Problem S21 (Signals and Systems)

Solution:

1. The signal is plotted below:

The signal is very smooth, almost like a Gaussian. Therefore, I expect that the duration bandwidth product will be close to the theoretical lower bound.

2. 

\[ \left( \frac{\Delta t}{2} \right)^2 = \frac{\int t^2 g^2(t) \, dt}{\int g^2(t) \, dt} \]

The two integrals are easily evaluated for the given \( g(t) \). The result is

\[ \int t^2 g^2(t) \, dt = \frac{7}{2} \]

\[ \int g^2(t) \, dt = \frac{5}{2} \]

Therefore,

\[ \Delta t = 2 \sqrt{\frac{7}{5}} \]

3. The time domain formula for the bandwidth is

\[ \left( \frac{\Delta \omega}{2} \right)^2 = \frac{\int \dot{g}^2(t) \, dt}{\int g^2(t) \, dt} \]

The numerator integral is

\[ \int \dot{g}^2(t) \, dt = \frac{1}{2} \]
Therefore,

\[ \Delta \omega = \frac{2}{\sqrt{5}} \]

4. The duration-bandwidth product is

\[ \Delta t \Delta \omega = \frac{4\sqrt{7}}{5} \approx 2.1166 \]

which is very close to the theoretical lower limit of 2. This is not surprising, since the shape of \( g(t) \) is close to a gaussian.
Problem S22  (Signals and Systems)

Solution:

I used Mathematica to find some of the integrals, although you could use tables or integrate by parts.

(a)  
\[ \bar{t} = \int t g^2(t) \, dt = \int_0^\infty t^7 e^{-2t/\tau} \, dt = \frac{315}{16} \tau^8 \]
\[ \bar{t} = \int g^2(t) \, dt = \int_0^\infty t^6 e^{-2t/\tau} \, dt = \frac{45}{8} \tau^7 \]

Therefore,
\[ \bar{t} = \frac{7}{2} \tau \]

(b)  
\[ \int (t - \bar{t})^2 g^2(t) \, dt = \frac{315}{32} \tau^9 \]

Therefore,
\[ \Delta t = \sqrt{7} \tau \]

(c)  
\[ \int g^2(t) \, dt = \frac{9}{8} \tau^5 \]

Therefore,
\[ \Delta \omega = \frac{2}{\sqrt{5} \tau} \]

(d) The duration-bandwidth product is
\[ \Delta t \Delta \omega = 2 \sqrt{\frac{7}{5}} \approx 2.366 \]

which compares favorably with the theoretical lower bound
\[ \Delta t \Delta \omega \geq 2 \]
Unified Engineering Problem Set
Week 14  Spring, 2007

SOLUTIONS

M 14.1  Spring modeled as a rod in uniaxial tension

\[ P \rightarrow \epsilon \rightarrow P \]

(a) Total energy of "rod" (spring) in terms of stress and strain

\[ U = \int_0^\infty \sigma \, d\epsilon \] per unit volume

strain prior to yielding (and assuming linear behavior):

\[ \sigma = E \epsilon \]

Combine these two equations to fit:

\[ U = \int_0^\infty E \epsilon \, d\epsilon \]

\[ \Rightarrow U = \frac{1}{2} E \epsilon^2 \bigg|_0^\infty = \frac{1}{2} E \epsilon^2 \]

Place this in terms of \( \sigma \) and \( E \) using the stress-strain relation in the form:

\[ \epsilon = \frac{\sigma}{E} \]

This gives:
\[ U = \frac{1}{2} \varepsilon \left( \frac{\sigma}{\varepsilon} \right)^2 \]

\[ \Rightarrow U = \frac{1}{2} \frac{\sigma^2}{\varepsilon} \text{ per unit volume} \]

(b) Need to place the total energy in terms of the pertinent parameters.

(i) For a given volume, from (a) we see it depends on \( \sigma \) and \( \varepsilon \). We maximize \( \sigma \) to the point of yielding (\( \sigma_y \)).

\[ U = \frac{1}{2} \frac{\sigma_y^2}{\varepsilon} \text{ per unit volume} \]

For a rod, we know that:

\[ \sigma = \frac{F}{A} \]

\[ \text{and volume } = AL = \text{constant} \]

So for a given volume, maximize \( \frac{\sigma_y^2}{\varepsilon} \)

Note that \( \varepsilon \) is just a constant.
(ii) For a given mass of material... this means that the volume will...  

