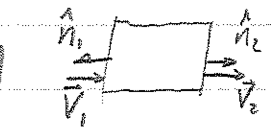
OE Fluids Problem F11 + 12 + 13 Solution Spring '07

a) $\mathcal{V} = \ell \cdot A$, where $\ell(t) = x_2 - x_1$ = length of CV.

$$\boxed{d\mathcal{V}/dt = d\ell/dt \cdot A = (dx_2/dt - dx_1/dt) \cdot A = (V_2 - V_1)A}$$

b) $\dot{W} = \oint\!\!\oint -p\vec{V}\cdot\hat{n}\, dA = -\left(p_1 \vec{V}_1\cdot\hat{n}_1 + p_2\vec{V}_2\cdot\hat{n}_2\right)A = -\left(-p_1V_1 + p_2V_2\right)A$



$$\boxed{\dot{W} = (p_1V_1 - p_2V_2)A}$$

c) $p_1 = p_{avg} + \frac{\Delta p}{2}$, $p_2 = p_{avg} - \frac{\Delta p}{2}$, $V_1 = V_{avg} + \frac{\Delta V}{2}$, $V_2 = V_{avg} - \frac{\Delta V}{2}$

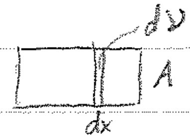
Subsititute into $\dot{W}$ expression:

$$\dot{W} = \left[\left(p_{avg} + \frac{\Delta p}{2}\right)\left(V_{avg} + \frac{\Delta V}{2}\right) - \left(p_{avg} - \frac{\Delta p}{2}\right)\left(V_{avg} - \frac{\Delta V}{2}\right)\right]A$$

$$= \left[\cancel{p_{avg}V_{avg}} + \frac{\Delta p}{2}V_{avg} + p_{avg}\frac{\Delta V}{2} + \cancel{\frac{\Delta p \Delta V}{4}} - \cancel{p_{avg}V_{avg}} + \frac{\Delta p}{2}V_{avg} + p_{avg}\frac{\Delta V}{2} - \cancel{\frac{\Delta p\Delta V}{4}}\right]A$$

$$\boxed{\dot{W} = \left[\Delta p\, V_{avg} + p_{avg}\,\Delta V\right]A}$$

d) $V(x) = V_{avg} - \Delta V \frac{x}{\ell}$, $d\mathcal{V} = A\,dx$



$$E_{kin} = \int_{-\ell/2}^{\ell/2} \tfrac{1}{2}\rho\left(V_{avg} - \Delta V\frac{x}{\ell}\right)^2 A\,dx$$

$$= \tfrac{1}{2}\rho A \int_{-\ell/2}^{\ell/2}\left(V_{avg}^2 - 2V_{avg}\frac{x}{\ell} + \Delta V^2\left(\frac{x}{\ell}\right)^2\right)dx$$

$$E_{kin} = \tfrac{1}{2}\rho A\left(V_{avg}^2 x - V_{avg}\frac{x^2}{\ell} + \tfrac{1}{3}\Delta V^2\frac{x^3}{\ell^2}\right)\Bigg|_{-\ell/2}^{\ell/2} = \tfrac{1}{2}\rho A\ell\left(V_{avg}^2 + \tfrac{1}{24}\Delta V^2\right)$$

or $E_{kin} = \frac{1}{2}m\left(V_{avg}^2 + \frac{1}{24}\Delta V^2\right)$; $m = \rho\mathcal{V} = \rho A\ell$

e) Force on CV: $F = (p_1 - p_2)A = \Delta p A$

The $\dot{W}$ from c) can be written as $\dot{W} = F\cdot V_{avg} + p_{avg}\frac{d\mathcal{V}}{dt}$

Clearly, the $F\cdot V_{avg}$ looks like mechanical power, going into E_{kin}.

Also, the $p_{avg}\cdot\frac{d\mathcal{V}}{dt}$ is "$p\,dv$" work rate, going into E_{int}.

Hence, $\frac{dE_{kin}}{dt} \simeq \Delta p\, V_{avg} A$, $\frac{dE_{int}}{dt} \simeq p_{avg}\,\Delta V A$

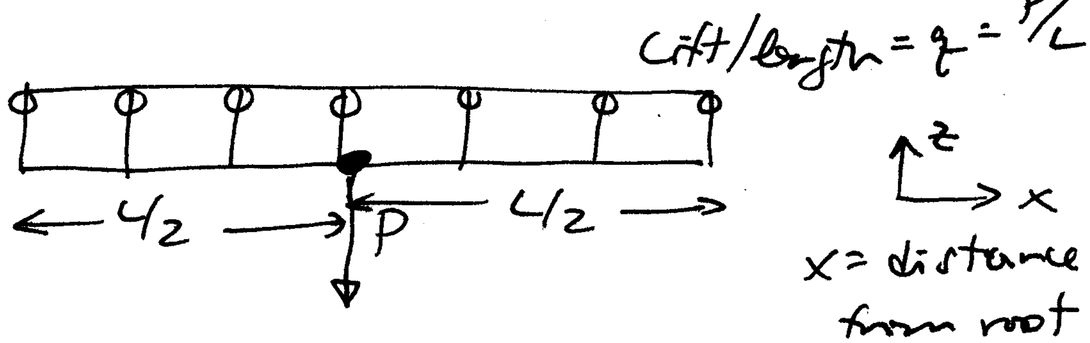
f) $E_{int} = \iiint \rho c_v T\, d\mathcal{V} = \rho\mathcal{V}c_vT$, $\frac{dE_{int}}{dt} = \rho\mathcal{V}c_v\frac{dT}{dt}$; since $\rho\mathcal{V} = m$ = constant here

$$\rightarrow \boxed{\frac{dT}{dt} = \frac{1}{\rho\mathcal{V}c_v}\cdot p_{avg}\,\Delta V A = \frac{p_{avg}\,\Delta V}{\rho\ell c_v}}$$

PAL
3/20/07

Unified Engineering Problem Set
Week 7 Spring, 2007
SOLUTIONS

M7.1 We continue considering the wing configuration of M6:



We use the results from M6 to continue our analysis of the configuration:

$F(x) = 0$ everywhere

$$S(x) = P\left(\frac{x}{L} - \frac{1}{2}\right) \qquad 0 < x < L/2$$

$$= P\left(\frac{x}{L} + \frac{1}{2}\right) \qquad -L/2 < x < 0$$

$$M(x) = \frac{P}{2}\left(\frac{x^2}{L} - x + \frac{L}{4}\right) \qquad 0 < x < L/2$$

$$\frac{P}{2}\left(\frac{x^2}{L} + x + \frac{L}{4}\right) \qquad -L/2 < x < 0$$

(a) The axial stress is related to the moment via:

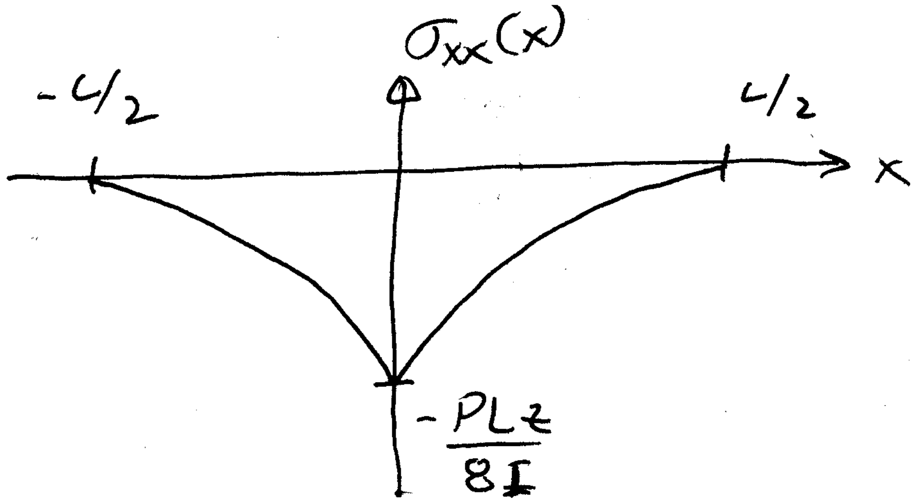
$$\sigma_{xx} = -\frac{M(x)z}{I}$$

$$\Rightarrow \sigma_{xx} = \frac{-Pz}{2I}\left(\frac{x^2}{L} - x + \frac{L}{4}\right) \quad 0 < x < L/2$$

$$= -\frac{Pz}{2I}\left(\frac{x^2}{L} + x + \frac{L}{4}\right) \quad -L/2 < x < 0$$

This stress varies at any point, x, along the beam with distance from the axis z. We do not know what the cross-section looks like (we just quantify its ability as I), but it does not affect the distribution of σ_{xx} if it is a constant cross-section and thus value. So the maximum stress occurs as far away from the axis that the cross-section goes. The moment is always positive, so the stress will be negative (compressive) for +z and positive (tensile) for -z. This is consistent for a beam that bends up.

The distribution of σ_{xx} with x will be the same as M(x) with the value modified by $-\frac{z}{I}$:



To find where along x the maximum value (think magnitude) occurs, one can just look at the diagram and see it occurs at the root (x=0)

$$\boxed{\text{Maximum } \sigma_{xx} \text{ at } x=0,\ \& \text{ maximum}}$$

$$= -\frac{PLz}{8I}$$

If one were to use $\frac{\partial \sigma_{xx}}{\partial x} = 0$ to do this, one gets:

$$0 = -\frac{Pz}{2I}\left(\frac{2x}{L} - 1\right) \qquad 0 < x < L/2$$

$$0 = -\frac{Pz}{2I}\left(\frac{2x}{L} + 1\right) \qquad -L/2 < x < 0$$

This gives $x = L/2, -L/2$ which is the <u>minimum</u>

$(\sigma_{xx} = 0)$

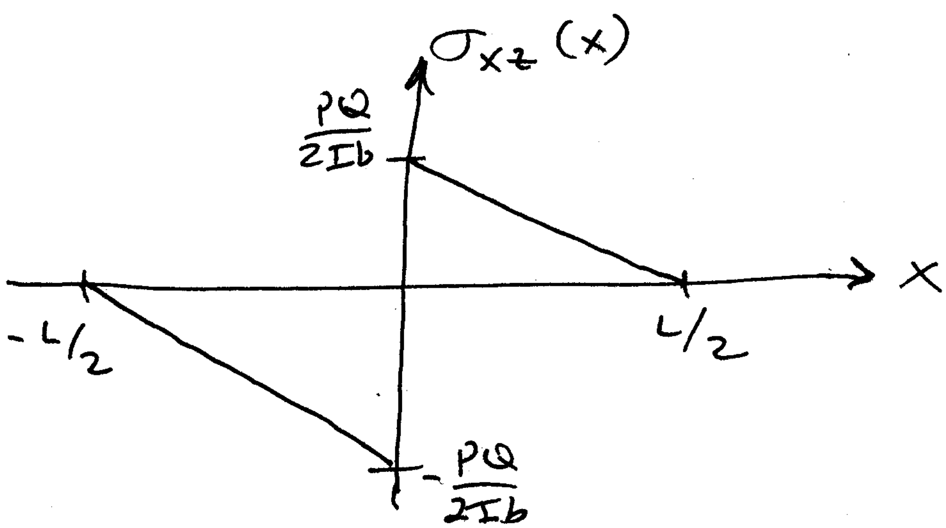
(b) The shear stress is related to shear force via:

$$\sigma_{xz} = -\frac{S(x)\,Q}{Ib}$$

This gives:

$$\boxed{\begin{aligned} \sigma_{xz} &= -\frac{PQ}{Ib}\left(\frac{x}{L}-\frac{1}{2}\right) && 0<x<L/2 \\ &= -\frac{PQ}{Ib}\left(\frac{x}{L}+\frac{1}{2}\right) && -L/2<x<0 \end{aligned}}$$

Again, we do not know the specifics of the cross-section, but can say the maximum in z occurs where Q/b is a maximum (the same along x if the cross-section is a constant). And the distribution of σ_{xz} is the same as $S(x)$ with the value changed by $-Q/Ib$:



It is clear from examination that the maximum in x occurs at the root (x = 0)

$$\boxed{\text{maximum } \sigma_{xz} \text{ at } x=0} \rightarrow \sigma_{xz} = \frac{PQ}{2Ib}$$

at $\frac{Q}{b}$ maximum

(c) The deflection of a beam, w, is related to the moment via:

$$EI \frac{d^2w}{dx^2} = M(x)$$

M(x) is symmetric in x, so w(x) must be as well. Using the equations:

$$\frac{d^2w}{dx^2} = \frac{P}{2EI}\left(\frac{x^2}{L} - x + \frac{L}{4}\right) \qquad 0<x<L$$

$$\Rightarrow \frac{dw}{dx} = \frac{P}{2EI}\left(\frac{x^3}{3L} - \frac{x^2}{2} + \frac{Lx}{4}\right) + C_1$$

$$\Rightarrow w(x) = \frac{P}{2EI}\left(\frac{x^4}{12L} - \frac{x^3}{6} + \frac{Lx^2}{8}\right) + C_1x + C_2$$

Now we need 2 Boundary Conditions. At the root, this is the reference point where the wing is attached to the fuselage. So the deflection relative to the plane is zero:

$$@ x=0,\ w=0 \Rightarrow C_2 = 0$$

The other boundary condition comes from

symmetry. Since the wing is continuous, it must have the same slope on each side of the fuselage at $x=0$. Due to symmetry, the slope must be zero, so:

$$@\, x=0,\ \frac{dw}{dx}=0 \Rightarrow C_1=0$$

This gives:

$$w(x)=\frac{P}{2EI}\left(\frac{x^4}{12L}-\frac{x^3}{6}+\frac{Lx^2}{8}\right)$$

check units:

$$\frac{[F]}{[F/L^2][L^4]}[L^3]=[L]\ \checkmark \text{ checks}$$

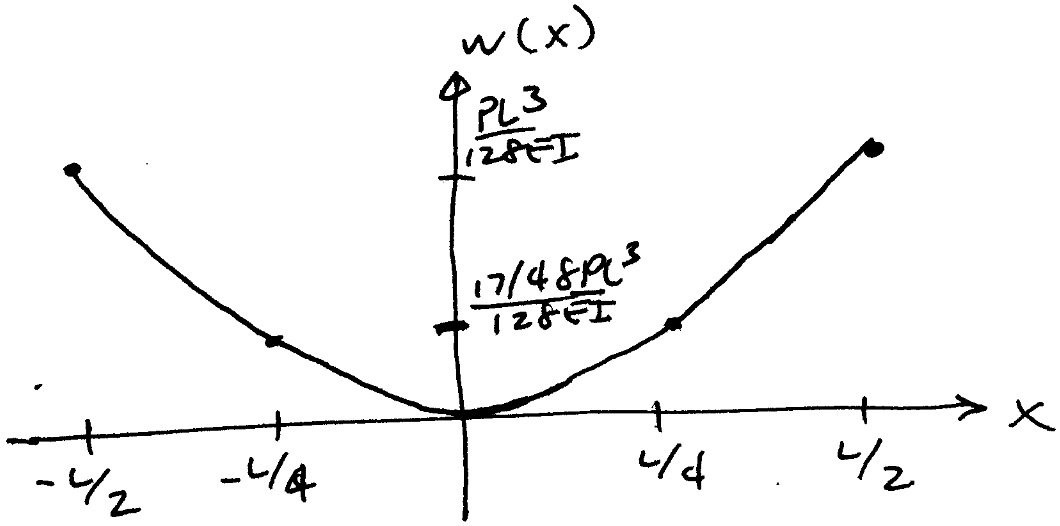
Looking at $-L/2<x<0$ the only difference in $M(x)$ is in the sign of the second term. The same boundary conditions apply, so the only difference is in the sign of the second term and that gives us symmetry as for $x<0$, this results in the same value for an odd power and a different sign! Thus:

$$w(x)=\frac{P}{2EI}\left(\frac{x^4}{12L}-\frac{x^3}{6}+\frac{Lx^2}{8}\right) \quad 0<x<L/2$$

$$=\frac{P}{2EI}\left(\frac{x^4}{12L}+\frac{x^3}{6}+\frac{Lx^2}{8}\right) \quad -L/2<x<0$$

Sketching this, it is important to get some points. The most important is the maximum value which will occur at the tip:

$$\boxed{w_{max} = w_{tip}\left(x = \pm L/2\right) = \frac{PL^3}{128EI}}$$



(for thought)

The maximum for the stresses will not change in z as this is only based on the cross-section. The question deals with x and that location and magnitude. In both cases, maximums occur at the root, so location ... do not change. For magnitude, L goes to 1.1L (or each half-span to $1.1L/2$) and the distributed load also changes. Our result from last time showed that shear was stronger, but the maximum value stayed the same at $\frac{P}{2}$, so

$$\boxed{\text{the } \sigma_{xz} \text{ maximum stays the same, equal to } \frac{PQ}{2Ib} \text{ at } x = 0}$$

There is a change in the moment as the maximum moment increased by 10% to $\frac{1.1 PL}{8}$ so:

$\boxed{\sigma_{xx} \text{ increases in magnitude by } 10\% \text{ with a maximum value of } -\frac{1.1 PLz}{8I} \text{ at } x=0}$

The big change comes in the deflection. The maximum will still be at the tip ($x = \pm L/2$), but the value increases. The overall shape is also the same. We can use the same resulting expressions as we have the same overall weight and boundary conditions but L goes to $1.1L$, so:

$$w_{tip} = \frac{P(1.1)^3 L^3}{128EI}$$

$$\Rightarrow \boxed{w_{tip} = w_{max}(x = \pm 1.1L) = \frac{1.331 PL^3}{128EI}}$$

This goes up by a factor of 1/3. It is the percentage increase in length to the third power.

