Unified Engineering II **Spring 2007**

Problem S17 Solution (Signals and Systems)

Problem Statement: Consider the *quadrature modulation/demodulation* system shown below. The purpose of the system is to transmit two signals, $x_1(t)$ and $x_2(t)$, over the same frequency band simultaneously. $x_1(t)$ and $x_2(t)$ are bandlimited signals, with bandwidth W. That is, their Fourier transforms $X_1(f)$ and $X_2(f)$ satisfy

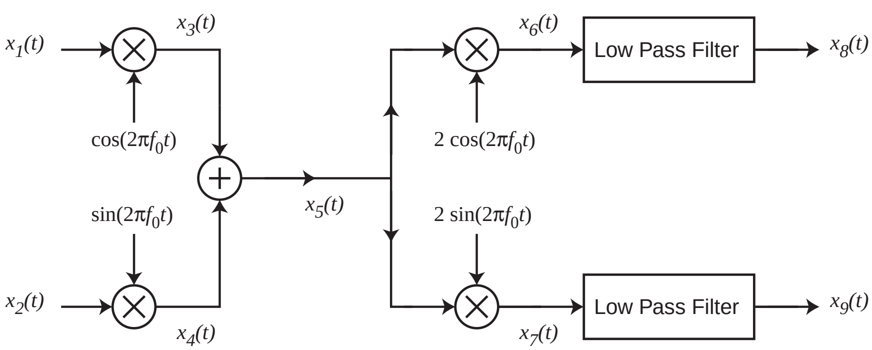
$$X_1(f) = 0, \quad |f| \geq W$$

$$X_2(f) = 0, \quad |f| \geq W$$

The bandwidth is much less than the modulation frequency, f_0. The lowpass filters shown in the diagram are ideal, with transfer function

$$L(f) = \begin{cases} 1, & |f| < W \\ 0, & |f| > W \end{cases}$$

Find the Fourier transforms of the signals $x_3(t)$, $x_4(t)$, $x_5(t)$, $x_6(t)$, $x_7(t)$, $x_8(t)$, and $x_9(t)$ in terms of $X_1(f)$ and $X_2(f)$.



Solution: Define

$$\begin{aligned} w_1(t) &= \cos(2\pi f_0 t) \\ w_2(t) &= \sin(2\pi f_0 t) \\ w_3(t) &= 2\,\cos(2\pi f_0 t) \\ w_4(t) &= 2\,\sin(2\pi f_0 t) \end{aligned}$$

The FTs are

$$\begin{aligned} W_1(f) &= \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \\ W_2(f) &= \frac{-j}{2}\delta(f - f_0) + \frac{j}{2}\delta(f + f_0) \\ W_3(f) &= \delta(f - f_0) + \delta(f + f_0) \\ W_4(f) &= -j\delta(f - f_0) + j\delta(f + f_0) \end{aligned}$$

Therefore,

$$\begin{aligned} X_3(f) &= X_1(f) * W_1(f) \\ &= \frac{1}{2}X_1(f - f_0) + \frac{1}{2}X_1(f + f_0) \end{aligned}$$

$$\begin{aligned} X_4(f) &= X_2(f) * W_2(f) \\ &= \frac{-j}{2}X_2(f - f_0) + \frac{j}{2}X_2(f + f_0) \end{aligned}$$

$$\begin{aligned} X_5(f) &= X_3(f) + X_4(f) \\ &= \frac{1}{2}X_1(f - f_0) + \frac{1}{2}X_1(f + f_0) + \frac{-j}{2}X_2(f - f_0) + \frac{j}{2}X_2(f + f_0) \end{aligned}$$

$$\begin{aligned} X_6(f) &= X_5(f) * W_3(f) \\ &= X_5(f - f_0) + X_5(f + f_0) \\ &= \frac{1}{2}X_1(f - 2f_0) + \frac{1}{2}X_1(f) + \frac{-j}{2}X_2(f - 2f_0) + \frac{j}{2}X_2(f) \\ &\quad + \frac{1}{2}X_1(f) + \frac{1}{2}X_1(f + 2f_0) + \frac{-j}{2}X_2(f) + \frac{j}{2}X_2(f + 2f_0) \\ &= X_1(f) + \frac{1}{2}X_1(f - 2f_0) + \frac{1}{2}X_1(f + 2f_0) \\ &\quad + \frac{-j}{2}X_2(f - 2f_0) + \frac{j}{2}X_2(f + 2f_0) \end{aligned}$$

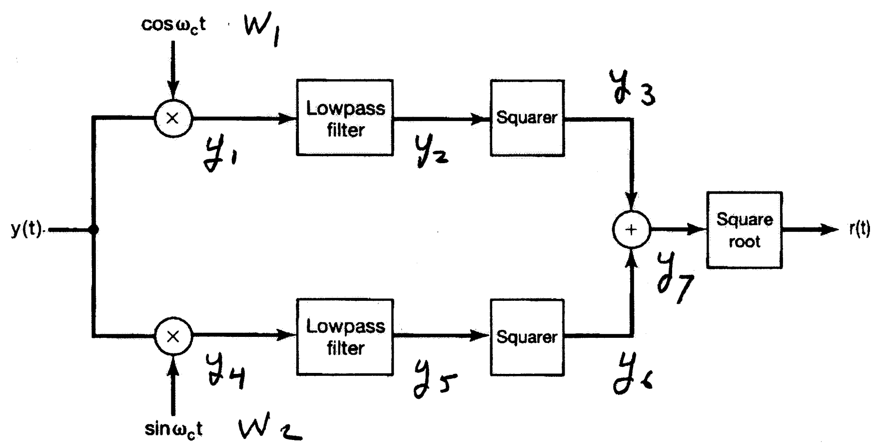
$$\begin{aligned} X_7(f) &= X_5(f) * W_4(f) \\ &= -jX_5(f - f_0) + jX_5(f + f_0) \\ &= \frac{-j}{2}X_1(f - 2f_0) + \frac{-j}{2}X_1(f) - \frac{1}{2}X_2(f - 2f_0) + \frac{1}{2}X_2(f) \\ &\quad + \frac{j}{2}X_1(f) + \frac{j}{2}X_1(f + 2f_0) + \frac{1}{2}X_2(f) - \frac{1}{2}X_2(f + 2f_0) \\ &= X_2(f) - \frac{1}{2}X_2(f - 2f_0) - \frac{1}{2}X_2(f + 2f_0) \\ &\quad + \frac{-j}{2}X_1(f - 2f_0) + \frac{j}{2}X_1(f + 2f_0) \end{aligned}$$