In (i) noted that volume = \( L A \).

Knowing density \( \rho \), then:

\[ \text{Mass} = \rho L A = \text{Constant} \]  

(1)

To determine total energy, need:

\[ \text{(Energy per unit volume)} \times \text{(Volume)} = \text{Energy} \]

Use equation (1) to get an expression for the volume \( V \):  

\[ \text{Volume} = L A = \frac{\text{Constant}}{\rho} \]

Use with the expression for \( A \) per unit volume:

\[ \Rightarrow \text{Energy} = \frac{1}{2} \frac{\sigma_T^2}{\varepsilon} \frac{\text{Constant}}{\rho} \]

Disregarding the constant:

\[ \Rightarrow \max \frac{\sigma_T^2}{\varepsilon \rho} \]

(iii) For a given cost of material... this (again) means that the volume will change.  

Operate with the material parameter of cost per mass \( c \). So:

\[ \text{Total cost} = (\text{Cost per mass})(\text{mass}) \]
with the total cost as a constant, we must also use an expression for mass involving material parameters as we did in (ii)

\[ \text{Mass} = \rho \cdot A \]

so:

\[ \text{Cost} = C \left( \rho \cdot A \right) = \text{constant} \]

we can thus once again fit an expression for the volume in terms of pertinent material parameters:

\[ \text{Volume} : \quad A = \frac{\text{constant}}{\rho \cdot C} \]

Use this in the expression for total energy:

\[ \text{Energy} = \frac{1}{2} \frac{\sigma^2}{\varepsilon} \quad \frac{\text{constant}}{\rho \cdot C} \]

Again, corresponding constants

\[ \Rightarrow \maximize \frac{\sigma^2}{\varepsilon \cdot \rho \cdot C} \]

(c) Looking at these six materials, calculate the pertinent combinations of parameters (i.e., the criterion for each case) and compare:
<table>
<thead>
<tr>
<th>Material</th>
<th>Given Volume: $\frac{\sigma_y^2}{\varepsilon}$ [10^6 Pa]</th>
<th>Given Mass: $\frac{\sigma_y^2}{\varepsilon_p}$ [Pa/(g/m^3)]</th>
<th>Given Cost: $\frac{\sigma_y^2}{\varepsilon_{pc}}$ [10^3 N/m^2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al alloy</td>
<td>3.57</td>
<td>1.32</td>
<td>0.66</td>
</tr>
<tr>
<td>Spring steel</td>
<td>27.43</td>
<td>3.43</td>
<td>1.14</td>
</tr>
<tr>
<td>Rubber</td>
<td>18.0</td>
<td>20.0</td>
<td>15.9</td>
</tr>
<tr>
<td>Titanium</td>
<td>16.90</td>
<td>3.76</td>
<td>0.37</td>
</tr>
<tr>
<td>Nickel</td>
<td>18.69</td>
<td>2.10</td>
<td>0.49</td>
</tr>
<tr>
<td>Graphite/epoxy</td>
<td>4.23</td>
<td>2.82</td>
<td>0.014</td>
</tr>
</tbody>
</table>

Note: Being clear and consistent on units is important. Look for each case.

\[
\frac{\sigma_y^2}{\varepsilon} = \frac{[10^6 \text{ Pa}]}{[10^8 \text{ Pa}]} = [10^3 \text{ Pa}] \quad \text{with } 10^3 \text{ used for numbers}
\]

\[
\frac{\sigma_y^2}{\varepsilon_p} : \text{Start from } \frac{\sigma_y^2}{\varepsilon} \text{ and add } \frac{1}{\varepsilon_p}
\]

\[
[10^6 \text{ Pa}] \times \frac{1}{[10^8 \text{ g/m}^3]} = [\text{Pa/(g/m}^3)]
\]

Could also go to:

\[
[\text{N/m}^2/(\text{g/m}^3)] = \frac{\text{N/m}^2}{\text{g}}
\]

\[
\frac{\sigma_y^2}{\varepsilon_{pc}} : \text{Start from } \frac{\sigma_y^2}{\varepsilon_p} \text{ and add } \frac{1}{\varepsilon_c}
\]
\[
\left[\frac{\text{Pa}}{\text{g/m}^3}\right] \cdot \left[\frac{1}{5 \times 10^3 \text{g}}\right] = \left[\frac{10^3 \text{ Pa} \cdot \text{m}^2}{\text{s}}\right]
\]

