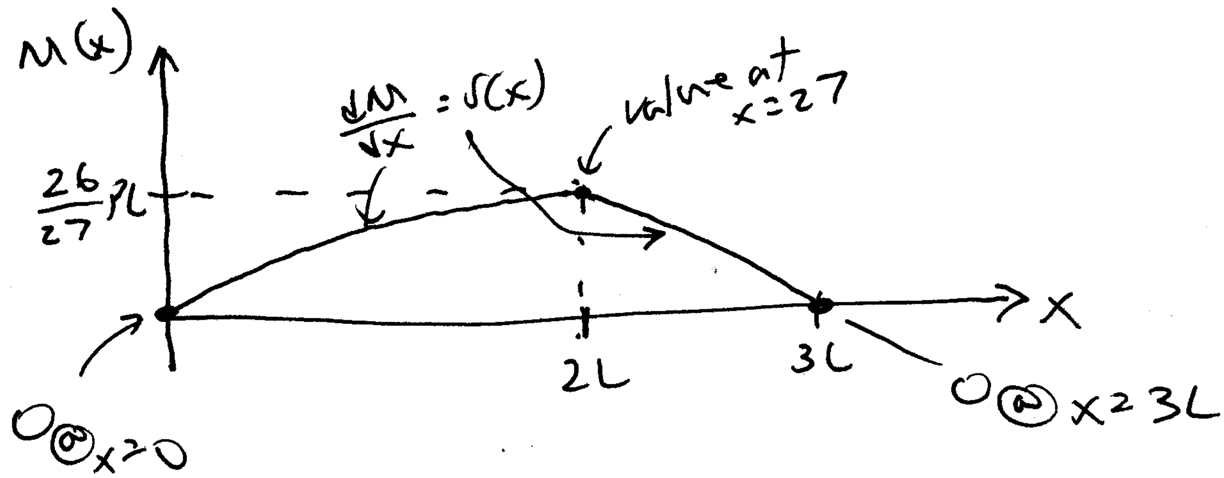
$F(x) = 0$ everywhere ... no need to plot

Loading:

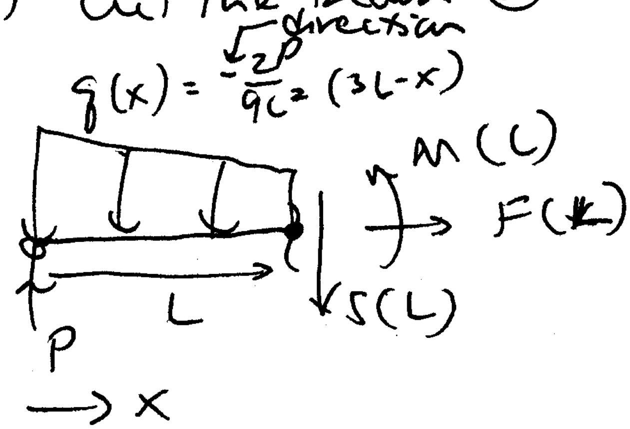


Point forces
$V_A = +P$ @ $x = 0$
$V_B = -P$ @ $x = 2L$
P @ $x = 3L$





(c) Cut the beam @ $x = L$:



Use equilibrium:

$\sum F_x = 0 \; \overset{+}{\rightarrow} \Rightarrow F(L) = 0$ ✓ check

$\sum F_z = 0 \; \uparrow + \Rightarrow P - \int_0^L \frac{2P}{9L^2}(3L - x)\,dx - S(L) = 0$

$$\Rightarrow P + \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right)\Big]_0^L = S(L)$$

$$\text{so: } S(L) = P + \frac{2P}{9L^2}\left(\frac{L^2}{2} - 3L^2\right)$$

$$= \frac{4}{9}P$$

$$\text{Check: } S(x) = \frac{2P}{9L^2}\left(\frac{x^2}{2} - 3Lx\right) + P$$

$$\Rightarrow S(L) = \frac{2P}{9L^2}\left(\frac{L^2}{2} - 3L^2\right) + P$$

$$= \frac{4}{9}P \quad ✓$$

Finally:

$$\sum M_L = 0 \; \circlearrowleft + \Rightarrow -PL + \int_0^L \frac{2P}{9L^2}(3L - x)\underbrace{(L - x)}_{*}\,dx + M(L) = 0$$