Low-pass filtering $x_6(t)$ and $x_7(t)$ eliminates all but the low-freqiency terms, so that

$$\begin{aligned} X_8(f) &= X_1(f) \\ X_9(f) &= X_2(f) \end{aligned}$$

Problem S18 (Signals and Systems)

To begin, label the signals as shown below:



From the problem statement,

$$y(t) = [x(t) + A] \cos(2\pi f_c t + \theta_c)$$

Define

$$\begin{aligned} z(t) &= x(t) + A \\ w(t) &= \cos(2\pi f_c t + \theta_c) \end{aligned}$$

The factor $w(t)$ can be expanded as

$$w(t) = \cos(2\pi f_c t + \theta_c) = \cos\theta_c \, \cos 2\pi f_c t - \sin\theta_c \, \sin 2\pi f_c t$$

The Fourier transform of $w(t)$ is then given by

$$\begin{aligned} W(f) &= \mathcal{F}[\cos(2\pi f_c t + \theta_c)] \\ &= \frac{1}{2}\cos\theta_c \, [\delta(f - f_c) + \delta(f + f_c)] - \frac{1}{2}\sin\theta_c \, [-j\delta(f - f_c) + j\delta(f + f_c)] \\ &= \frac{1}{2}(\cos\theta_c + j\sin\theta_c)\,\delta(f - f_c) + \frac{1}{2}(\cos\theta_c - j\sin\theta_c)\,\delta(f + f_c) \end{aligned}$$

The Fourier transform of $z(t) = x(t) + A$ is given by

$$Z(f) = \mathcal{F}[z(t)] = X(f) + A\delta(f)$$

$Z(f)$ is bandlimited, because $X(f)$ is, and of course the impulse function is bandlimited. So the FT of $y(t)$ is given by the convolution

$$\begin{aligned} Y(f) &= Z(f) * W(f) \\ &= \frac{1}{2}\left[(\cos\theta_c + j\sin\theta_c)\, Z(f - f_c) + (\cos\theta_c - j\sin\theta_c)\, Z(f + f_c)\right] \end{aligned}$$

Next, compute the spectra of $y_1(t)$ and $y_2(t)$. To do so, we need the spectra of $w_1(t)$ and $w_2(t)$:

$$\begin{aligned}
W_1(f) = \mathcal{F}[w_1(t)] &= \mathcal{F}[\cos 2\pi f_c t] \\
&= \frac{1}{2}[\delta(f-f_c) + \delta(f+f_c)] \\
W_2(f) = \mathcal{F}[w_2(t)] &= \mathcal{F}[\sin 2\pi f_c t] \\
&= \frac{1}{2}[-j\delta(f-f_c) + j\delta(f+f_c)]
\end{aligned}$$

Then

$$\begin{aligned}
Y_1(f) &= W_1(f) * Y(f) \\
&= \frac{1}{2}[Y(f-f_c) + Y(f-f_c)] \\
&= \frac{1}{4}[(\cos\theta_c + j\sin\theta_c) Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c) Z(f)] \\
&\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c) Z(f) + (\cos\theta_c - j\sin\theta_c) Z(f+2f_c)] \\
&= \frac{1}{2}\cos\theta_c\, Z(f) \\
&\quad + \frac{1}{4}[(\cos\theta_c + j\sin\theta_c) Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c) Z(f+2f_c)]
\end{aligned}$$

Similarly,

$$\begin{aligned}
Y_4(f) &= W_2(f) * Y(f) \\
&= \frac{1}{2}[-jY(f-f_c) + jY(f-f_c)] \\
&= \frac{-j}{4}[(\cos\theta_c + j\sin\theta_c) Z(f-2f_c) + (\cos\theta_c - j\sin\theta_c) Z(f)] \\
&\quad + \frac{j}{4}[(\cos\theta_c + j\sin\theta_c) Z(f) + (\cos\theta_c - j\sin\theta_c) Z(f+2f_c)] \\
&= -\frac{1}{2}\sin\theta_c\, Z(f) \\
&\quad + \frac{1}{4}[(-j\cos\theta_c + \sin\theta_c) Z(f-2f_c) + (j\cos\theta_c + \sin\theta_c) Z(f+2f_c)]
\end{aligned}$$

Now, when $y_1(t)$ and $y_4(t)$ are passed through the lowpass filters, the $Z(f-2f_c)$ and $Z(f+2f_c)$ terms are eliminated, and the $Z(f)$ terms are passed. Therefore,

$$\begin{aligned}
Y_2(f) &= \frac{1}{2}\cos\theta_c\, Z(f) \\
Y_5(f) &= -\frac{1}{2}\sin\theta_c\, Z(f)
\end{aligned}$$

and

$$\begin{aligned}
y_2(t) &= \frac{1}{2}\cos\theta_c\, z(t) \\
y_5(t) &= -\frac{1}{2}\sin\theta_c\, z(t)
\end{aligned}$$

After passing these signals through the squarers, we have

$$
\begin{aligned}
y_3(t) &= \frac{1}{4}\cos^2\theta_c\, z^2(t) \\
y_6(t) &= \frac{1}{4}\sin^2\theta_c\, z^2(t)
\end{aligned}
$$