Could also go to: \(\left[\frac{10^5 \text{ N} \cdot \text{m}}{\text{s}}\right]\)

Comments:

- For the volume criterion, spring steel is the best material by at least 50%.
- For the mass criterion, rubber is the best material by almost an order of magnitude.
- For the cost criterion, rubber is the best material by over an order of magnitude.
- Overall, rubber is easily the best in two of the three criteria and is second closest in the other, so it is the most likely choice overall.
\textbf{Matrix Condition A:} \hspace{1cm} \sigma_{11} = -p \quad \sigma_{12} = 0 \\
\sigma_{22} = -p \quad \sigma_{13} = 0 \\
\sigma_{33} = -p \quad \sigma_{23} = 0 \\

\textbf{Condition B:} \hspace{1cm} \sigma_{nn} = 0.5p \quad \sigma_{12} = 0 \\
\sigma_{22} = p \quad \sigma_{13} = 0 \\
\sigma_{33} = 2p \quad \sigma_{23} = 0 \\

\textbf{Condition C:} \hspace{1cm} \sigma_{nn} = 2p \quad \sigma_{12} = 0 \\
\sigma_{22} = -p \quad \sigma_{13} = 0 \\
\sigma_{33} = 0.5p \quad \sigma_{23} = 0 \\

\textbf{Condition D:} \hspace{1cm} \sigma_{nn} = p \quad \sigma_{12} = 0 \\
\sigma_{22} = 4p \quad \sigma_{13} = 0 \\
\sigma_{33} = 0.5p \quad \sigma_{23} = 0 \\

(a) Application of the Tresca condition requires knowledge of the principal stresses.

For Conditions A, B, and C, there are no applied shear stresses so the applied normal stresses are the principal stresses.

Put these in appropriate order based on magnitude:
For condition D, there is no applied stress in the $3$-axis since $\sigma_3 = \sigma_{23} = 0$ so $\sigma_{33}$ is a principal stress. However, $\sigma_3$ is non-zero, so the principal stresses in the $1-2$ plane need to be determined.

We'll call $\sigma_{33} = \sigma_{III}$ and label the two in the $1-2$ plane as $\sigma_I$ and $\sigma_{II}$. (We'll put them in order from least term for planes other than $1-2$.)

Principal stresses are roots of equation:

$$\tau^2 - \tau (\sigma_{II} + \sigma_{12}) + (\sigma_{II} \sigma_{12} - \sigma_{12}^2) = 0$$

For condition D:

$$\tau^2 - \tau (4\tau + 4\tau) + (4\tau(4\tau) - (2\tau)^2) = 0$$

$$\Rightarrow \tau^2 - 5\tau^2 + (4\tau^2 - 4\tau^2) = 0$$

$$\Rightarrow \tau (\tau - 5\tau) = 0$$

$$\Rightarrow \tau \sigma_{II} = 5\tau$$

$$\Rightarrow \sigma_{II} = 0$$

Finally, we have for condition D:

$$\sigma_I = 5\tau$$
$$\sigma_{II} = 0.5\tau$$
$$\sigma_{III} = 0$$
Now apply the Tresca criterion where yield occurs if:

\[ |\sigma_I - \sigma_II| = \sigma_y \]
\[ \text{or} \]
\[ |\sigma_II - \sigma_III| = \sigma_y \]
\[ \text{or} \]
\[ |\sigma_III - \sigma_I| = \sigma_y \]

In addition, the directionality associated with this is that yielding occurs via shear on the plane of maximum shear stress corresponding to the difference in these two principal stresses.

Here: \( \sigma_y = 1500 \text{ MPa} \)

Apply each condition...

**Condition A:**
Hydrostatic stress

\[ \Rightarrow \text{All differences} = 0 \quad \Rightarrow \text{No yielding} \]

**Condition B:**
\[ |\sigma_I - \sigma_II| = |2p - p| = p = \sigma_y \]
\[ \Rightarrow p = 1500 \text{ MPa} \]
\[ |\sigma_II - \sigma_III| = |p - 0.5p| = 0.5p = \sigma_y \]
\[ \Rightarrow p = 3000 \text{ MPa} \]
\[ |\sigma_III - \sigma_I| = |0.5p - 2p| = 1.5p = \sigma_y \]
\[ \Rightarrow p = 1000 \text{ MPa} \]