* Note: moment arm from point L is $(L - x)$
@ $x = L$, moment arm $= 0$.

$$\Rightarrow M(L) = PL - \frac{2P}{9L^2}\int_0^L \left(3L^2 - 4Lx + x^2\right)dx$$

$$= PL - \frac{2P}{9L^2}\left(3L^2x - 2Lx^2 + \frac{x^3}{3}\right)\Bigg]_0^L$$

$$= PL - \frac{2}{9}PL\left(3 - 2 + \frac{1}{3}\right)$$

$$= PL - \frac{2}{9}PL\left(\frac{4}{3}\right) = PL\left(1 - \frac{8}{27}\right) = \frac{19}{27}PL$$

Check: $M(x) = \frac{2P}{9L^2}\left(\frac{x^3}{6} - \frac{3}{2}Lx^2\right) + Px$

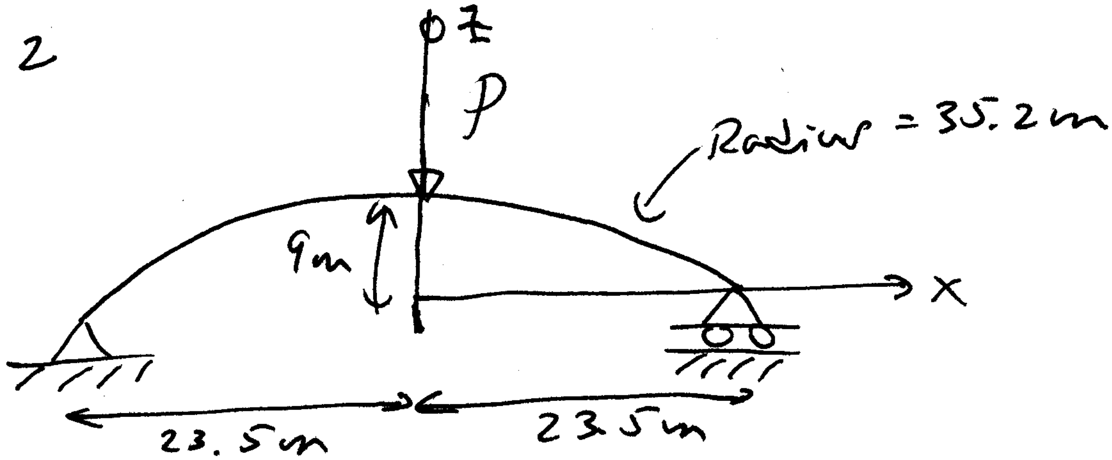
$$\Rightarrow M(L) = \frac{2PL}{9}\left(\frac{1}{6} - \frac{3}{2}\right) + PL$$

$$= \frac{2PL}{9}\left(-\frac{8}{6}\right) + PL$$

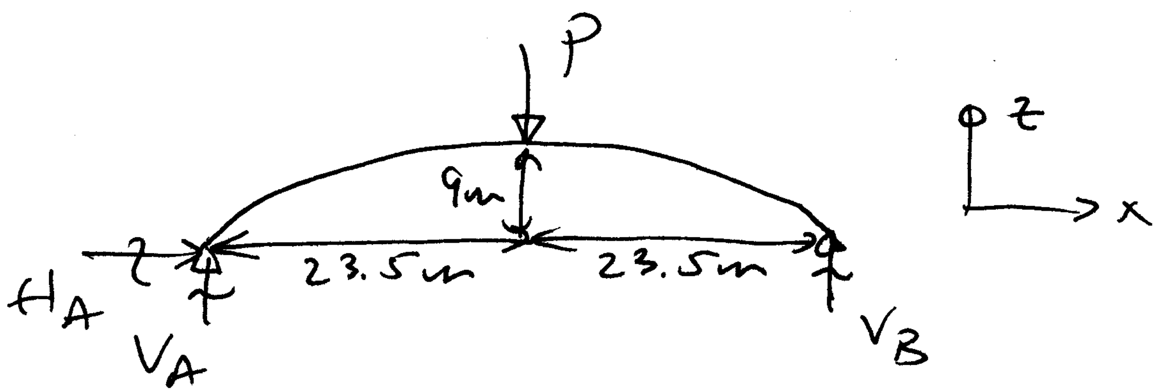
$$= -\frac{8}{27}PL + PL = \frac{19}{27}PL \quad \checkmark$$