$y_7(t)$ is the sum of these, so that

$$
\begin{aligned}
y_7(t) &= y_3(t) + y_7(t) \\
&= \frac{1}{4}\left[\cos^2\theta_c\, z^2(t) + \sin^2\theta_c\, z^2(t)\right] \\
&= \frac{1}{4} z^2(t)
\end{aligned}
$$

Finally, $r(t)$ is obtained by passing taking the square root of $y_7(t)$, so that

$$
\begin{aligned}
r(t) &= \sqrt{z^2(t)/4} \\
&= \frac{|z(t)|}{2}
\end{aligned}
$$

if the positive root is always taken. But $z(t) = x(t) + A$ is always positive, according to the problem statement. Therefore,

$$x(t) = 2r(t) - A$$

Unified Engineering II **Spring 2007**

Problem S19 Solution

Label the signals in the problem as below:

8.8. Consider the modulation system shown in Figure P8.8. The input signal $x(t)$ has a Fourier transform $X(j\omega)$ that is zero for $|\omega| > \omega_M$. Assuming that $\omega_c > \omega_M$, answer the following questions:

(a) Is $y(t)$ guaranteed to be real if $x(t)$ is real?

(b) Can $x(t)$ be recovered from $y(t)$?

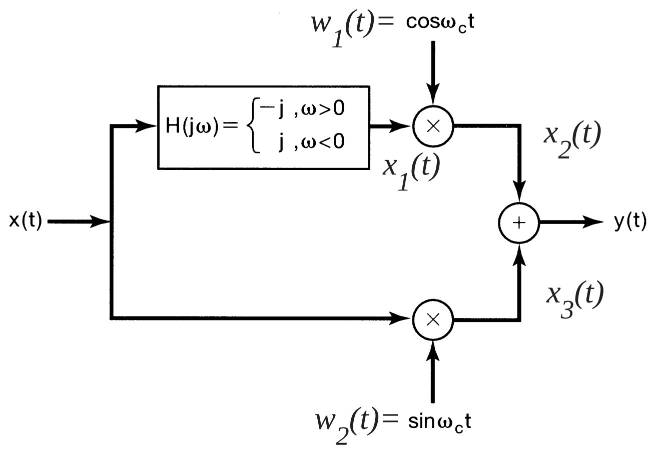


Figure P8.8

The Fourier transform of $x(t)$ is given by $X(f)$. Then the FT of $x_1(t)$ is given by

$$X_1(f) = H(f)X(f) = \begin{cases} -jX(f), & 0 < f < f_M \\ +jX(f), & -f_M < f < 0 \\ 0, & |f| > f_M \end{cases}$$

The signal $x_2(t)$ is given by

$$x_2(t) = w_1(t)x_1(t)$$

where $w_1(t) = \cos 2\pi f_c t$. The FT of $w_1(t)$ is

$$W_1(f) = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

The FT of $x_2(t)$ is then

$$
\begin{aligned}
X_2(f) &= X_1(f) * W_1(f) \\
&= \frac{1}{2}[X_1(f-f_c) + X_1(f+f_c)] \\
&= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ +\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ -\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

The signal $x_3(t)$ is given by

$$x_3(t) = w_2(t)x(t)$$

where $w_2(t) = \sin 2\pi f_c t$. The FT of $w_2(t)$ is

$$W_2(f) = \frac{1}{2}[-j\delta(f-f_c) + j\delta(f+f_c)]$$

The FT of $x_3(t)$ is then

$$
\begin{aligned}
X_3(f) &= X(f) * W_2(f) \\
&= \frac{1}{2}[-jX(f-f_c) + jX(f+f_c)] \\
&= \begin{cases} -\frac{j}{2}X(f-f_c), & f_c < f < f_c + f_M \\ -\frac{j}{2}X(f-f_c), & f_c - f_M < f < f_c \\ +\frac{j}{2}X(f+f_c), & -f_c < f < -f_c + f_M \\ +\frac{j}{2}X(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

Finally, the FT of $y(t)$ is given by

$$
\begin{aligned}
Y(f) &= X_2(f) + X_3(f) \\
&= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ 0, & f_c - f_M < f < f_c \\ 0, & -f_c < f < -f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases} \\
&= \begin{cases} -jX(f-f_c), & f_c < f < f_c + f_M \\ +jX(f+f_c), & -f_c - f_M < f < -f_c \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

First, $y(t)$ is guaranteed to be real if $x(t)$, because if $x(t)$ real, $X(f)$ has conjugate symmetry, and then $Y(f)$ has conjugate symmetry, which implies $y(t)$ real.

Second, $x(t)$ can be recovered from $y(t)$ as follows. If $y(t)$ is modulated by $2\sin 2\pi f_c t$, the resulting signal is $z(t) = 2y(t)\sin 2\pi f_c t$, which has FT

$$
\begin{aligned}
Z(f) &= -jY(f-f_c) + jY(f+f_c) \\
&= \begin{cases} -X(f-2f_c), & 2f_c < f < 2f_c + f_M \\ +X(f), & -f_M < f < 0 \\ +X(f), & 0 < f < f_M \\ -X(f+2f_c), & -2f_c - f_M < f < -2f_c \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

If $z(t)$ is then passed through a lowpass filter, with cutoff at $f = \pm f_M$, then the resulting signal is identical to $x(t)$.

Problem 520 SOLUTION

We can work in f or ω.
I prefer f — a similar solution will result using ω.