Finally look at $\frac{dw}{dx}$ at the tip

regular L: $\left(\frac{dw}{dx}\right)_{tip} = \frac{PL^2}{48EI}$

10% longer: $\boxed{\left(\frac{dw}{dx}\right)_{tip} = \frac{P(1.1)^2 L^2}{12EI} = \frac{1.21 PL^2}{12EI} \Rightarrow \text{up by} \sim 20\%}$

M7.2 Given a statically determinate beam made of titanium, we consider four cross-sections with different shapes but of the same area of 30 in^2.

First note that $S(x)$ and $M(x)$ are not affected by the cross-section. Thus we will be looking at the geometrical contribution of the cross-sections to the various cases.

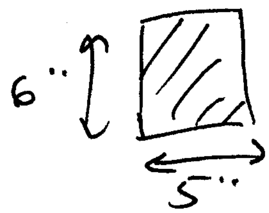
(a) The deflection is determined via:

$$\frac{M}{EI} = \frac{d^2 w}{dx^2}$$

To minimize w, one must minimize $\frac{M}{EI}$ which implies maximizing I.

Consider each cross-section:

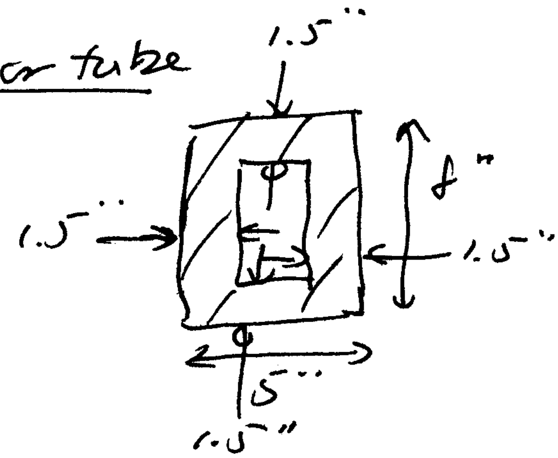
(A) Solid rectangle



$$I_{(A)} = \frac{1}{12} b h^3 = \frac{1}{12}(5")(6")^3$$

$$\Rightarrow \boxed{I_{(A)} = 90 \text{ in}^4}$$

(B) The rectangular tube



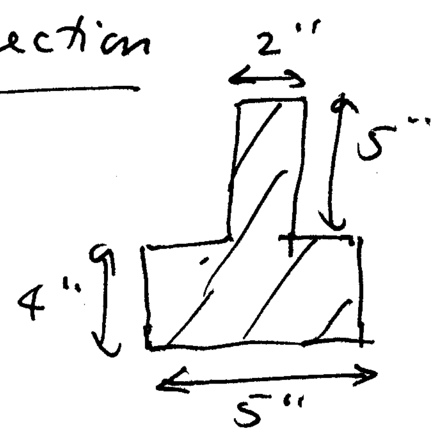
This is symmetric about the centerpoint, so the centroid is the centerpoint. We can find the moment of inertia by subtracting the moment of inertia of the inner material removed. Use $I = \frac{bh^3}{12}$ for a rectangle:

$$I_{B} = \frac{(5'')(8'')^3}{12} - \frac{(2'')(5'')^3}{12}$$

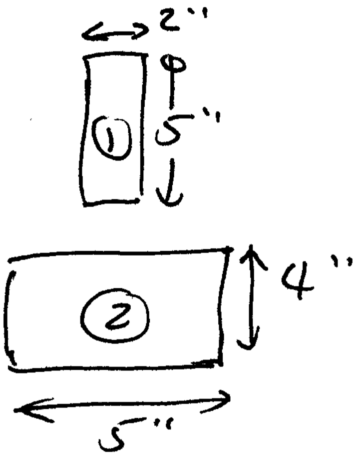
$$= 213.33\ in^4 - 20.8(3)\ in^4$$

$$\Rightarrow \boxed{I_{B} = 192.5\ in^4}$$

(C) The "T" cross-section



This is not symmetric about the centerpoint, so we must find the centroid and work from there. Break up the piece into two rectangular section and work our table after finding the centroid:



The centroid (center of area) is found via:

$$z_c = \frac{\iint z\,dA}{\iint dA}$$

Define an initial axis system to be midway in the cross-section horizontally and at the bottom of it (NOTE: This can be defined anywhere but is chosen here for convenience)

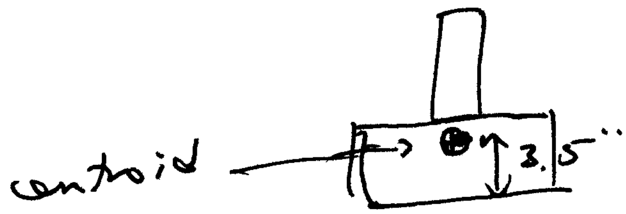
So:

$$z_c = \frac{\int_{4''}^{9''} (2\,in)\,z\,dz + \int_{0}^{4''} (5\,in)\,z\,dz}{A_1 + A_2}$$

$$= \frac{(2\,in)\,z^2/2\Big]_{4''}^{9''} + (5\,in)\frac{z^2}{2}\Big]_0^{4''}}{10\,in^2 + 20\,in^2}$$

$$= \frac{65\,in^3 + 40\,in^3}{30\,in^2} = 3.5\,in = z_c$$

Now put the table together:



Section	$A\ [in^2]$	$z_c\ [in]$	$Az_c\ [in^3]$	$Az_c^2\ [in^4]$	$I_0\ [in^4]$
1	10	3.0	30	90	20.8(3)
2	20	-1.5	-30	45	26.6(7)

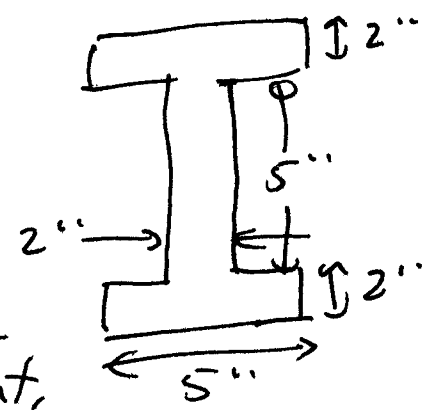
Summing these items via Parallel Axis Theorem:

$$I_{©} = \sum_{\text{sections}} \left(I_0 + A z_c^2 \right)$$

$$= \left\{ (20.83 + 26.67) + (90 + 45) \right\} \text{ in}^4$$

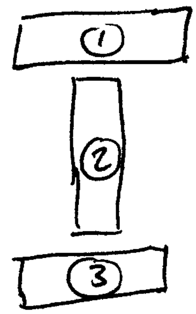
$$\Rightarrow \boxed{I_{©} = 182.5 \text{ in}^4}$$

(D) <u>The I-beam</u>



This is symmetric about the centerpoint, so that is the centroid.

