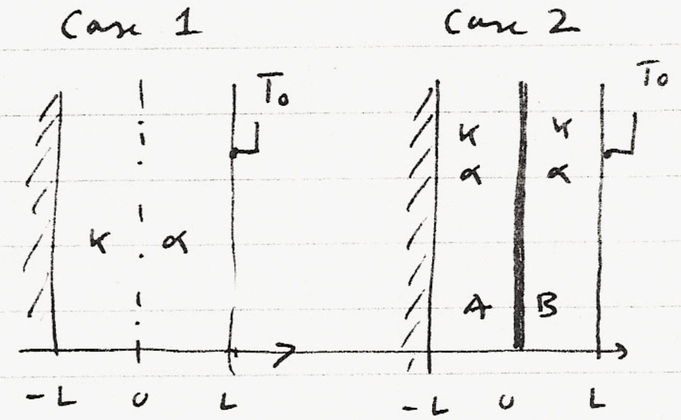
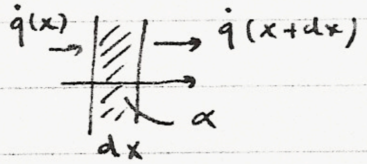
T21



concepts: - conduction heat x-fer with sources
- 1st law
- Fourier's law

a) 1st law:

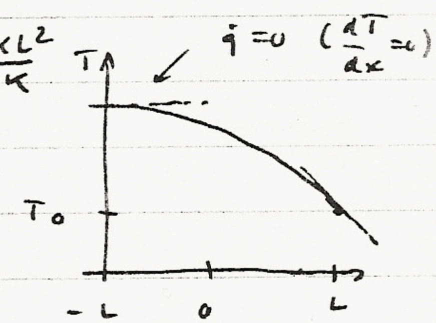


$$0 = \dot q(x) - \left(\dot q(x) + \frac{d\dot q}{dx}dx\right) + \alpha\, dx \; ; \qquad -\frac{d\dot q}{dx} + \alpha = 0 \; , \quad \dot q = -k\frac{dT}{dx}$$

$$\frac{d^2T}{dx^2} + \frac{\alpha}{k} = 0 \qquad \text{B.C.: } T(L) = T_0 \; , \quad \left.\frac{dT}{dx}\right|_{x=-L} = 0$$

b) $$\frac{dT}{dx} = -\frac{\alpha}{k}x + c_1 \; ; \quad c_1 = -\frac{\alpha L}{k}$$

$$T(x) = -\frac{\alpha}{2k}x^2 - \frac{\alpha L}{k}x + c_2 \; ; \quad c_2 = T_0 + \frac{3}{2}\frac{\alpha L^2}{k}$$



$$T(x) = \frac{\alpha L^2}{k}\left(\frac{3}{2} - \frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right) + T_0$$

$$\boxed{T(x) = -\frac{\alpha L^2}{2k}\left(\frac{x}{L} + 1\right)^2 + T_0 + 2\frac{\alpha L^2}{k}}$$

c) max temp @ $x = -L$ $\qquad T_{max} = T_0 + 2\frac{\alpha L^2}{k}$

d) domain A: $\frac{d^2T}{dx^2} + \frac{\alpha}{k} = 0 \;\; \forall\, x \in [-L, 0]$; $\left.\frac{dT}{dx}\right|_{x=-L} = 0$, $T(0) = T_{sA}$

domain B: $\frac{d^2T}{dx^2} + \frac{\alpha}{k} = 0 \;\; \forall\, x \in [0, L]$; $T(0) = T_{sB}$, $T(L) = T_0$

dielectric strip: $\dot q_{strip} = \frac{T_{sA} - T_{sB}}{R_{strip}}$; 1st law: $\dot q$ = const across strip

$$\rightarrow \left.\frac{dT}{dx}\right|^{A}_{x=0} = -\frac{\dot q_{strip}}{k} \; ; \quad \left.\frac{dT}{dx}\right|^{B}_{x=0} = -\frac{\dot q_{strip}}{k}$$

A: $T(x) = \frac{-\alpha}{2k_1}x^2 + C_{A1}x + C_{A2}$

$\frac{dT}{dx}(x) = -\frac{\alpha}{k}x + C_{A1}$; $C_{A1} = -\frac{\alpha L}{k}$

$T(0) = T_{SA} = C_{A2}$ $\longrightarrow$ $T(x) = -\frac{\alpha}{2k}x^2 - \frac{\alpha L}{k}x + T_{SA}$

B: $T(x) = -\frac{\alpha}{2k}x^2 + C_{B1}x + C_{B2}$

$T(0) = T_{SB} = C_{B2}$; $T(L) = T_0 = -\frac{\alpha}{2k}L^2 + C_{B1}L + T_{SB}$