(a) We must first find $P(f)$. Since $p(t)$ is periodic, it has a Fourier series

$$p(t) = \sum_{n=\infty}^{\infty} a_n e^{j2\pi n t/2\Delta}$$

since the period is $T = 2\Delta$. The a_n are given by

$$a_n = \frac{1}{T} \int_0^T p(t)\, e^{-j2\pi n t/2\Delta}\, dt$$

$$= \frac{1}{T} \int_{0^-}^{T^-} \left[\delta(t) - \delta(t-\Delta)\right] e^{-j\pi n t/\Delta}$$

$$= \frac{1}{T}\left(1 - e^{-j\pi n}\right)$$

$$= \begin{cases} 2/T, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore,

$$a_n = \begin{cases} 1/\Delta, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Therefore,

$$p(t) = \sum_{n \text{ odd}} \frac{1}{\Delta} e^{j2\pi nt/2\Delta}$$

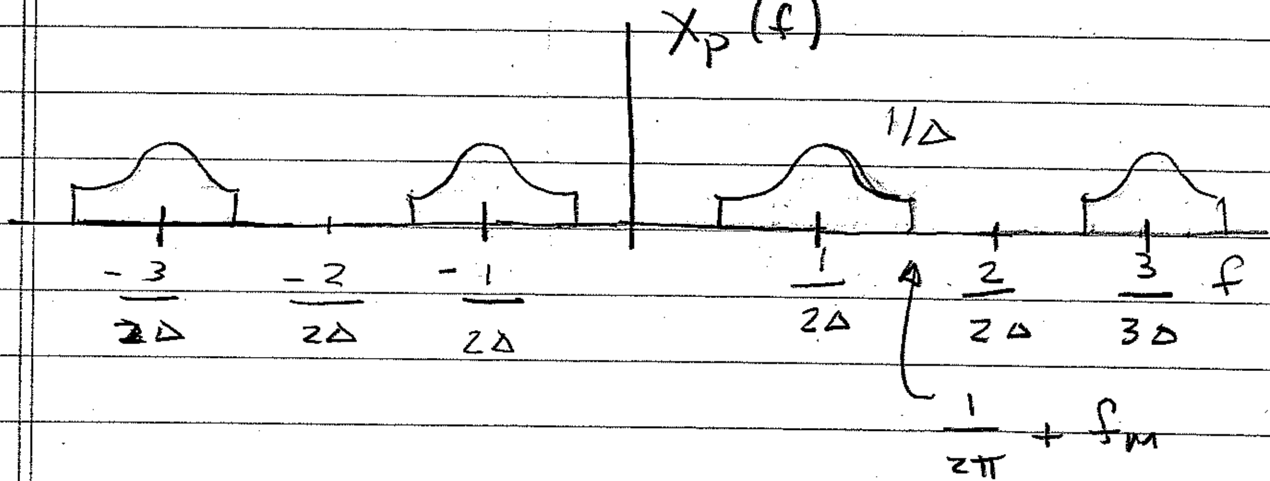
Taking the FT,

$$P(f) = \sum_{n \text{ odd}} \frac{1}{\Delta} \delta(f - n/2\Delta)$$

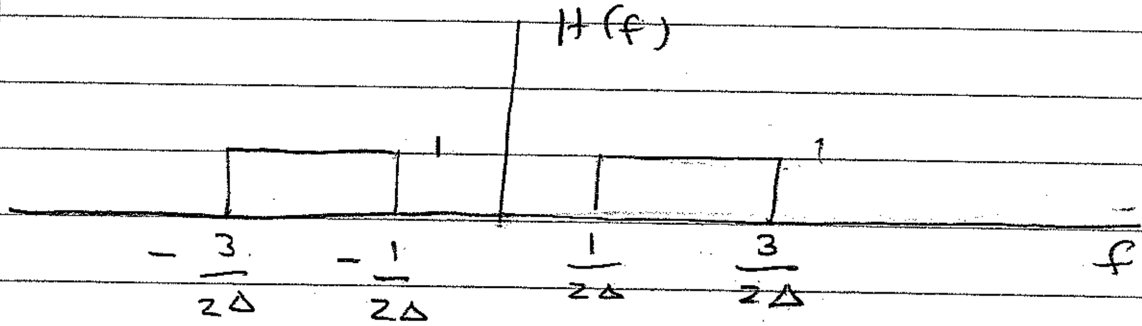
Note that since $x_p(t) = x(t) \cdot p(t)$,

$$X_p(f) = X(f) * P(f)$$

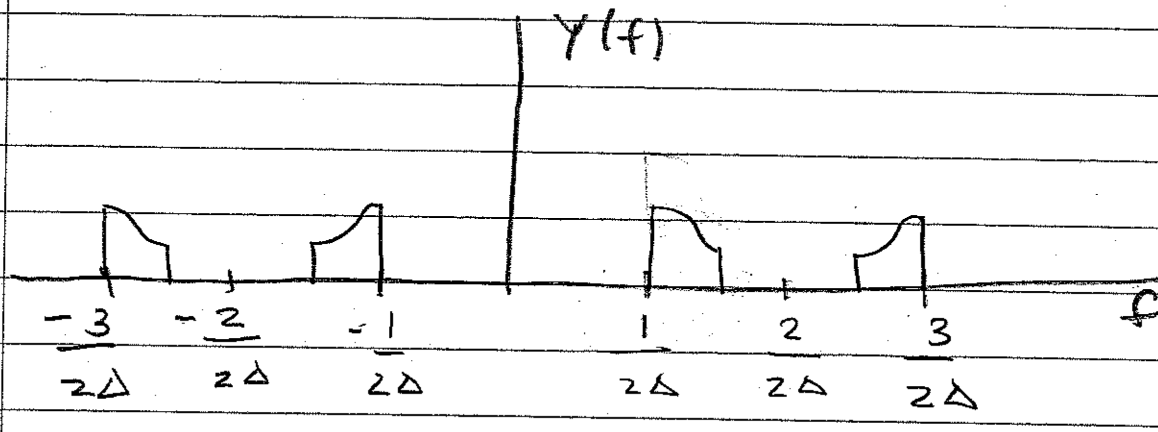
$$= \sum_{n \text{ odd}} \frac{1}{\Delta} X(f - n/2\Delta)$$