Break this up into three sections and use the Parallel Axis Theorem. The sections are the 2 flanges and the web:



Assemble the table:

Section	$A\,[in^2]$	$z_c\,[in]$	$A z_c^2\,[in^4]$	$I_o\,[in^4]$
1	10	3.5	122.5	3.3(3)
2	10	0	0	20.8(3)
3	10	-3.5	122.5	3.3(3)

Use:
$$I_{D} = \sum_{\#\ \text{sections}} (I_o + A z^2)$$
$$= \{2 \times 3.3(3) + 20.8(3) + (2 \times 122.5)\}\ in^4$$
$$\Rightarrow \boxed{I_{D} = 272.5\ in^4}$$

Summarizing for all 4 sections:

I_{A} : 90 in^4 solid rectangle

I_{B} : 192.5 in^4 rectangular tube

I_{C} : 182.5 in^4 T-beam

I_{D} : 272.5 in^4 I-beam

we want to maximize I for the minimum deflection, so this occurs for the:

$$\boxed{\text{I-beam} \quad - \quad \text{Section } (D)}$$

(b) The axial stress is determined via:

$$\sigma_{xx} = -\frac{M z}{I}$$

first look at each cross-section

Thus, the maximum magnitude (i.e. absolute value) occurs by <u>maximizing</u> $\left|\frac{z}{I}\right|$ where z is the distance from the centroid. I is a constant for any particular cross-section. Looking at the results in (a) and the geometries:

(D): $I_{(D)} = 272.5 \text{ in}^4 \qquad |z_{max}| = 4.5 \text{ in}$

$$\Rightarrow \left|\frac{z}{I}\right|_{max_{(D)}} = 0.0165 \text{ in}^{-3}$$

(C): $I_{(C)} = 182.5 \text{ in}^4 \qquad |z_{max}| = 5.5 \text{ in}$

$$\Rightarrow \left|\frac{z}{I}\right|_{max_{(C)}} = 0.0301 \text{ in}^{-3}$$

(B): $I_{(B)} = 192.5 \text{ in}^4 \qquad |z_{max}| = 4.0 \text{ in}$

$$\Rightarrow \left|\frac{z}{I}\right|_{max_{(B)}} = 0.0208 \text{ in}^{-3}$$

(A): $I_{(A)} = 90 \text{ in}^4$ $\quad |z_{max}| = 3.0 \text{ in}$

$$\Rightarrow \left|\frac{z}{I}\right|_{max_{(A)}} = 0.0333 \text{ in}^{-3}$$

Find the minimum of the maximum values to find the smallest value of the maximum σ_{xx}.
So the cross-section with the smallest value of the maximum magnitude of σ_{xx} is the

$$\boxed{\text{I-beam - Section (D)}}$$

this occurs at both the top and bottom of the beam.

(c) The shear stress is determined via:

$$\sigma_{xz} = \frac{-SQ}{Ib}.$$

S is a constant (the same) for all cross-sections.
To find the smallest of the maximum magnitudes of σ_{xz} first find the maximum value for each cross-section via maximizing $\frac{Q}{Ib}$. I is a constant but Q and b can change with z.

(A): <u>solid rectangle</u>

$$Q = \int_{z}^{z_{top}} b\, z\, dz$$

This will be maximized at the centerpoint ($z = 0$). The parameter b is a constant, so that does not need to be considered.

$$Q_{max_{A}} = \int_0^{3\,in} (5\,in)\, z\, dz$$
$$= (5\,in)\, z^2/2 \Big]_0^{3\,in} = 22.5\,in^3$$

So the value is:

$$\left|\frac{Q}{Ib}\right|_{max_{A}} = 0.050\ in^{-2}$$

B: <u>rectangular tube</u>

By inspection we see that Q_{max} is at the centroid (as it should be) and this is the point of minimum width (the 2 edges = 3"), so the maximum value of $\frac{Q}{b}$ is at $z=0$.

Find this:

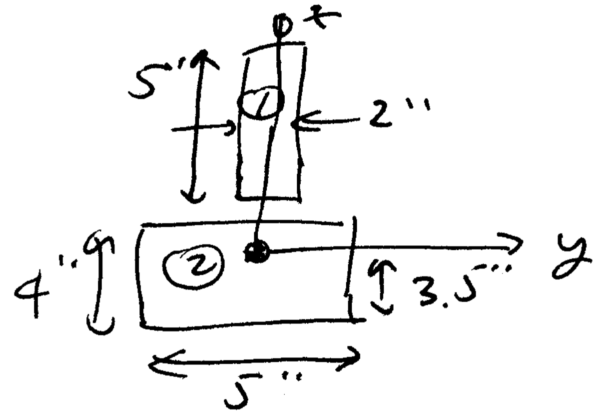
$$Q(z=0) = \int_0^{2.5''} (3\,in)\, z\, dz + \int_{2.5''}^{4''} (5\,in)\, z\, dz$$
$$= \left\{ (3\,in) \frac{z^2}{2} \Big]_0^{2.5''} + (5\,in) \frac{z^2}{2} \Big]_{2.5''}^{4''} \right.$$
$$= \left\{ 9.375 + 40 - 15.625 \right\} in^3$$
$$\Rightarrow Q(z=0) = 33.75\,in^3$$