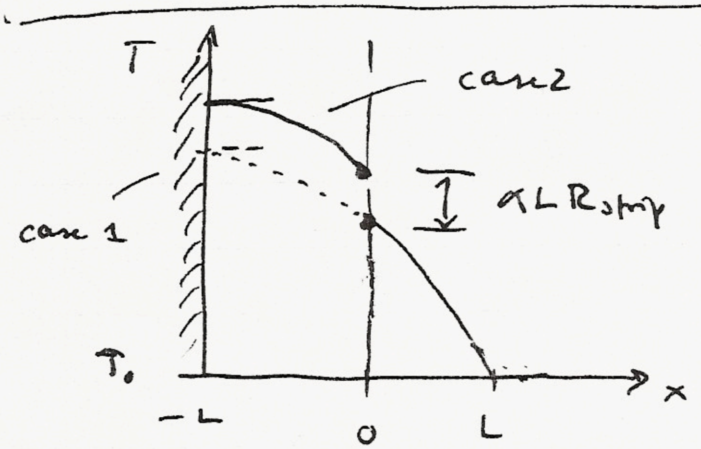
$\rightarrow T(x) = -\frac{\alpha}{2k}x^2 + \left[\frac{T_0 - T_{SB}}{L} + \frac{\alpha L}{2k}\right]x + T_{SB}$

$\rightarrow$ $-\frac{\alpha L}{k} = -\frac{1}{k}\frac{T_{SA} - T_{SB}}{R_{strip}}$; $\frac{T_0 - T_{SB}}{L} + \frac{\alpha L}{2k} = -\frac{1}{k}\frac{T_{SA} - T_{SB}}{R_{strip}}$

$\rightarrow$ $T_{SB} = T_0 + \frac{3}{2}\frac{\alpha L^2}{k}$; $T_{SA} = T_{SB} + \alpha L R_{strip}$

Domain A: $T(x) = \frac{-\alpha}{2k}x^2 - \frac{\alpha L}{k}x + \frac{3}{2}\frac{\alpha L^2}{k} + T_0 + \alpha L R_{strip}$ $\quad x \in [-L, 0]$

Domain B: $T(x) = \frac{-\alpha}{2k}x^2 - \frac{\alpha L}{k}x + \frac{3}{2}\frac{\alpha L^2}{k} + T_0$ $\quad x \in [0, L]$



zero slope at wall (adiabatic), temperature in domain A increased due to resistance in strip at <u>fixed</u> T_0!

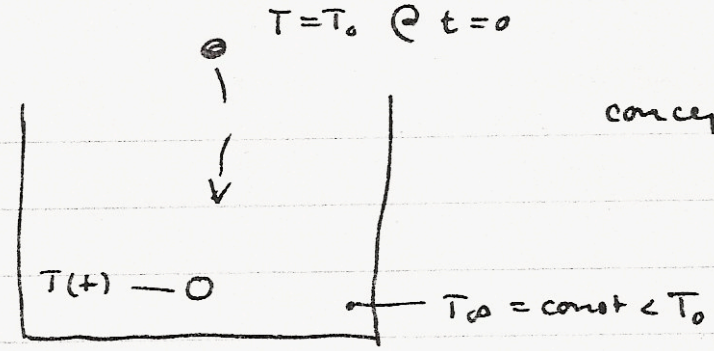
e) $\Delta T_{strip} = \alpha L R_{strip}$

f) max temp. @ $x = -L$

$T_{max} = T_0 + \frac{2\alpha L^2}{k} + \alpha L R_{strip}$

Spring 2007, ZSP

T22



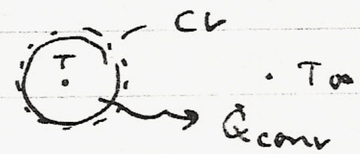
concepts:
- unsteady heat xfer
- convection
- 1st law

a) convection heat transfer is dominant because

$$\frac{T_{centre} - T_{wall}}{T_{wall} - T_\infty} \propto \frac{hL}{K} \ll 1$$ → neglect temp. non-uniformity inside sphere and hence conduction