Note that $f_m = \omega_m / 2\pi$. As drawn above, $\Delta = \frac{\pi}{2\omega_m} = \frac{1}{4 f_m}$

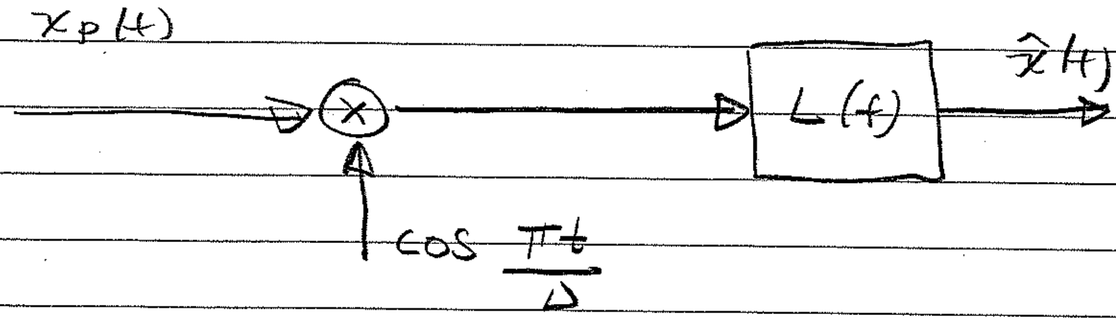


$Y(f)$ is just $H(f) \cdot X_p(f)$:



(b) To recover $x(t)$ from $x_p(t)$, first multiply $x_p(t)$ by $\cos \frac{\pi t}{\Delta}$.

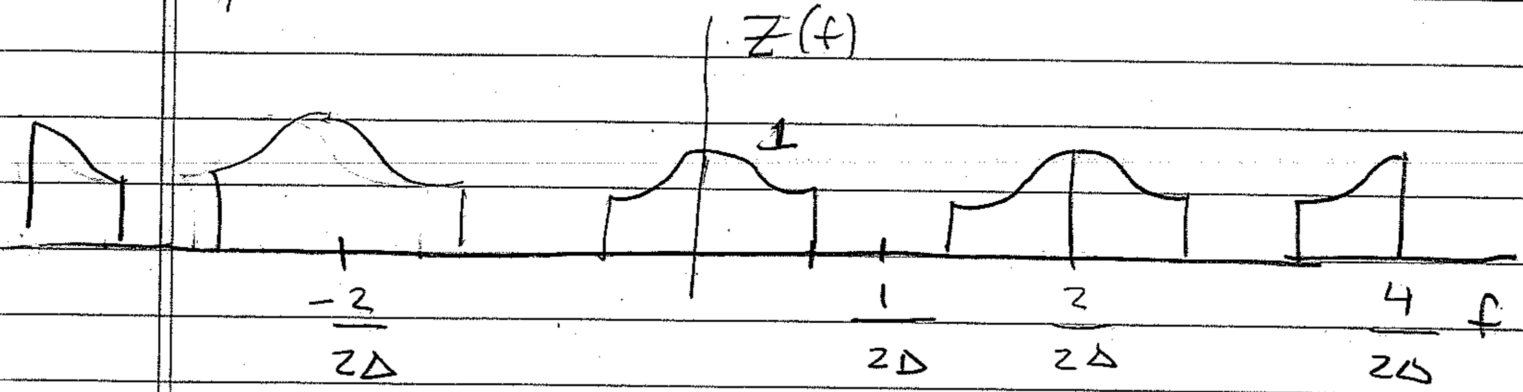
This process inverts every other sample, so that we end up with the samples of $x(t)$ as in a normal sampling process! So one approach that works is



where

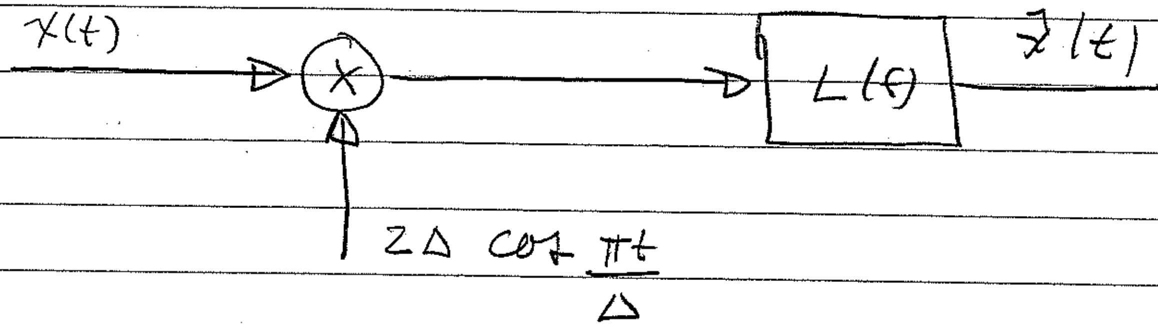
$$L(f) = \begin{cases} \Delta, & |f| < \frac{1}{2\Delta} \\ 0, & \text{else} \end{cases}$$

(c) To recover $x(t)$ from $y(t)$, multiply $y(t)$ by $2\Delta \cos \pi t/\Delta$, to obtain $z(t)$. The resulting spectrum is



So passing through an ideal low pass will recover $x(t)$:

So the solution is



where

$$L(f) = \begin{cases} 1, & |f| < \frac{1}{2\Delta} \\ 0, & \text{else} \end{cases}$$

— almost the same as in (b)