<u>ALL</u> <u>CHECK</u>

M6.2



(a) First draw an FBD (Free Body Diagram):



Use equilibrium:

$$\sum F_x = 0 \ \overset{+}{\rightarrow} \Rightarrow H_A = 0$$

$$\sum F_z = 0 \ \uparrow + \Rightarrow V_A + V_B - P = 0$$

$$\sum M_A = 0 \ \circlearrowleft + \Rightarrow -P(23.5m) + V_B(47m) = 0$$

$$\Rightarrow V_B = P/2$$

use symmetry

$$\Rightarrow V_A = P/2$$

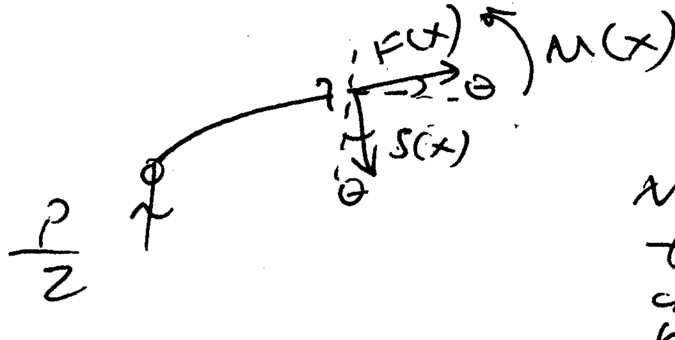
this makes sense since Kresge's configuration is symmetric, so the reactions should be symmetric.

Summarizing:

$$\boxed{\begin{aligned} H_A &= 0 \\ V_A &= P/2 \\ V_B &= P/2 \end{aligned}}$$

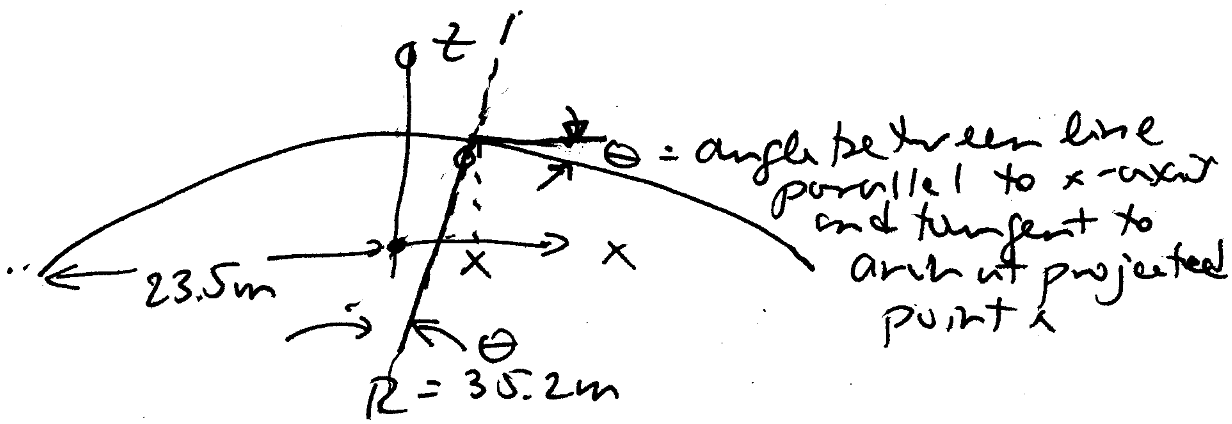
(b) we will analyze this in two sections: "prior to the load $(x<0)$ and after the load $(x>0)$ [on each side of the center of the front face].

As we cut the Kresge arched beam we note that the reactions have an angle to them since the tangent to the beam is not parallel to the x-axis except at the peak:



NOTE: Shear and axial force and perpendicular and parallel to the cut beam face (90° to its tangent) at any point

The key is to do some geometry and determine the x- and z- components (via the angle θ) and any point x projected onto the beam:

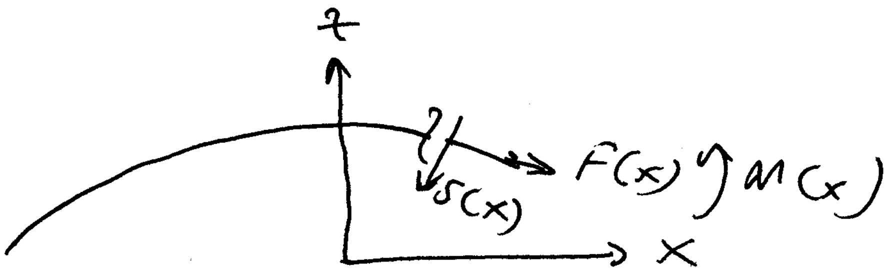


The tangent line will be perpendicular to the intersection of the radius with the arched beam. So the angle θ is the same as that between the z-axis and the radial line to the point on the arched beam.
Finally the distance from the z-axis to the arched beam parallel to the x-axis is x and is equal to R sin θ. So:

$$x = R\sin\theta \quad \Rightarrow \sin\theta = \frac{x}{R}$$

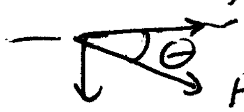
$$\Rightarrow \theta = \sin^{-1}\frac{x}{R}$$

Now look at the beam resultants:



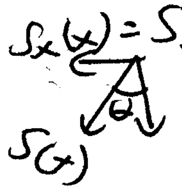
Consider F(x) and S(x) and resolve components along x and z:

$$F_x(x) = F(x)\cos\theta$$



$$F_z(x) = F(x)\sin\theta$$

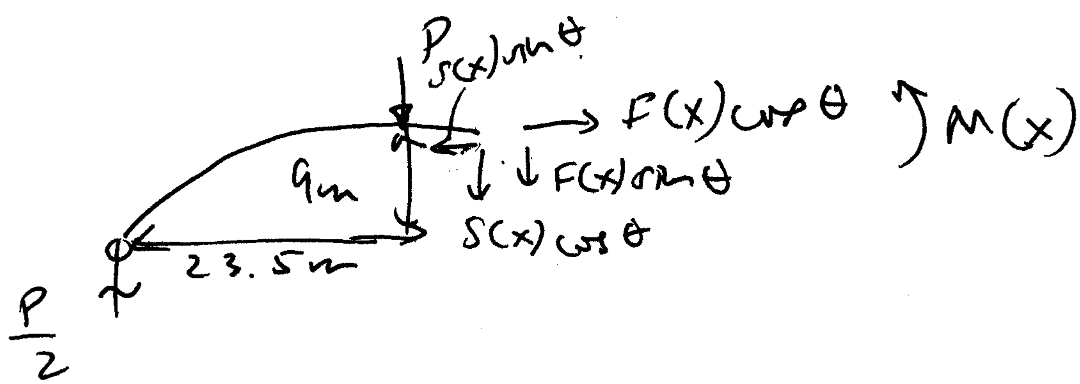
$$S_x(x) = S_x \sin\theta$$



$$S_z(x) = S(x)\cos\theta$$

Now use equilibrium. We'll start after the load (midpoint)

<u>$0 < x < 23.5\,m$</u>



$$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(x)\cos\theta - S(x)\sin\theta = 0$$

$$\Rightarrow F(x) = S(x)\tan\theta$$

$$\sum F_z = 0 \uparrow + \Rightarrow \frac{P}{2} - P - S(x)\cos\theta - F(x)\sin\theta = 0$$

$$\text{so: } -S(x)\cos\theta - S(x)\frac{\sin^2\theta}{\cos\theta} = \frac{P}{2}$$

$$\Rightarrow S(x)\left[\cos^2\theta + \sin^2\theta\right] = -\frac{P}{2}\cos\theta$$

giving $S(x) = -\frac{P}{2}\cos\theta$

using in what we found for $F(x)$:

$$F(x) = -\frac{P}{2}\sin\theta$$

Now the moment:

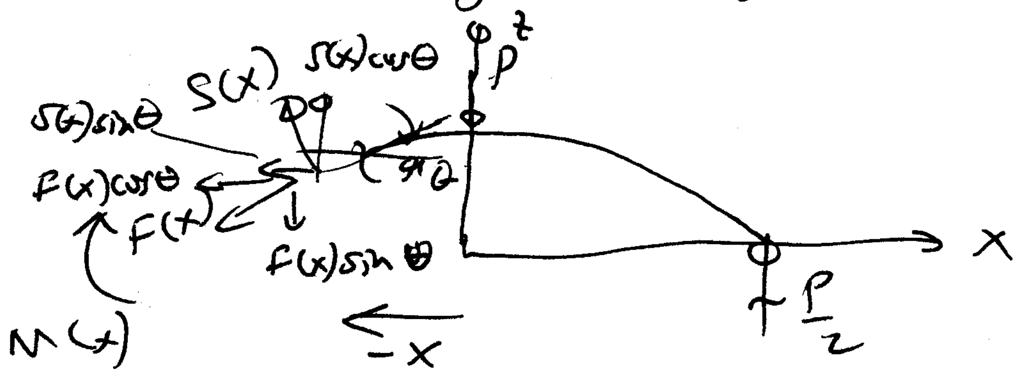
Take it about the point of cut in the Kreuge beam so one does not need to worry about $F(x)$ and $S(x)$. The moment arm for the reaction is $(x + 23.5\,m)$; for the applied load it is x:

$$\sum M_{x_{cut}} = 0 \; \circlearrowleft + \Rightarrow -\frac{P}{2}(23.5\,m + x) + Px + M(x) = 0$$

$$\text{so: } -\frac{Px}{2} + P(11.75\,m) = M(x)$$

If we consider the case for the other side of the centerpoint: $\underline{-23.5\text{in} < x < 0}$

we can use symmetry and draw:



The switching of directions including in the angle will account for many of the changes. However, we need to not doubly account for the negative aspect in using $\sin\theta$ since for $x<0$, $\sin\theta < 0$ and the picture above actually depicts $|\sin\theta|$. Account for this in the summation

$$\sum F_x = 0 \overset{+}{\rightarrow} \Rightarrow -F(x)\cos\theta + S(x)\sin\theta = 0$$

$$\Rightarrow F(x) = S(x)\tan\theta$$

$$\sum F_z = 0 \quad \uparrow + \Rightarrow \frac{P}{2} - P + S(x)\cos\theta + F(x)\sin\theta = 0$$

$$\text{so: } S(x)\cos\theta + S(x)\frac{\sin^2\theta}{\cos\theta} = \frac{P}{2}$$

$$\Rightarrow S(x) = \frac{P}{2}\cos\theta$$

with what we found for $F(x)$:

$$F(x) = \frac{P}{2}\sin\theta$$

finally:

account for sign

$$\sum M_{x_{cut}} = 0 \quad \circlearrowleft + \Rightarrow M(x) + P(-x) - \frac{P}{2}(23.5m - x) = 0$$

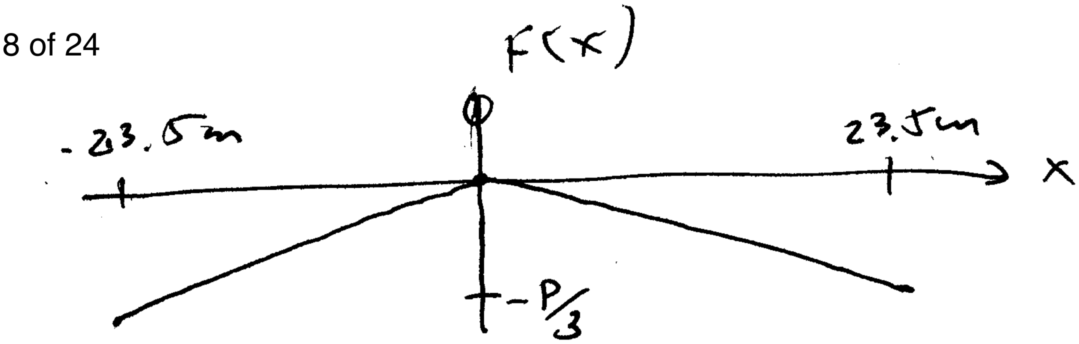
$$\Rightarrow M(x) = \frac{Px}{2} + P(11.75m)$$