Critical case is last one.
\[ \Rightarrow \text{yielding at } p = 1500 \text{ MPa} \]
\[ \text{on plane at } 45^\circ \text{ between } \sigma_{11} \text{ and } \sigma_{22} \]

**Condition C:** \[ |\sigma_1 - \sigma_2| = 12p - (-p) = \sigma_Y \]
\[ \Rightarrow 3p = 1500 \text{ MPa} \]
\[ \Rightarrow p = 500 \text{ MPa} \]
\[ |\sigma_2 - \sigma_3| = 1p - 0.5p = \sigma_Y \]
\[ \Rightarrow 0.5p = 750 \text{ MPa} \]
\[ \Rightarrow p = 1500 \text{ MPa} \]
\[ |\sigma_3 - \sigma_1| = 10.5p - 2p = \sigma_Y \]
\[ \Rightarrow 1.5p = 1500 \text{ MPa} \]
\[ \Rightarrow p = 1000 \text{ MPa} \]

critical case is first \( \sigma \)

\[ \Rightarrow \text{yielding at } p = 500 \text{ MPa} \]
\[ \text{on plane at } 45^\circ \text{ between } \sigma_{11} \text{ and } \sigma_{22} \]

**Condition D:** \[ |\sigma_1 - \sigma_2| = 15p - 0 \| = \sigma_Y \]
\[ \Rightarrow 5p = 1500 \text{ MPa} \]
\[ \Rightarrow p = 300 \text{ MPa} \]
\[ |\sigma_2 - \sigma_3| = 0 - 0.5p = \sigma_Y \]
\[ \Rightarrow 0.5p = 1500 \text{ MPa} \]
\[ \Rightarrow p = 3000 \text{ MPa} \]
\[ |\sigma_3 - \sigma_1| = 10.5p - 5p = \sigma_Y \]
\[ \Rightarrow 4.5p = 1500 \text{ MPa} \]
\[ \Rightarrow p = 333.3 \text{ MPa} \]
Critical case is that

\[ \Rightarrow \text{yielding at } p = 300 \text{ MPa} \]

on plane at 45° to direction of principal stress in 1-2 plane and \( \sigma_{33} \)

Find this angle in 1-2 plane by using:

\[ \Theta_p = \frac{1}{2} \tan^{-1} \left( \frac{20\sigma_{12}}{\sigma_{11} - \sigma_{22}} \right) \]

\[ = \frac{1}{2} \tan^{-1} \left( \frac{4p}{p - (-4p)} \right) \]

\[ = \frac{1}{2} \tan^{-1} \left( \frac{4}{5} \right) \]

\[ \Theta_p = \frac{1}{2} (38.7°) \]

\[ \Rightarrow \Theta_p = 19.3° \]

(b) The von Mises criterion is:

\[ (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 \]

Look at each condition again.

\underline{Condition A:} Hydrostatic stress

All principal stresses are zero, so again

\[ \Rightarrow \text{no yielding} \]

\underline{Condition B:}

\[ (2p - p)^2 + (p - 0.5p)^2 + (0.5p - 2p)^2 = 2\sigma_y^2 \]

\[ \Rightarrow p^2 + 0.25p^2 + 2.25p^2 = 2\sigma_y^2 \]
\[
3.5p^2 = (1500 \text{ MPa})^2 \times 2
\]
\[
p = \sqrt{\frac{2}{13.5}} (1500 \text{ MPa}) = 1134 \text{ MPa}
\]

for (B) \[p = 1134 \text{ MPa}\]

Condition C:
\[
2p + (p - 0.5p)^2 + (0.5p - 2p)^2 = 2\sigma_y^2
\]
\[
p^2 + 2.25p^2 + 2.25p^2 = 2\sigma_y^2
\]
\[
p = \sqrt{\frac{2}{13.5}} \sigma_y
\]
\[
\Rightarrow \text{ for (C)}: p = 577 \text{ MPa}
\]

Condition D:
\[
(5p - 0.5p)^2 + (0.5p - 0)^2 + (0 - 5p)^2 = 2\sigma_y^2
\]
\[
20.25p^2 + 0.25p^2 + 25p^2 = 2\sigma_y^2
\]
\[
35.5p^2 = 2\sigma_y^2
\]
\[
p = \sqrt{\frac{2}{43.5}} \sigma_y^2
\]
\[
\Rightarrow \text{ for (D)}: p = 314 \text{ MPa}
\]