(d) In either case, the system will work if

$$\frac{1}{2\Delta} > f_m = \frac{\omega_m}{2\pi}$$

$$\Rightarrow \quad \Delta > 2f_m = \frac{\omega_m}{\pi}$$

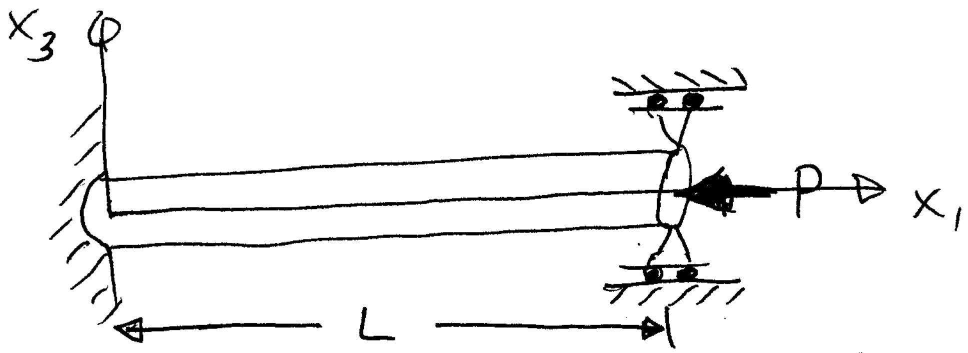
This is consistent with the Nyquist sampling theorem.

PAL
5/5/07

Unified Engineering Problem Set
Week 13 Spring, 2007

SOLUTIONS

M 13.1



Cross-Section = A
Moment of Inertia = I
Modulus = E

The basic governing equation is:

$$\frac{d^2 u_3}{dx_1^2} + \frac{P}{EI} u_3 = 0 \qquad (1)$$

with the general homogeneous solution:

$$u_3 = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B \cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C + D x_1 \qquad (2)$$

• At the clamped end $(x_1 = 0)$: $u_3 = 0$

$$\frac{du_3}{dx_1} = 0$$

- At the roller-supported end with applied load ($x_1 = L$):

$$u_3 = 0$$

$$M = 0 \Rightarrow \frac{d^2 u_3}{d x_1^2} = 0$$

→ To facilitate writing the solution let us represent:

$$\lambda = \sqrt{\frac{P}{EI}}$$

So (1) becomes: $\frac{d^2 u_3}{dx_1^2} + \lambda^2 u_3 = 0 \qquad (1)'$

and (2) becomes: $u_3 = A \sin \lambda x_1 + B \cos \lambda x_1 + C + D x_1 \qquad (2)'$

To use the boundary conditions in the basic solution need derivatives of (2). So:

$$\frac{du_3}{dx_1} = \lambda A \cos \lambda x_1 - \lambda B \sin \lambda x_1 + D$$

$$\frac{d^2 u_3}{dx_1} = -\lambda^2 A \sin \lambda x_1 - \lambda^2 B \cos \lambda x_1$$

Now apply each of the 4 Boundary Conditions to get 4 equations:

@ $x_1 = 0$, $u_3 = 0 \Rightarrow B + C = 0 \qquad (3)$

@ $x_1 = 0$, $\frac{du_3}{dx_1} = 0 \Rightarrow \lambda A + D = 0 \qquad (4)$

@ $x_1 = L$, $u_3 = 0 \Rightarrow A \sin\lambda L + B\cos\lambda L + C + DL = 0$ (5)

@ $x_1 = L$, $\frac{d^2 u_3}{dx_1^2} = 0 \Rightarrow -\lambda^2 A \sin\lambda L - \lambda^2 B \cos\lambda L = 0$

giving: $A \sin\lambda L + B\cos\lambda L = 0$ (6)

→ Now manipulate these equations.....

Use (6) in (5) $\Rightarrow C + DL = 0$

giving: $\boxed{C = -DL}$

use this in (3) to get: $\boxed{B = DL}$

working directly with (4) gives: $\boxed{A = -D/\lambda}$

→ All constants are now in terms of one constant, D. Use these expressions in the overall solution (2)':

$$u_3 = -D/\lambda \sin\lambda x_1 + DL\cos\lambda x_1 - DL + Dx_1$$

and thus:

$$u_3 = D(-1/\lambda \sin\lambda x_1 + L\cos\lambda x_1 - L + x_1)$$

This gives 2 possible solutions:

$D = 0$ (trivial)

or

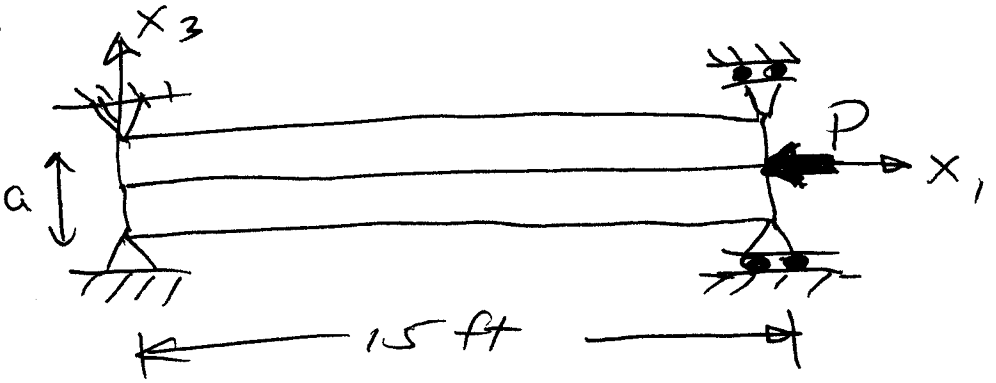
$$-\frac{1}{\lambda}\sin\lambda x_1 + L\cos\lambda x_1 - L + x_1 = 0$$

bringing back $\sqrt{\frac{P}{EI}} = \lambda$ yields:

$$\boxed{-\sqrt{\frac{EI}{P}}\sin\left(\sqrt{\frac{P}{EI}}\,x_1\right) + L\cos\left(\sqrt{\frac{P}{EI}}\,x_1\right) - L + x_1 = 0}$$

This is the expression to determine the items. The loads P that satisfy this are the eigenvalues and thus the buckling load(s). And with these value(s) back in the governing expression, we have the eigenvectors and thus the buckling mode(s).