Introduce: h

b) $$\frac{dE_{cv}}{dt} = -\dot{Q}_{conv}$$



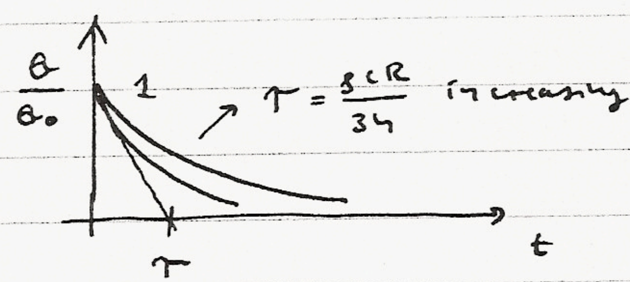
$$mc\frac{dT}{dt} = -Ah(T - T_\infty)$$ transient heat xfer

c) $$\frac{dT}{dt} = -\frac{4\pi R^2 h}{\rho c \frac{4}{3}\pi R^3}(T - T_\infty)$$ $$\frac{dT}{dt} = -\frac{3h}{\rho c R}(T - T_\infty)$$

d) $\theta = T - T_\infty$: $$\frac{d\theta}{dt} = -\frac{1}{\tau}\theta$$ $$\theta(t) = \theta_0 e^{-\frac{t}{\tau}}$$

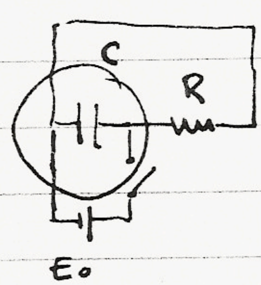
$$\tau = \frac{\rho c R}{3h}$$

e) $$\tau = \frac{\rho c R}{3h}$$



τ↑ for R↑, h↓, c↑, ρ↑

f) R-C circuit



$$\frac{dE}{dt} + \frac{E}{RC} = 0$$ so $\tau = RC$

$$E(t) = E_0 e^{-t/RC}$$

Unified Engineering II **Spring 2007**

Problem S15 (Signals and Systems) SOLUTION

Find the Fourier transforms of the following signals:

1.
$$g(t) = \delta(t - T)$$

Solution: Simply perform the FT to obtain

$$G(j\omega) = \int_{-\infty}^{\infty} \delta(t - T) e^{-j\omega t}\, dt = e^{-j\omega T}$$

where we have used the sifting property of the FT to find the result.

2.
$$g(t) = \begin{cases} 1, & |t| \leq T \\ 0, & |t| > T \end{cases}$$

Solution: Perform the FT integral to obtain

$$\begin{aligned} G(j\omega) &= \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt \\ &= \int_{-T}^{T} e^{-j\omega t}\, dt \\ &= \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega} \\ &= 2\frac{\sin(\omega T)}{\omega} \\ &= 2T \mathrm{sinc}(\omega T) \end{aligned}$$

3.
$$g(t) = \frac{1}{t^2 + T^2}$$

Solution: We want to find

$$\mathcal{F}[g(t)] = G(j\omega) = f(\omega)$$

From duality, we know that

$$\mathcal{F}[f(t)] = 2\pi g(-\omega) = 2\pi \frac{1}{\omega^2 + T^2}$$

Find the inverse FT:

$$f(t) = \mathcal{F}^{-1}\left[2\pi \frac{1}{\omega^2 + T^2}\right] = \mathcal{L}^{-1}\left[2\pi \frac{1}{-s^2 + T^2}\right]$$

where we have used $\omega = s/j$. Do a partial fraction expansion:

$$\begin{aligned} 2\pi \frac{1}{-s^2 + T^2} &= \frac{-2\pi}{(s + T)(s - T)} \\ &= \frac{\pi}{T} \frac{1}{s + T} + \frac{\pi}{T} \frac{1}{s - T} \end{aligned}$$

To take the inverse LT, note that since we are dealing with FTs, we must assume that the ROC includes the imaginary axis, so that the ROC is

$$-T < \mathrm{Re}(s) < T$$

and the inverse LT is

$$f(t) = \frac{\pi}{T} e^{-tT} \sigma(t) + \frac{\pi}{T} e^{tT} \sigma(-t) = \frac{\pi}{T} e^{-|tT|}$$

Finally, we need $f(\omega)$, not $f(t)$, so that

$$G(j\omega) = f(\omega) = \frac{\pi}{T} e^{-|\omega T|}$$

4.

$$g(t) = \frac{\sin \pi t/T}{\pi t/T}$$

Solution: We want to find

$$\mathcal{F}[g(t)] = G(j\omega) = f(\omega)$$

From duality, we know that

$$\mathcal{F}[f(t)] = 2\pi g(-\omega) = \frac{\sin \pi\omega/T}{\pi\omega/T} = 2\pi \mathrm{sinc}(\pi\omega/T)$$

Find the inverse FT:

$$f(t) = \mathcal{F}^{-1}\left[2\pi \mathrm{sinc}(\pi\omega/T)\right]$$

We can use the results of part (2) to obtain

$$f(t) = \begin{cases} T, & |t| \le \pi/T \\ 0, & |t| > \pi/T \end{cases}$$

Finally, exchange ω for t to obtain

$$G(j\omega) = f(\omega) = \begin{cases} T, & |\omega| \le \pi/T \\ 0, & |\omega| > \pi/T \end{cases}$$

That this is indeed the correct transform can be confirmed by taking the inverse transform of $G(j\omega)$.