So for the rectangular tube the maximum $\left|\frac{Q}{Ib}\right|$ is at $z=0$ with:

$$\left|\frac{Q}{Ib}\right|_{max_{B}} = 0.058\ in^{-2}$$

Ⓒ: T-beam

Again the Q's must be found for the two separate sections



for section ① $0.5\text{ in} < z < 5.5\text{ in}$:

$$Q = \int_z^{5.5''} (2\text{ in})\, z\, dz = (2\text{ in}) \frac{z^2}{2} \Big]_z^{5.5\text{ in}}$$

$$= 30.25\text{ in}^3 - z^2 (\text{in})$$

This is maximized at $z = 0.5\text{ in}$

$= 30\text{ in}^3$

This is maximized with division by width for

$$\left|\frac{Q}{b}\right|_{max①} = 15\text{ in}^2$$

for section ② $-3.5\text{ in} < z < 0.5\text{ in}$

$$Q_② = \int_z^{0.5''} (5\text{ in})\, z\, dz + Q_{①\, z=0.5''}$$

$$= (5\text{ in}) \frac{z^2}{2} \Big]_z^{0.5''} + 30\text{ in}^3$$

this is maximized at the centroid $(z=0)$

$$Q_{②max} = 30.625\text{ in}^3$$

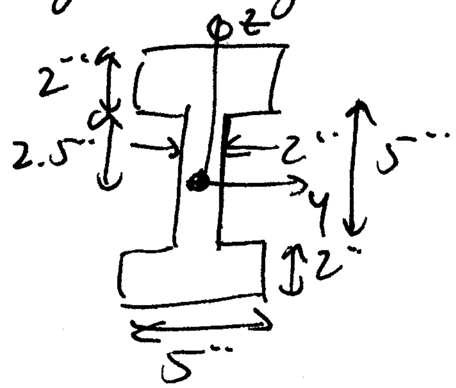
Divide through by the width to find:

$$\left|\frac{Q}{b}\right|_{max_{(2)}} = \frac{30.625\text{ in}^3}{5\text{ in}} = 6.125\text{ in}^2$$

So, for the T-beam the maximum $\left|\frac{Q}{Ib}\right|$ is at $z = 0.5$ in with the value $\left|\frac{Q}{Ib}\right|_{max_{(C)}} = 0.0822\text{ in}^{-2}$

(D) : <u>I-beam</u>

The I-beam is symmetric, so maximum value of Q occurs at the centroid. Find this by looking at the individual pieces



$$Q_{max_{(D)}} = \int_{2.5''}^{4.5''} (5'')\,z\,dz + \int_0^{2.5''} (2'')\,z\,dz$$

$$= (5\text{ in})\,\frac{z^2}{2}\Big]_{2.5''}^{4.5''} + (2\text{ in})\,\frac{z^2}{2}\Big]_0^{2.5''}$$

$$= \frac{5}{2}(20.25 - 6.25)\text{ in}^3 + 6.25\text{ in}^3$$

$$= 41.25\text{ in}^3$$

Now divide by the width and I:

$$\left|\frac{Q}{Ib}\right|_{max_{(D)}} = 0.0757\text{ in}^{-2}$$

occurring at $z = 0$

Summarizing for $\left|\frac{Q}{Ib}\right|_{max}$:

A	0.050 in^{-2} at $z=0$	solid rectangle
B	0.058 in^{-2} at $z=0$	rectangular tube
C	0.0822 in^{-2} at $z=0.5 \text{ in}$	T-beam
D	0.0757 in^{-2} at $z=0$	I-beam

so the cross-section with the smallest value of its maximum magnitude of σ_{xz} is the

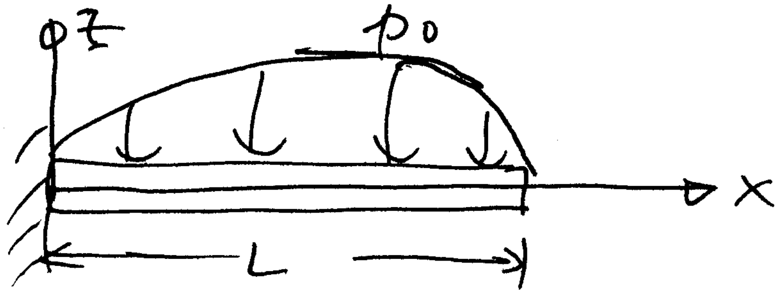
$$\boxed{\text{solid rectangle - section (A)}}$$

and this occurs at $z=0$

(d) An I-beam is the most efficient configuration with regard to bending stiffness and thus to get the minimum deflection. This shows that getting the most material as far away from the centroid is most effective in this regard. It also helps in minimizing stresses as this moves stresses farther from the centroid giving them a longer moment arm. With this and there being more material that is stressed, the moment is carried more effectively and thus with less stress.

However, this correlation between resisting deformation and carrying stress is not always the same (see sections (B) and (C)). This becomes emphasized when looking at shear stresses and must be considered when later assessing failure.

M7.3

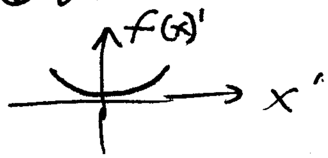


First need an expression for the distributed loading $q(x)$.

It is a quadratic, negative in sign and displaced, and the center is at $x = L/2$.

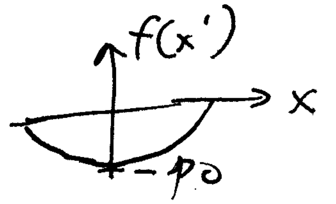
→ So begin with a quadratic function in x':

$$f(x') = ax'^2$$



→ Displace by $-p_0$:

$$f(x') = ax'^2 - p_0$$



→ Transforming to x:

$$x = (x' + L/2)$$

$$\Rightarrow x' = x - L/2$$

$$\Rightarrow f(x) = a(x - L/2)^2 - p_0$$

$$= a(x^2 - Lx + L^2/4) - p_0$$

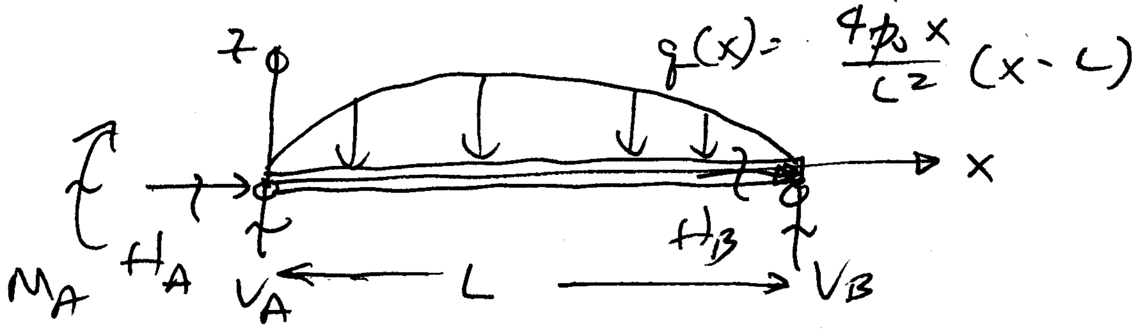
→ To get a, @ $x = 0$, $f(x) = 0 \Rightarrow a = \frac{4p_0}{L^2}$

check @ $x = L$, $f(x) = 0$ ✓ checks

and @ $x = L/2$, $f(x) = -p_0$ ✓ checks

So: $q(x) = \frac{4p_0 x}{L^2}(x - L)$

Now draw the Free Body Diagram:



There are no external forces in the x-direction, so can say the reactions in the x-direction are zero: $H_A = H_B = 0$.
With that there are still:

~~3~~ 4 Reaction forces ⇒ Statically
~~3~~ 2 degrees of freedom   In determinate

Thus, the basic procedure is the same, but not able to get numbers to each part in sequence, but rather have a number of simultaneous equations to solve.

Start by finding the equations for the reactions:

$$\sum F_z = 0 \ \uparrow + \Rightarrow V_A + V_B + \int_0^L \frac{4p_0 x}{L^2}(x-L)\,dx = 0$$

$$V_A + V_B + \frac{4p_0}{L^2}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right)\Big]_0^L = 0$$

$$\Rightarrow V_A + V_B = \frac{2p_0L}{3} \qquad (1)$$

$$\sum M_A = 0 \ \circlearrowleft + \Rightarrow M_A - V_BL - \int_0^L \frac{4p_0x}{L^2}(x-L)\,x\,dx = 0$$

$$M_A - V_BL = \frac{4p_0}{L^2}\left(\frac{x^4}{4} - \frac{Lx^3}{3}\right)\Big]_0^L$$

$$M_A - V_B L = -\frac{p_o L^2}{3} \qquad (2)$$

move on to the Shear and Moment Resultants

use: $S(x) = \int q(x)\,dx$

so: $S(x) = \int \frac{4p_o x}{L^2}(x-L)$

$$= \frac{4p_o}{L^2}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right) + C_1$$

at $x=0$ the shear is equal to V_A

$$S(0) = V_A \Rightarrow C_1 = V_A$$

$$S(x) = \frac{4p_o}{L^2}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right) + V_A \qquad (3)$$

Now use: $M(x) = \int S(x)\,dx$

$$\Rightarrow M(x) = \int\left[\frac{4p_o}{L^2}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right) + V_A\right]dx$$

$$= \frac{4p_o}{L^2}\left(\frac{x^4}{12} - \frac{Lx^3}{6}\right) + V_A x + C_2$$

at $x=0$, the moment is equal to M_A

or at $x=L$, the moment is zero.

using the latter:

$$0 = \frac{4p_o}{L^2}\left(-\frac{L^4}{12}\right) + V_A L + C_2$$

$$\Rightarrow C_2 = \frac{p_o L^2}{3} - V_A L$$

giving:

$$M(x) = \frac{4p_o}{L^2}\left(\frac{x^4}{12} - \frac{Lx^3}{6}\right) + V_A(x-L) + \frac{p_o L^2}{3} \qquad (4)$$

Now move on to get an expression for the deflection via the Moment-Curvature Relation:

$$M = EI \frac{d^2 w}{dx^2}$$

$$\Rightarrow \frac{d^2 w}{dx^2} = \frac{1}{EI}\left\{\frac{4p_0}{L^2}\left(\frac{x^4}{12} - \frac{Lx^3}{6}\right) + V_A(x-L) + \frac{p_0 L^2}{3}\right\}$$

p_0 and V_A do not vary with x, so performing one integral:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{\frac{4p_0}{L^2}\left(\frac{x^5}{60} - \frac{Lx^4}{24}\right) + V_A\left(\frac{x^2}{2} - Lx\right) + \frac{p_0 L^2 x}{3}\right\} + C_3$$

At $x = 0$, the end is clamped, so $\frac{dw}{dx} = 0$.

To help, write the constant of integration as $\frac{C_3}{EI}$ (as EI is a constant). Nevertheless:

$$C_3 = 0$$

So:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{\frac{4p_0}{L^2}\left(\frac{x^5}{60} - \frac{Lx^4}{24}\right) + V_A\left(\frac{x^2}{2} - Lx\right) + \frac{p_0 L^2 x}{3}\right\} \quad (5)$$

Integrate again to get:

$$w = \frac{1}{EI}\left\{\frac{4p_0}{L^2}\left(\frac{x^6}{360} - \frac{Lx^5}{120}\right) + V_A\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{p_0 L^2 x^2}{6}\right\} + C_4$$

At $x = 0$, the clamped end provides $w = 0$

$$\Rightarrow C_4 = 0$$

Finally:

$$w = \frac{1}{EI}\left\{\frac{p_0}{L^2}\left(\frac{x^6}{90} - \frac{Lx^5}{30}\right) + V_A\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{p_0 L^2 x^2}{6}\right\} \quad (6)$$