(c) For each of the conditions, the Tresca criterion gives a more conservative estimate of the yielding load characteristic \(p\) (see summary table that follows). The one case where this is not true is Condition A which is a hydrostatic case and both criteria predict the same yielding pressure. This is a fundamental basis for each.
Summary:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Tresca (MPa)</th>
<th>von Mises (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>B</td>
<td>1000</td>
<td>0.34</td>
</tr>
<tr>
<td>C</td>
<td>500</td>
<td>577</td>
</tr>
<tr>
<td>D</td>
<td>300</td>
<td>314</td>
</tr>
</tbody>
</table>

The Tresca criterion considers yielding on a single plane, and thus the two principal stresses acting on that plane. In contrast, the von Mises criterion involves and intercepts all the applied stresses and thus slightly higher values.

M14.3 Airplane fuselage

\[ t = 0.035 \text{ in} \]

At limit \( p = 10 \text{ psi} \) (pressure differential):

\[ \sigma_{\text{hoop}} = \sigma_{22} = \frac{pR^2}{t} \]

\[ \sigma_{\text{long}} = \sigma_{11} = \frac{pR}{2t} \]
(a) Stress of material is sum of stress due to pressure and stress from superimposed Loads accounting for 50% load-carrying factor of skin.

So:
\[
\sigma_{11} = \frac{1}{2} \left( \sigma_{11} (due\ to) + \sigma_{11} (applied) \right)
\]
\[
\sigma_{22} = \frac{1}{2} \left( \sigma_{22} (due\ to) \right)
\]
\[
\sigma_{12} = \frac{1}{2} \left( \sigma_{12} (applied) \right)
\]

Using the pressure equations. At this limit condition:
\[
\sigma_{11} (due\ to) = \frac{10 \text{ psi} \cdot (6 \text{ ft}) \cdot (12 \text{ in})}{2 \cdot 0.035 \text{ in}} = 10,286 \text{ psi}
\]
\[
\sigma_{22} (due\ to) = \frac{10 \text{ psi} \cdot (6 \text{ ft}) \cdot (12 \text{ in})}{0.035 \text{ in}} = 20,572 \text{ psi}
\]

Using in the above:
\[
\sigma_{11} = 5143 + \sigma_{11} (applied) \text{ [psi]}
\]
\[
\sigma_{22} = 10,286 \text{ psi}
\]
\[
\sigma_{12} = (\sigma_{12} (applied)) \text{ [psi]}
\]

Recognizing that \( \sigma_{11} \) and \( \sigma_{22} \) are half of the applied load condition.

Now use the Tresca condition. As before:
\[
|\sigma_1 - \sigma_3| = \sigma_y \quad \Rightarrow \quad |\sigma_1 - \sigma_3| = \sigma_y \quad \Rightarrow \quad \sigma_1 - \sigma_3 = \sigma_y
\]
Here we have plane stress with \( \sigma_{III} = 0 \), so this becomes:

\[
\begin{align*}
|\sigma_{I} - \sigma_{II}| &= \sigma_Y \\
|\sigma_{II}| &= \sigma_Y \\
|\sigma_{III}| &= \sigma_Y
\end{align*}
\]

with \( \sigma_Y = 50 \text{ ksi} \)

It is necessary to find the principal stresses for the plane stress case. Again (as in first problem), use:

\[ \tau^2 = \tau (\sigma_{11} + \sigma_{22}) + (\sigma_{11} \sigma_{22} - \sigma_{12}^2) = 0 \]

and find roots. Do so in this form. Use quadratic solution for:

\[
\tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
\Rightarrow \quad \sigma_{I, II} = \frac{1}{2} \left\{ (\sigma_{11} + \sigma_{22}) \pm \left[ (\sigma_{11} + \sigma_{22})^2 - 4(\sigma_{11} \sigma_{22} - \sigma_{12}^2) \right]^{1/2} \right\}
\]

write out:

\[
\Rightarrow = \frac{1}{2} \left\{ (\sigma_{11} + \sigma_{22}) \pm \left[ (\sigma_{11} + \sigma_{22})^2 - 4(\sigma_{11} \sigma_{22} + \sigma_{12}^2) \right]^{1/2} \right\}
\]

\[
= \frac{1}{2} \left\{ (\sigma_{11} + \sigma_{22}) \pm \left[ \sigma_{11}^2 - 2\sigma_{11} \sigma_{22} + \sigma_{22}^2 + 4\sigma_{12}^2 \right]^{1/2} \right\}
\]

so:

\[
\sigma_{I} = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) + \sqrt{\frac{\sigma_{12}^2 + (\frac{\sigma_{11} - \sigma_{22}}{2})^2}{2}}
\]

\[
\sigma_{II} = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) - \sqrt{\frac{\sigma_{12}^2 + (\frac{\sigma_{11} - \sigma_{22}}{2})^2}{2}}
\]