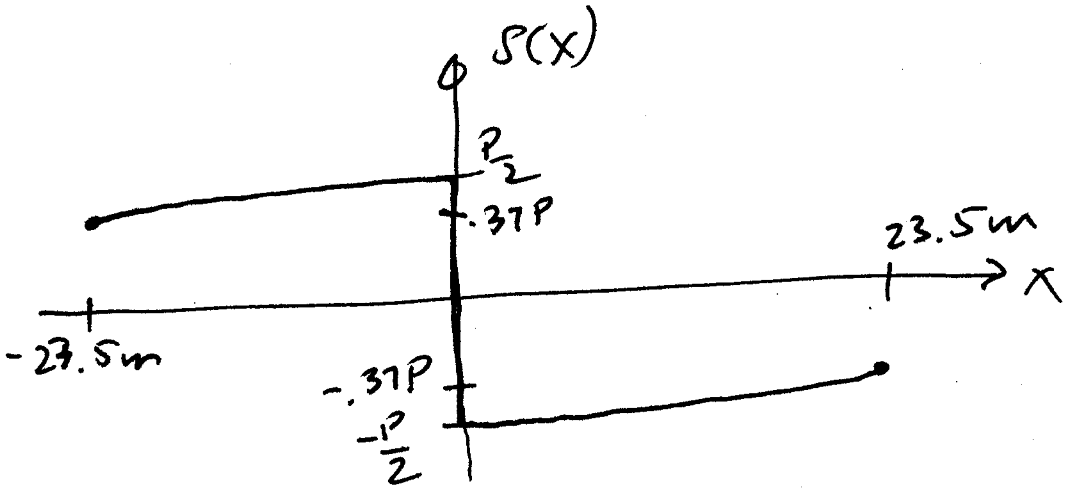
Summarizing

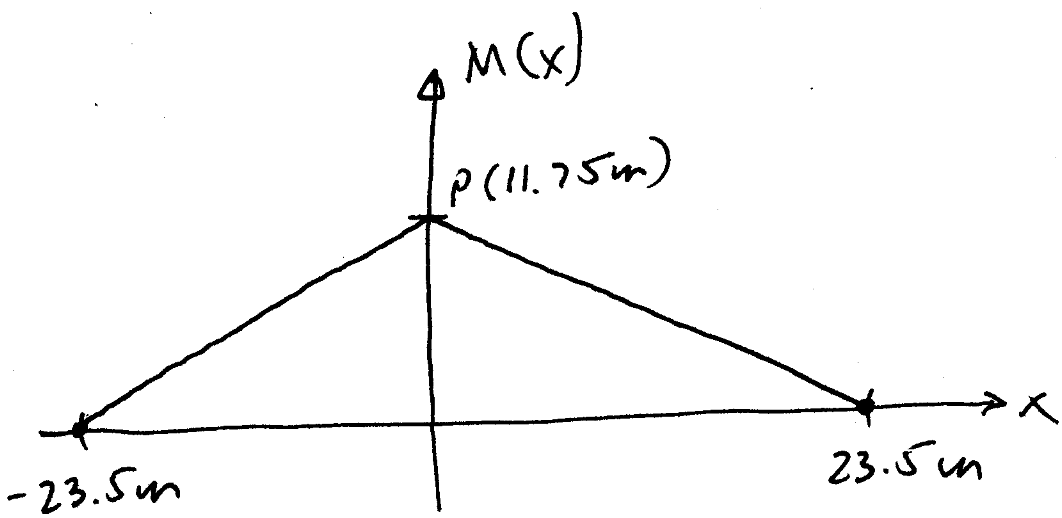
center

$$\boxed{\begin{array}{ll} -23.5m < x < 0 & 0 < x < 23.5m \\ F(x) = \frac{P}{2}\sin\theta = \frac{Px}{2R} & F(x) = -\frac{P}{2}\sin\theta = \frac{-Px}{2R} \\ S(x) = \frac{P}{2}\cos\theta & S(x) = -\frac{P}{2}\cos\theta \\ M(x) = \frac{Px}{2} + P(11.75m) & M(x) = -\frac{Px}{2} + P(11.75m) \end{array}}$$