5. Find the inverse transform of

$$G(j\omega) = \left(\frac{\sin \omega T}{\omega T}\right)^2$$

Solution: Note that

$$\mathcal{F}^{-1}\left[\frac{\sin \omega T}{\omega T}\right] = f(t) = \begin{cases} 1/2T, & |t| \le T \\ 0, & |t| > T \end{cases}$$

from part (2). Therefore,

$$g(t) = \mathcal{F}^{-1}[G(j\omega] = f(t) * f(t)$$

The convolution can be performed graphically or symbolically to obtain

$$g(t) = \begin{cases} \frac{1}{4T^2}\left(t + 2T\right), & -2T < t \leq 0 \\ \frac{1}{4T^2}\left(-t + 2T\right), & 0 < t \leq 2T \\ 0, & \text{otherwise} \end{cases}$$

The result can be confirmed by taking the direct FT of $g(t)$.

Problem S16 (Signals and Systems) SOLUTION

1. Define

$$g_a(t) = e^{j\omega_0 t}e^{-a|t|} = \begin{cases} e^{j\omega_0 t}e^{-at}, & t > 0 \\ e^{j\omega_0 t}e^{at}, & t < 0 \end{cases}$$

   The FT of $g_a(t)$ is

$$G_a(j\omega) = \frac{1}{j\omega - j\omega_0 + a} - \frac{1}{j\omega - j\omega_0 - a}$$

   Clearing the fraction, we have that

$$G_a(j\omega) = \frac{-2a}{(j\omega - j\omega_0 + a)(j\omega - j\omega_0 - a)} = \frac{2a}{(\omega - \omega_0)^2 + a^2}$$

   Note that $G_a(j\omega)$ is real, so we need worry only about the real part. Also, note that in the limit as $a \to 0$, $G_a(j\omega)$ is zero everywhere, except at $\omega = \omega_0$, where it goes to infinity. So $G(j\omega)$ looks like an impulse at $\omega = \omega_0$. Its area is given by

$$A = \int_{-\infty}^{\infty} \frac{2a}{(\omega - \omega_0)^2 + a^2}\, d\omega$$

   This integral may be evaluated by making the change of variables $u = \omega - \omega_0$. Then $du = d\omega$, and

$$A = \int_{-\infty}^{\infty} \frac{2a}{u^2 + a^2}\, du = 2\tan^{-1}\left(\frac{u}{a}\right)\Big|_{-\infty}^{\infty} = 2\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 2\pi$$

   Therefore,

$$G(j\omega) = 2\pi\delta(\omega - \omega_0)$$

2. The FT of a cosine is given by

$$\mathcal{F}[\cos\omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

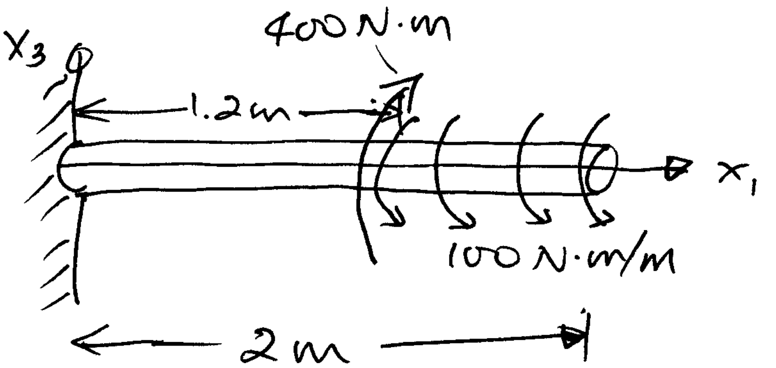
   The FT of a sine is given by

$$\mathcal{F}[\sin\omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] = \pi[-j\delta(\omega - \omega_0) + j\delta(\omega + \omega_0)]$$