M13.2



(a) Model this as a simply-supported column.

For a simply-supported configuration:

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

for a square cross-section with a side length of a, use:

$$I = \frac{bh^3}{12} \text{ to get: } I = \frac{a^4}{12}$$

The value of E for aluminum as per this case is 10.3×10^6 lbs/in². $L = 15$ ft and converting this to inches: $L = 180$ in

using these in the expression for P_{cr} gives:

$$P_{cr} = \frac{\pi^2 (10.3 \times 10^6 \text{ lbs/in}^2)(a^4/12)}{(180 \text{ in})^2}$$

working this through gives:

$$\boxed{P_{cr} = 261.5\ a^4}$$

a in [in]
P in [lb]

(b) To determine the squashing load, the material compressive ultimate is needed.

For the aluminum: $\sigma_{cu} = 63$ ksi

Have: $$\frac{P_{squash}}{A} = \sigma_{cu}$$

Here... $A = a^2$

So: $$\boxed{P_{sq} = 63{,}000\ a^2}$$

a in [in]
P in [lb]

one can also determine the start of a "transition" zone via:

$$\frac{P_{transition}}{A} = \sigma_{cY}$$

here: $\sigma_{cY} = 55$ ksi

$$\Rightarrow \boxed{\underset{(yield)}{P_{trans}} = 55{,}000\ a^2}$$

a in [in]
P in [lb]

(c) The key to drawing the design chart is to determine the points (P and a) where the mode of failure goes from "buckling" to "transition" to "crushing/squashing". Do this by equating the buckling curves with the latter two, solving for a, and substituting the result to get P. Then plot each curve.

Summarizing:

(A) Buckling: $P_{cr} = 261.4\, a^4$

(B) Transition (yielding): $P_{tran} = 55{,}000\, a^2$

(C) Squashing: $P_{sq} = 63{,}000\, a^2$

All: a in [in], P in [lb]

going from (A) to (B):

$$261.4\, a^4 = 55{,}000\, a^2$$

$$\Rightarrow a^2 = 210.4$$

$$\Rightarrow a = 14.5 \text{ in}$$

giving $P = 1.16 \times 10^7$ lbs

going from (A) to (C):