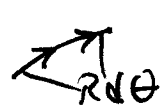
Now draw sketches



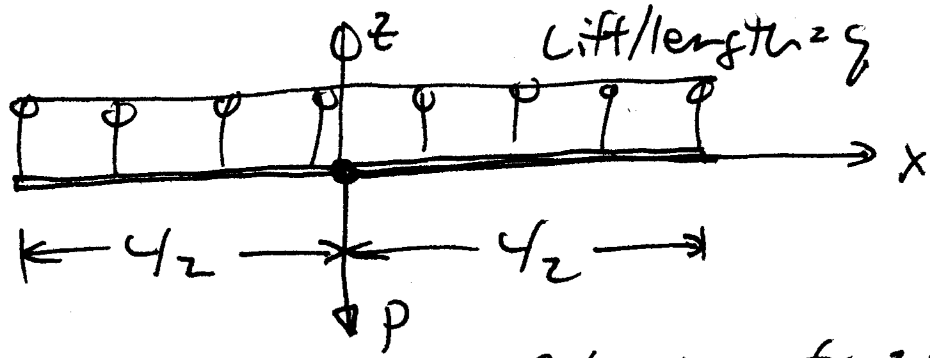




(<u>NOTE</u>: The equations $\frac{dS}{dx} = q$ and $\frac{dM}{dx} = S$ are not exactly followed since one must consider the length <u>along</u> the arch for such)



M6.3



- Put origin of x-z system at fuselage
- Assign magnitude of lift/length as q

(a) The model is basically the free body diagram.

There are no reaction forces since the wing has no external supports that can carry the load.

NOTE: It is not dynamic because of symmetry (balanced moment) and because of the special condition that the integrated lift force equals total plane weight in stable flight

(b) For steady level flight, the total lift must be equal to the weight, i.e.:

$$\sum F_z = 0 \quad \oplus\uparrow \Rightarrow \; -P + \int_{-L/2}^{+L/2} q\,dx = 0$$

$$\Rightarrow q x \Big]_{-L/2}^{+L/2} = P \Rightarrow \boxed{q = \frac{P}{L}}$$

PAL

There are no axial forces so $F(x) = 0$

We have two sections of the wing:

$$0 < x < L/2$$

$$-L/2 < x < 0$$

There is symmetry so our results should be the same, but let's be sure:

For: $\underline{0 < x < L/2}$

$$q(x) = P/L$$

use: $\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int \frac{P}{L} dx$

$$= \frac{Px}{L} + C_1$$

Go to the tip and see $S = 0$ at $x = L/2$

$$\Rightarrow S(L/2) = 0 = \frac{P}{2} + C_1 \Rightarrow C_1 = -P/2$$

$$\Rightarrow S(x) = P\left(\frac{x}{L} - \frac{1}{2}\right)$$

progress to:

$$\frac{dM}{dx} = S$$

$$\Rightarrow M(x) = \int P\left(\frac{x}{L} - \frac{1}{2}\right) dx$$

$$= \frac{Px^2}{2L} - \frac{Px}{2} + C_2$$

Again, at the tip: $M = 0$ at $x = L/2$

so: $M\left(\frac{L}{2}\right) = 0 = \frac{PL}{8} - \frac{PL}{4} + C_2$

$$\Rightarrow C_2 = \frac{PL}{8}$$

finally: $M(x) = \frac{P}{2}\left(\frac{x^2}{L} - x + \frac{L}{4}\right)$

Now for: $\underline{-L/2 < x < 0}$

again $q(x) = \frac{P}{L}$

using: $\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int \frac{P}{L}\,dx$

$$= \frac{Px}{L} + C_3$$

Go to that tip $(x = -L/2)$ where $S = 0$ and:

$$S(-L/2) = -\frac{P}{2} + C_3 \Rightarrow C_3 = +\frac{P}{2}$$

$$\Rightarrow S(x) = P\left(\frac{x}{L} + \frac{1}{2}\right)$$

Note that the value must change by the weight P at the root $\left(\frac{dS}{dx}(x=0) = -P\right)$ and it does!

and finally with: $\frac{dM}{dx} = S(x)$

$$\Rightarrow M(x) = \int P\left(\frac{x}{L} + \frac{1}{2}\right)dx$$

$$= \frac{Px^2}{2L} + \frac{Px}{2} + C_4$$

Again, at this tip $(x = -L/2)$, the moment is zero. So:

$$M(-L/2) = 0 = \frac{PL}{8} - \frac{PL}{4} + C_4$$

$$\Rightarrow C_4 = \frac{PL}{8}$$

resulting in: $M(x) = \frac{P}{2}\left(\frac{x^2}{L} + x + \frac{L}{4}\right)$

This is symmetric about the root as x switches from + to − in value.