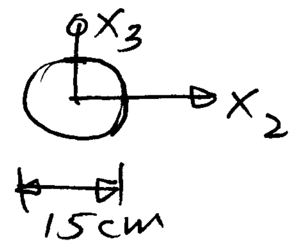
PAL
4/26/07

Unified Engineering Problem Set
Week 12 Spring, 2007
SOLUTIONS

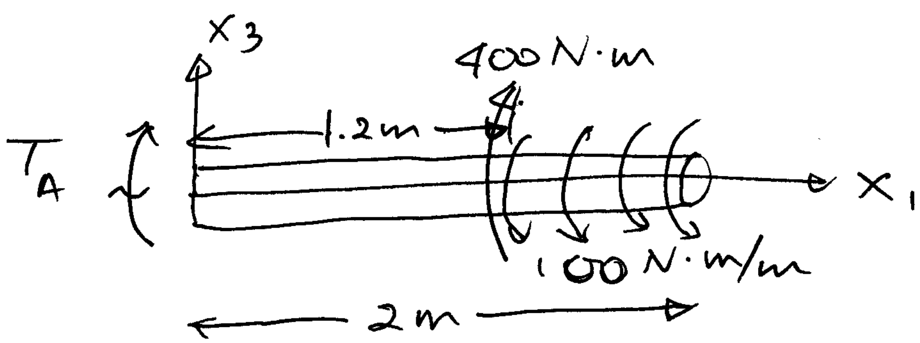
M12.1 Steel rod



Cross-Section



(a) Draw the Free Body Diagram



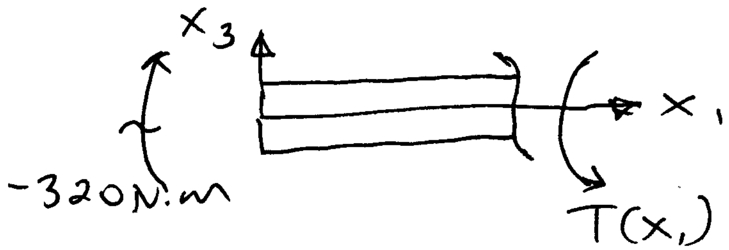
Take equilibrium of moments/torques about x_1. Use right hand rule for + (RHR)

$$\sum T_{x_1} = 0 \Rightarrow -T_A - 400\,N\cdot m + \left(100\,\frac{N\cdot m}{m}\right)(0.8m) = 0$$

$$\Rightarrow T_A = -320\,N\cdot m$$

To determine the torque distribution $T(x_1)$, make cuts in the shaft/rod in the regions of different loading

<u>First region</u> $0 < x_1 < 1.2m$



Can take equilibrium:

$$\sum T_{x_1} = 0 \quad \text{RHR} \; + \Rightarrow 320\,N\cdot m + T(x_1) = 0$$

$$\Rightarrow \boxed{T(x_1) = -320\,N\cdot m}$$

$0 < x_1 < 1.2m$

<u>OR</u> use $\frac{dT}{dx_1} = -t(x_1)$

$= 0$

$\Rightarrow T = C_1$

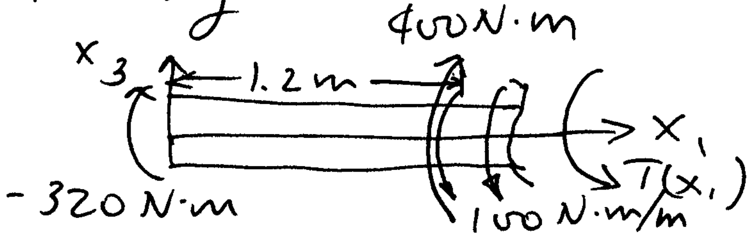
Boundary condition is that at $x_1 = 0$, T opposes the reaction torque $\Rightarrow T(0) = -320\,N\cdot m$

$$\Rightarrow \boxed{T(x_1) = -320\,N\cdot m}$$

$0 < x_1 < 1.2m$ same result as before (as it must be)

Now look at region 2 $\qquad 1.2\,m < x_1 < 2\,m$

taking a cut:



by taking equilibrium:

$\sum T_{x_1} = 0$ (RHR, + direction) $\Rightarrow 320\,N\cdot m - 400\,N\cdot m + \left(100\,\frac{N\cdot m}{m}\right)(x_1 - 1.2\,m) + T(x_1) = 0$

$$\Rightarrow T(x_1) = 80\,N\cdot m - \left(100\,\frac{N\cdot m}{m}\right)(x_1 - 1.2\,m)$$

$$\boxed{T(x_1) = 200\,N\cdot m - \left(100\,\frac{N\cdot m}{m}\right)(x_1)}$$

$1.2\,m < x_1 < 2\,m$

or use:
$$\frac{dT}{dx_1} = -t(x_1) = -100\,\frac{N\cdot m}{m}$$

$$\Rightarrow T(x_1) = -100\,\frac{N\cdot m}{m}\,x_1 + C_1$$

Look at boundary of region to get C_1. Easiest is $x_1 = 2\,m$ ..... this is free end $\Rightarrow T(2\,m) = 0$

using $\Rightarrow 0 = -200\,\frac{N\cdot m}{m} + C_1$

gives: $C_1 = 200\,\frac{N\cdot m}{m}$

and $\Rightarrow \boxed{T(x_1) = \left(-100\,\frac{N\cdot m}{m}\right)(x_1) + 200\,N\cdot m}$