There is one other boundary condition on the displacement. At the pinned end ($x = L$), $w = 0$. Use this to find V_A via (6):

$$0 = \frac{1}{EI}\left\{\frac{p_0}{L^2}\left(\frac{L^6}{90} - \frac{L^6}{30}\right) + V_A\left(\frac{L^3}{6} - \frac{L^3}{2}\right) + \frac{p_0 L^4}{6}\right\}$$

$$\Rightarrow \frac{p_0}{L^2}\left(-\frac{2L^6}{90}\right) + \frac{p_0 L^4}{6} + V_A\left(-\frac{2L^3}{6}\right) = 0$$

gives: $-\frac{2 p_0 L^4}{15} + p_0 L^4 - 2 V_A L^3 = 0$

$$\Rightarrow \quad \boxed{V_A = \frac{13}{30} p_0 L}$$

using (1):

$$V_B = p_0 L\left(\frac{2}{3} - \frac{13}{30}\right)$$

$$\Rightarrow \boxed{V_B = \frac{7}{30} p_0 L}$$

Note: Not needed to get $w(x)$

using (2):

$$M_A - \frac{7}{30} p_0 L^2 = -\frac{p_0 L^2}{3}$$

$$\Rightarrow \boxed{M_A = -\frac{3}{30} p_0 L^2}$$

Note: Not needed to find $w(x)$

Return to (6) and find the deflection:

$$w = \frac{1}{EI}\left\{\frac{p_0}{L^2}\left(\frac{x^6}{90} - \frac{Lx^5}{30}\right) + \frac{13 p_0 L}{30}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{p_0 L^2 x^2}{6}\right\}$$

Rewrite as:

$$w(x) = \frac{p_o L^4}{30EI}\left\{ \frac{1}{3}\left(\frac{x}{L}\right)^6 - \left(\frac{x}{L}\right)^5 + \frac{13}{6}\left(\frac{x}{L}\right)^3 - \frac{13}{2}\left(\frac{x}{L}\right)^2 + 5\left(\frac{x}{L}\right)^2 \right\}$$

$$\boxed{w(x) = \frac{p_o L^4}{30EI}\left\{ \frac{1}{3}\left(\frac{x}{L}\right)^6 - \left(\frac{x}{L}\right)^5 + \frac{13}{6}\left(\frac{x}{L}\right)^3 - \frac{3}{2}\left(\frac{x}{L}\right)^2 \right\}}$$

Check the units:

$$[L] \stackrel{?}{=} \frac{[F/L][L^4]}{[F/L^2][L^4]} \quad \checkmark \quad \underline{\text{yes}}$$

To find the maximum magnitude of deflection and its location, take the derivative and set it to zero. We know in this case that the maximum will not occur at the boundaries since the deflection is zero at both locations. Thus:

$$\frac{dw}{dx} = 0 = 2\left(\frac{x}{L}\right)^5 - 5\left(\frac{x}{L}\right)^4 + \frac{13}{2}\left(\frac{x}{L}\right)^2 - 3\left(\frac{x}{L}\right)$$

Find the value that satisfies this. Either explicitly solve (YIKES!) or try values ($0 < \frac{x}{L} < 1$) until you get the answer. I chose the latter. I know this will occur somewhere near the middle and probably closer to the pin as this is less stiff. I'll start with $\left(\frac{x}{L}\right) = 0.6$ giving 0.047 ... not bad

This is a positive slope and from the problem I see:

So I need to come back from where the slope is positive.

Try $\frac{x}{L} = 0.58 \Rightarrow 0.0121$

Keep going with $\frac{x}{L} : 0.57 \Rightarrow -0.0056$

The sign changed and I'm at 3 decimals...
Good enough.

Maximum magnitude at $x = 0.57L$. Plug into expression for $w(x)$:

$$\boxed{w_{max} = -0.00450 \frac{p_0 L^4}{EI} \quad \text{at } \frac{x}{L} = 0.57}$$

(b) To find the maximum magnitude of the axial stress, start with:

$$\sigma_{xx} = -\frac{Mz}{I}$$

z and I do not vary in x, so we look for the maximum of $\frac{\sigma_{xx}}{(z/I)} = -M(x)$

From equation (4) and with results for V_A:

$$M(x) = \frac{4p_0}{L^2}\left(\frac{x^4}{12} - \frac{Lx^3}{6}\right) + \frac{13p_0L}{30}(x-L) + \frac{p_0L^2}{3}$$

Manipulate, pull out common factors and nondimensionalize (again) using (x/L):

$$M(x) = p_0L^2\left[\frac{1}{3}\left(\frac{x}{L}\right)^4 - \frac{2}{3}\left(\frac{x}{L}\right)^3 + \frac{13}{30}\left(\frac{x}{L}\right) - \frac{1}{10}\right]$$

Find the location of maximum moment by taking the first derivative and setting it to zero:

$$\frac{dM(x)}{dx} = \frac{4}{3}\left(\frac{x}{L}\right)^3 - 2\left(\frac{x}{L}\right)^2 + \frac{13}{30} = 0$$

Working numbers as before, the solution is ~~so~~ maximum within $0 < x < L$ ~~so~~ at $\frac{x}{L} = 0.6$ with a value of $0.0592 p_o L^2$

check if it occurs at the boundary $x = 0$

$M_A = -0.1 p_o L^2$ yes it does

$$\Rightarrow \boxed{|\sigma_{xx\,max}| = \frac{-0.1\, p_o L^2 z_{max}}{I} \quad \text{at } x = 0 \text{ (clamped end)}}$$

(c) To find the maximum shear stress, we:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

As for the case of σ_{xx}, we see that Q, I, and b do not vary in x. So we look for the maximum magnitude of S:

$$S(x) = \frac{4p_o}{L^2}\left(\frac{x^3}{3} - \frac{Lx^2}{2}\right) + \frac{13}{30} p_o L$$

As before manipulate to get:

$$S(x) = p_o L\left[\frac{4}{3}\left(\frac{x}{L}\right)^3 - \frac{4}{2}\left(\frac{x}{L}\right)^2 + \frac{13}{30}\right]$$

Taking the derivative and setting it to zero gives:

$$\frac{dS(x)}{dx} = 0 \Rightarrow 0 = \left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)$$

$$\Rightarrow \frac{x}{L} = 0, 1$$

So it occurs at one of the boundaries:

@ $x = 0$, $S(0) = \frac{13}{30} p_o L$

@ $x = L$, $S(L) = \frac{7}{30} p_o L$

$\Rightarrow$ maximum shear at the clamped end and

$$\Rightarrow \boxed{|\sigma_{xz\,max}| = \frac{13}{30}\frac{p_o L}{L}\left|\frac{Q}{b}\right|_{max} \quad \text{at } x = 0}$$