Summarizing:

$F(x) = 0$ everywhere

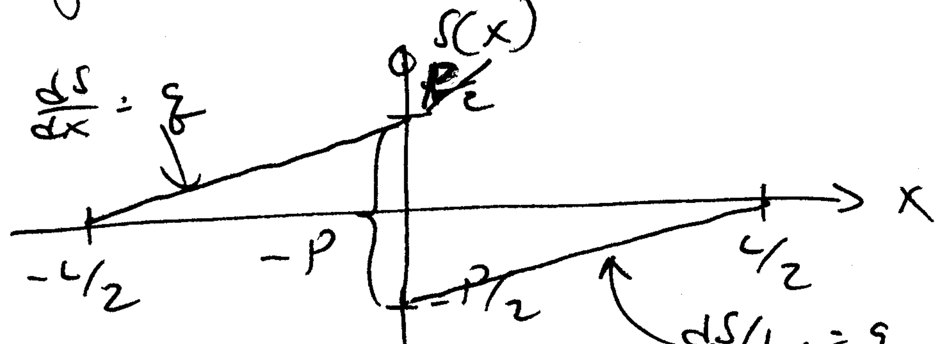
$S(x) = P\left(\frac{x}{L} - \frac{1}{2}\right) \quad 0 < x < L/2$

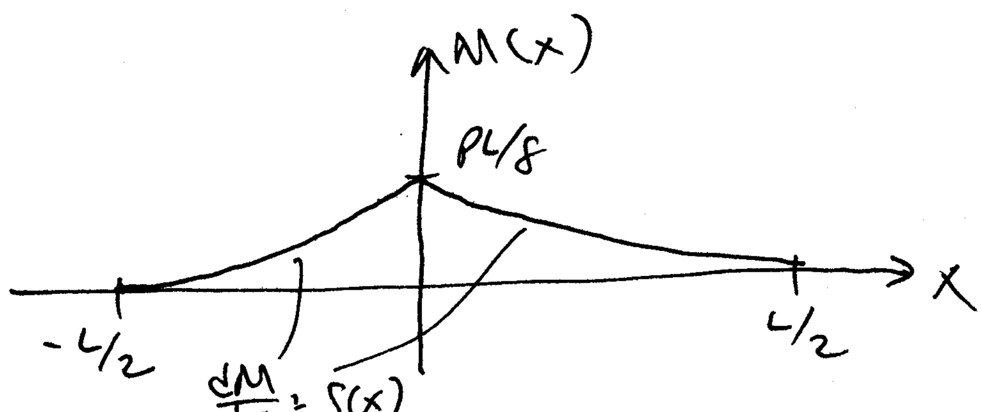
$\quad = P\left(\frac{x}{L} + \frac{1}{2}\right) \quad -L/2 < x < 0$

$M(x) = \frac{P}{2}\left(\frac{x^2}{L} - x + \frac{L}{4}\right) \quad 0 < x < L/2$

$\quad = \frac{P}{2}\left(\frac{x^2}{L} + x + \frac{L}{4}\right) \quad -L/2 < x < 0$

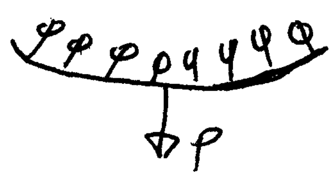
Sketching the last two:





(c) The highest moment is at the root, so that is the location of greatest loading or in the shear.

A commonsense inspection of the loading conditions suggests that the wing will deform in the following manner:



(symmetric about the fuselage)

Note also that the load is transferred at the attachment to the fuselage

(forethought)

Now have $L \rightarrow 1.1L$

What happens? First q decreases to $q = \frac{P}{1.1L}$

(still integrates to P, but longer length)

So the shear will be "strung out" but still have the same maximum value of $P/2$:

$$S(x) = P\left(\frac{x}{1.1L} - \frac{1}{2}\right) \qquad 0 < x < \frac{1.1L}{2}$$

$$= P\left(\frac{x}{1.1L} + \frac{1}{2}\right) \qquad -\frac{1.1}{2}L < x < 0$$

The key change comes in the moment. There are longer moment arms, so the maximum moment at the root will increase. Via the equation:

$$M(x) = \frac{P}{2}\left(\frac{x^2}{1.1L} - x + \frac{1.1L}{4}\right) \quad 0 < x < \frac{1.1}{2}L$$

$$= \frac{P}{2}\left(\frac{x^2}{1.1L} + x + \frac{1.1L}{4}\right) \quad -\frac{1.1}{2}L < x < 0$$

This gives a maximum value of

$$M(0) = \frac{1.1\,PL}{8}$$

it increases by 10%!