$1.2\,m < x_1 < 2\,m$

same result

PAL

$$\phi(x_1 = 0) = 0$$
$$\Rightarrow C_1 = 0$$

So: $\phi(x_1) = -\dfrac{320 N\cdot m}{GJ} x_1 \qquad 0 < x_1 < 1.2m$

Use this to determine the twist at $x_1 = 1.2m$ to use as the boundary condition for region (section) 2 (i.e. matching):

$$\Rightarrow \phi(1.2m) = -\frac{320 N\cdot m}{GJ}(1.2m)$$

Proceeding to Region 2: $(1.2m < x_1 < 2m)$

where $T(x_1) = 200 N\cdot m - (100 \frac{N\cdot m}{m})(x_1)$

so: $\dfrac{d\phi}{dx_1} = \dfrac{1}{GJ}\left(200 N\cdot m - 100 \frac{N\cdot m}{m} x_1\right)$

$$\Rightarrow \phi(x_1) = \frac{1}{GJ}\int\left(200 N\cdot m - 100 \frac{N\cdot m}{m} x_1\right)dx_1$$

$$= \frac{1}{GJ}\left(200 N\cdot m\, x_1 - 50 \frac{N\cdot m}{m} x_1^2\right) + C_2$$

Now use $\phi(1.2m) = \dfrac{1}{GJ}(-384 N\cdot m^2)$

matching gives:

$$-384 N\cdot m^2 = 200 N\cdot m(1.2m) - 50 \frac{N\cdot m}{m}(1.2m)^2 + C_2 GJ$$

$$\Rightarrow -384 N\cdot m^2 = 240 N\cdot m^2 - 72 N\cdot m^2 + C_2 GJ$$

giving: $c_2 = \frac{-552\ N \cdot m^2}{GJ}$

so:

$$\phi(x_1) = \frac{1}{GJ}\left(200\ N \cdot m\, x_1 - 50\frac{N \cdot m}{m} x_1^2 - 552\ N \cdot m^2\right)$$

And at the tip:

$$\phi(2m) = \frac{1}{GJ}\left(400\ N \cdot m^2 - 200\ N \cdot m^2 - 552\ N \cdot m^2\right)$$

$$\Rightarrow \phi(2m) = \frac{-352\ N \cdot m^2}{GJ}$$

Finally, need to calculate GJ to get the torsional stiffness of the rod.

In general: $J = \iint (x_2^2 + x_3^2)\, dA$

We know that for a circular cross-section:

$$J = \frac{\pi R^4}{2}$$

Here, $R = 7.5\ cm = 0.075\ m$

$$\Rightarrow J = 4.97 \times 10^{-5}\ m^4$$

We are given $E = 200\ GPa$, $\nu = 0.3$ for steel.

For isotropic materials:

$$G = \frac{E}{2(1+\nu)}$$

$$\Rightarrow G = \frac{200 \times 10^9 \ N/m^2}{2(1+0.3)} = 76.9 \times 10^9 \ N/m^2$$

$$\Rightarrow GJ = (76.9 \times 10^9 \ N/m^2)(4.97 \times 10^{-5} m^4)$$
$$= 3.82 \times 10^6 \ N \cdot m^2$$

Use this in the expression for the tip deflection:

$$\phi(2m) = \frac{-352 \ N \cdot m^2}{3.82 \times 10^6 \ N \cdot m^2}$$

$$\Rightarrow \phi_{tip} = -9.21 \times 10^{-5} \text{ radians}$$

with 2π radians $= 360°$

$$\Rightarrow 57.3°/\text{radian}$$

$$\Rightarrow \boxed{\phi_{tip} = -5.28° \times 10^{-3}}$$

(c) To find the shear stress use:

$$\tau_{res} = \frac{Tr}{J}$$

To determine the maximum magnitude find the maximum magnitude of $T(x_1)$ and r. The value of r is maximized along the outer surface of the rod ($r = 7.5$ cm)

$T(x_1)$ has a maximum magnitude of -320 N·m for $0 < x_1 < 1.2$ m.