$$261.4\, a^4 = 63{,}000\, a^2$$

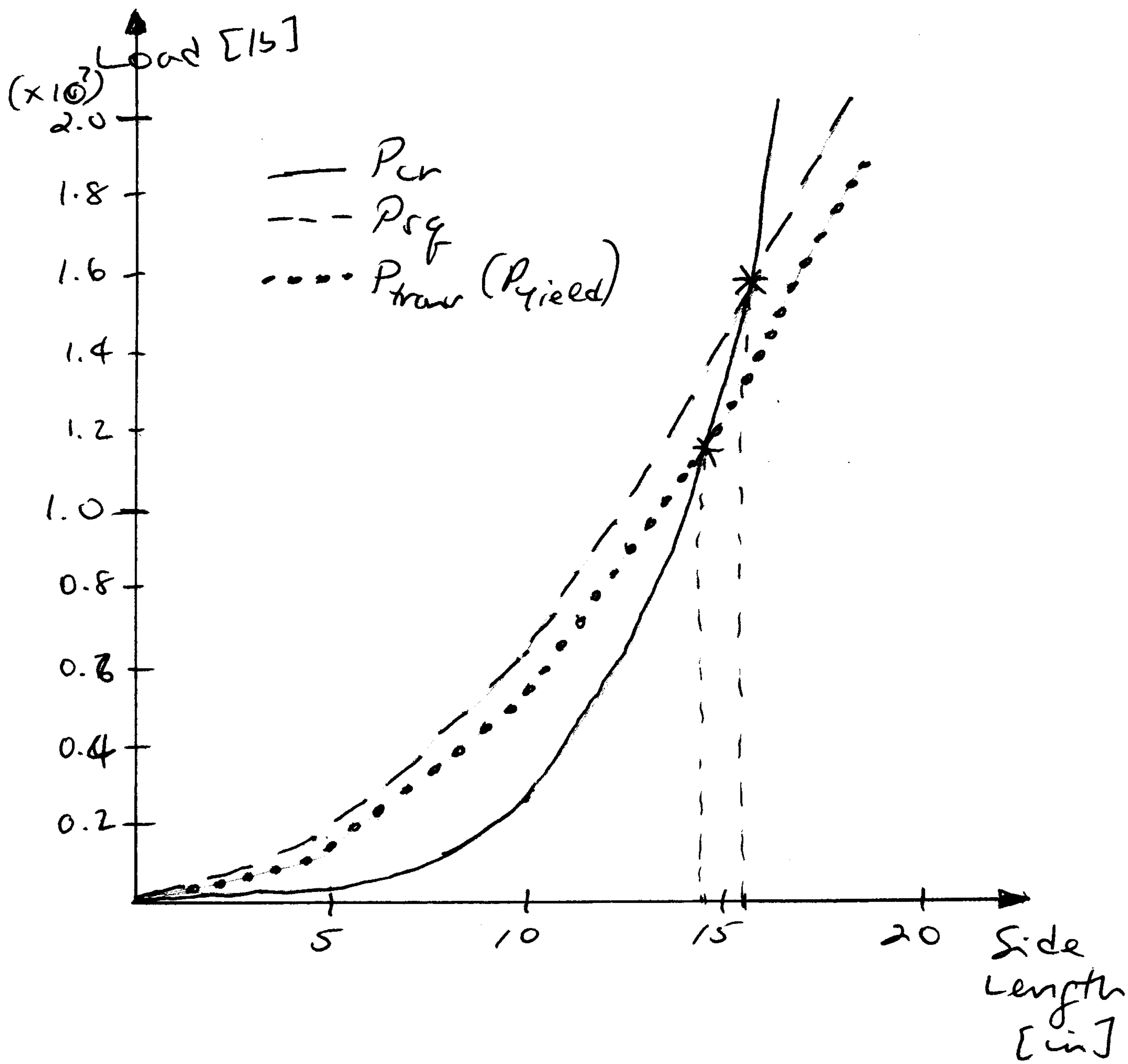
$$\Rightarrow a^2 = 241.0$$

$$\Rightarrow a = 15.5 \text{ in}$$

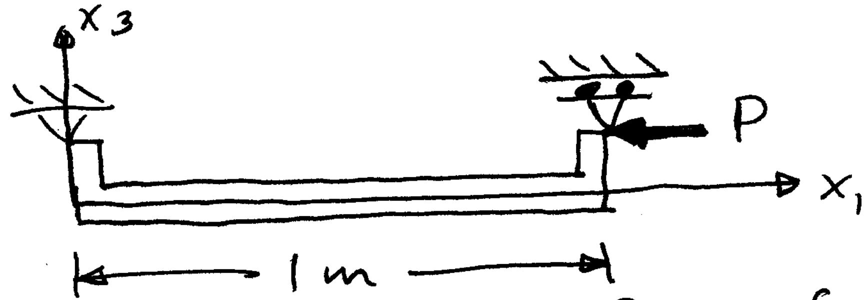
giving $P = 1.52 \times 10^7$ lbs

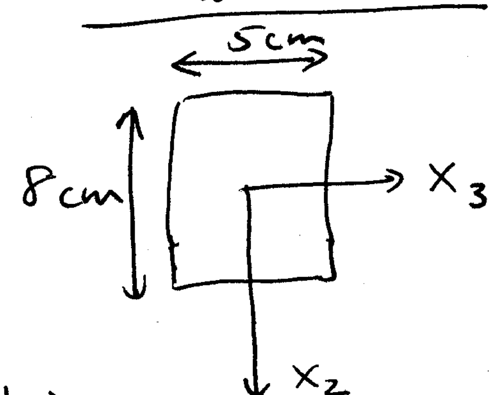
Now draw the plots of each curve and label these key points

Design Chart



M13.3





(a) The maximum load is the limit placed by the buckling load. This does not change due to the eccentric loading. This is a simply supported configuration, so:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Have for the steel: $E = 200\,GPa = 200 \times 10^9 \frac{N}{m^2}$

Need moment of inertia I. For a rectangular cross-section:

$$I = \frac{bh^3}{12}$$

The structure will buckle in the direction with the lowest I, so where "h" is smallest:

$$\Rightarrow I = \frac{(8\,cm)(5\,cm)^3}{12}$$

$$= \frac{(8\times10^{-2}\,m)(5\times10^{-2}\,m)^3}{12} = 8.33\times10^{-7}\,m^4$$

This gives:

$$P_{cr} = \frac{\pi^2 (200\times10^9 \frac{N}{m^2})(8.33\times10^{-7}\,m^4)}{(1\,m)^2}$$

$$\Rightarrow \boxed{P_{cr} = 1.64\times10^6\,N}$$

Check σ_{cr} to see if it is below σ_{cy} and σ_{cu}:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1.64\times10^6\,N}{(8\times10^{-2}\,m)(5\times10^{-2}\,m)} = 4.1\times10^8\,\frac{N}{m^2}$$

$$= 411\,MPa$$

and this is well below the yield and ultimate stress

(b) For the case of a simply-supported configuration loaded eccentrically, the governing equation is:

$$u_3 = e\left[\frac{\left(1-\cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}x_1 + \cos\sqrt{\frac{P}{EI}}x_1 - 1\right]$$

Use the pertinent values of P_{cr}, E, I, and L, and to determine the deflection at the column center, set $x_1 = 0.5m$.

Normalize that deflection by the length and the applied load by the critical load.

To do this....

multiply P by $\frac{P_{cr}}{P_{cr}} = \frac{\pi^2 EI}{P_{cr} L^2}$

$$\Rightarrow \sqrt{\frac{P}{EI}} = \sqrt{\frac{P}{EI} \cdot \frac{\pi^2 EI}{P_{cr} L^2}} = \sqrt{\frac{P}{P_{cr}} \frac{\pi^2}{L^2}}$$

$$\text{so: } \sqrt{\frac{P}{EI}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}$$

Put this back into the earlier equation to get:

$$u_3 = e\left[\frac{1-\cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\,L\right)}{\sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\,L\right)} \sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\,x_1\right) + \cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\,x_1\right) - 1\right]$$

continuing on and dividing through by L:

$$\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\pi\sqrt{\frac{P}{P_{cr}}}}{\sin\pi\sqrt{\frac{P}{P_{cr}}}} \sin\left(\pi\sqrt{\frac{P}{P_{cr}}}\,\frac{x_1}{L}\right) + \cos\left(\pi\sqrt{\frac{P}{P_{cr}}}\,\frac{x_1}{L}\right) - 1\right]$$

and at the center, $x_1/L = 0.5$ giving:

$$\boxed{\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\pi\sqrt{\frac{P}{P_{cr}}}}{\sin\pi\sqrt{\frac{P}{P_{cr}}}} \sin\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) + \cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]}$$

This is the same expression for
all cases (just use specific
value of e)

(c) Use this relationship to make plots
for the five cases of $\frac{e}{L} = 0, 0.01, 0.02, 0.05, 0.1$.

Normalized Load vs. Normalized Center Deflection