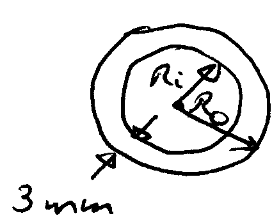
To get a value for τ_{res}, convert these to consistent units and proceed:

$$\text{maximum magnitude } \tau_{res} = \frac{(-320\ N\cdot m)(0.075 m)}{0.795\times 10^{-3}\ m^4}$$

$\Rightarrow$ maximum magnitude

$$\boxed{\tau_{res} = 0.030\ MPa \quad \text{at } r = 7.5\text{ cm},\ 0 < x_1 < 1.2\text{ m}}$$

(d) Rod is now a hollow tube with wall thickness of 3 mm:



$R_o = 7.5$ cm

$R_i = 7.2$ cm $= R_o - 3$ mm

The only thing that changes from the solid rod case is the cross-sectional polar moment of inertia. So ....

(a) Torque distribution does not change!

This is statically determinate and does not depend upon the cross-sectional properties

(b) All remains the same in considering the twist except J changes. Calculate J via superposition (remove inner section):

$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2}(R_o^4 - R_i^4)$$

$$= \frac{\pi}{2}(7.5^4 - 7.2^4)\ [m^4]\ (\times 10^{-8})$$

$$\Rightarrow J = 7.49 \times 10^{-6}\ m^4$$

so: ϕ will change by the change in the inverse of J since: $\phi = \frac{1}{GJ}\int T(x_1)dx_1$

$$J_{new}/J_{old} = \frac{7.49 \times 10^{-6}\ m^4}{4.97 \times 10^{-5}\ m^4} = 0.151$$

$$\Rightarrow \phi_{new}/\phi_{old} = J_{old}/J_{new} = 6.62$$

so: **ϕ_{tip} increases by 562%!**

(c) Here, the shear stress is $\tau_{rs} = \frac{Tr}{J}$

only the polar moment of inertia, J, changes here. The maximum value of r is the same and the maximum value of T and its location is the same. The change is via the inverse of J.

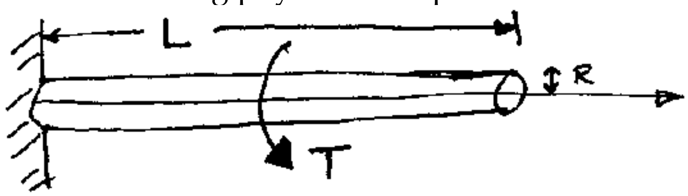
$\Rightarrow$ **maximum magnitude τ_{rs} increases by 562%**

stays at
$r = 7.5\ cm$
$0 < x_1 < 1.2\ m$

(M12.2) MEMORANDUM

TO: Company of interest
FROM: Paul A. Lagace
Re: Device to measure torque

The preliminary design of a circular rod to measure torque has been evaluated as per your request. The objective is to maximize the sensitivity of the measuring capability of the rod provided by a pointer at the far end of the rod calibrated to a linear angular scale. The rod is clamped at one end and the torque is applied through a gear, at the center of the shaft, connected by a chain. The rod has a circular cross-section. These considerations yield the following physical set-up for the shaft:



The equation to determine the twist at the tip of the rod is:

$$\phi_{tip} = TL / 2GJ$$

and was derived through the approach documented in the accompanying Appendix. The parameters are the applied torque, T, the total rod length, L, the material shear modulus, G, and the polar moment of inertia of the cross-section, J. The torque is applied and you have indicated the length is constrained. Thus, in order to maximize the sensitivity of the arrangement, one must maximize the twist measured for a given torque and this implies minimizing the shear modulus of the material, G, and the polar moment of inertia of the cross-section, J.

We would need more information and design constraints from your company to be better able to recommend a material to be used. We expect that you will use a typical isotropic material, such as a metal, for your application. The shear modulus is directly related to the longitudinal modulus, E, and inversely related to the Poisson's ratio. Poisson's ratio is equal to 0.3 for most materials, so we recommend you minimize the value of the longitudinal modulus, E, within your other constraints. In working to minimize the polar moment of inertia of the cross-section, J, we note that a hollow tube, as opposed to one with a solid cross section, produces a lower value of this parameter. For a hollow tube, the value of the parameter is directly related to the fourth power of the outer radius minus the fourth power of the inner radius. Thus, within other considerations, the cross-section should have a wall as thin as possible, and with the smallest outer radius possible.

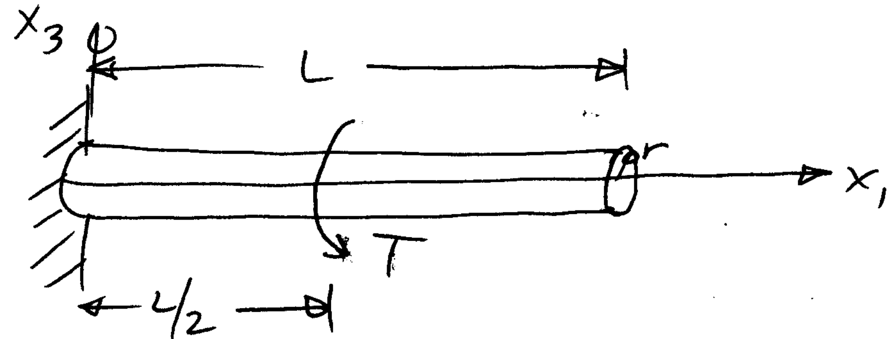
In summary, the recommendations for the rod are as follows:

- Given more design constraints by your company, we will be able to better downselect material for the rod
- The rod should be as long as possible.
- The cross-section of the rod should be hollow, designed to have walls as thin as possible, and with as small a radius as possible.

APPENDIX

Drawing of physical situation:

- Circular rod (length L)
- Clamped at one end
- Torque applied at half-length ($L/2$)



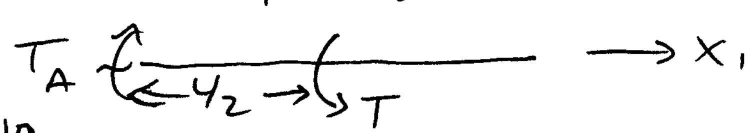
→ want to measure angle ϕ at rod tip

Twist of rod is determined via:

$$\frac{d\phi}{dx_1} = \frac{T(x_1)}{GJ} \qquad (1)$$

⇒ Determine $T(x_1)$:

Draw Free Body Diagram



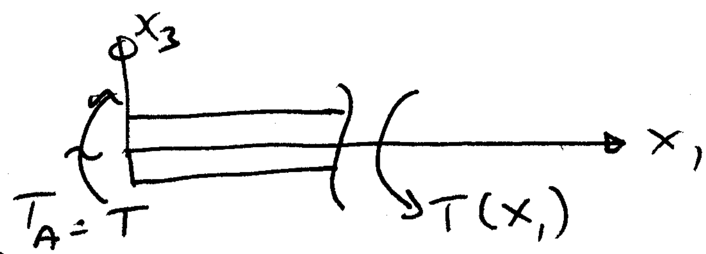
$\Sigma T = 0$ (RHR, +) $\Rightarrow -T_A + T = 0$

$\Rightarrow T_A = T$

Next make cuts along rod. Need two sections one to each side of concentrated applied torque.

<u>Section 1:</u>

$0 < x_1 < L/2$



$\sum T = 0$ (RHR, +) $\Rightarrow -T_A + T(x_1) = 0$

$\Rightarrow T(x_1) = T \quad 0 < x_1 < L/2$

<u>Section 2:</u>

$L/2 < x_1 < L$

$\sum T = 0$ (RHR, +) $\Rightarrow -T_A + T + T(x_1) = 0$

$\Rightarrow T(x_1) = 0 \quad L/2 < x_1 < L$

using (1):

$$0 < x_1 < L/2 \Rightarrow \frac{d\phi}{dx_1} = \frac{T}{GJ}$$

$$\Rightarrow \phi = \frac{T}{GJ}\int dx_1 = \frac{Tx_1}{GJ} + C_1$$

At the wall ($x_1 = 0$) the twist is zero ($\phi = 0$)

$$\Rightarrow C_1 = 0$$

So $\phi = \dfrac{Tx_1}{GJ} \quad 0 < x_1 < L/2$

evaluate at $x_1 = L/2$ for matching: $\phi = \dfrac{TL}{2GJ}$

Now for Section (2):

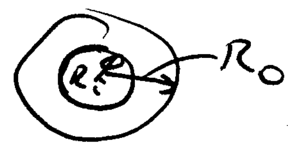
$$\frac{d\phi}{dx_1} = 0 \quad \Rightarrow \phi = C_2$$

match at $L/2$ $\Rightarrow C_2 = \frac{TL}{2GJ}$

so: $\boxed{\phi_{tip} = \frac{TL}{2GJ}}$

Now consider the shear modulus, G, and the polar moment of inertia, J:

Allowing for a hollow tube:



$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

For an isotropic material:

$$G = \frac{E}{2(1+\nu)}$$

We want to maximize ϕ_{tip} (reading) in order to maximize reading sensitivity for a given applied torque. Thus maximize:

$$\frac{2\phi_{tip}}{T} = \frac{L}{GJ}$$

(2: a constant)

with $G = \frac{E}{2(1+\nu)}$

$J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$