The Tresca condition can be rewritten using these expressions:
\[ S_0 \text{kSi} = \left| \sigma_2 \right| = \sqrt{\sigma_{12}^2 + \sigma_{22}^2} \]  

(4)

Now rewrite these in terms of the total stresses from equations (1'), (2'), and (3'):

\[ S_0 \text{kSi} = \left| \sigma_2 \right| = \sqrt{\sigma_{12}^2 + \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right)^2} \]  

(5')

\[ S_0 \text{kSi} = \left| \sigma_2 \right| = \sqrt{\sigma_{12}^2 + \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right)^2} \]  

(6')

Now apply different values of \( \sigma_2 \) for each case and determine the value of \( S_0 \text{kSi} \) that can be failure criterion plot.
From equation (4) :

<table>
<thead>
<tr>
<th>$\sigma_{11} \text{ A.C.}$</th>
<th>$\sigma_{12} \text{ A.T.}$</th>
<th>$\sigma_{13} \text{ A.T.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pm 24.9$</td>
<td>$\pm 24.9$</td>
</tr>
<tr>
<td>$+10, -10$</td>
<td>$\pm 23.8$</td>
<td>$\pm 23.8$</td>
</tr>
<tr>
<td>$+20, -20$</td>
<td>$\pm 21.6$</td>
<td>$\pm 21.6$</td>
</tr>
<tr>
<td>$+30, -30$</td>
<td>$\pm 17.8$</td>
<td>$\pm 17.8$</td>
</tr>
<tr>
<td>$+40, -40$</td>
<td>$\pm 10.7$</td>
<td>$\pm 10.7$</td>
</tr>
<tr>
<td>$+50, -50$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From equation (5) :

<table>
<thead>
<tr>
<th>$\sigma_{11} \text{ A.C.}$</th>
<th>$\sigma_{12} \text{ A.T.}$</th>
<th>$\sigma_{13} \text{ A.T.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pm 42.2$</td>
<td>$\pm 42.2$</td>
</tr>
<tr>
<td>$+10, -10$</td>
<td>$\pm 46.7$</td>
<td>$\pm 46.7$</td>
</tr>
<tr>
<td>$+20, -20$</td>
<td>$\pm 50.8$</td>
<td>$\pm 50.8$</td>
</tr>
<tr>
<td>$+30, -30$</td>
<td>$\pm 4.5$</td>
<td>$\pm 4.5$</td>
</tr>
<tr>
<td>$+40, -40$</td>
<td>$\pm 10.9$</td>
<td>$\pm 10.9$</td>
</tr>
<tr>
<td>$+50, -50$</td>
<td>$\pm 0$</td>
<td>$\pm 0$</td>
</tr>
</tbody>
</table>

From equation (6) :

<table>
<thead>
<tr>
<th>$\sigma_{11} \text{ A.C.}$</th>
<th>$\sigma_{12} \text{ A.T.}$</th>
<th>$\sigma_{13} \text{ A.T.}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pm 42.2$</td>
<td>$\pm 42.2$</td>
</tr>
<tr>
<td>$+10, -10$</td>
<td>$\pm 37.2$</td>
<td>$\pm 37.2$</td>
</tr>
<tr>
<td>$+20, -20$</td>
<td>$\pm 31.4$</td>
<td>$\pm 31.4$</td>
</tr>
<tr>
<td>$+30, -30$</td>
<td>$\pm 21.3$</td>
<td>$\pm 21.3$</td>
</tr>
<tr>
<td>$+40, -40$</td>
<td>$\pm 10.9$</td>
<td>$\pm 10.9$</td>
</tr>
<tr>
<td>$+50, -50$</td>
<td>$\pm 0$</td>
<td>$\pm 0$</td>
</tr>
</tbody>
</table>

Plot the limiting cases
"Operating stress envelope" for tensile material via Tresca condition with limit pressure already accounted for.
(b) With the "damage tolerant" approach, use the basic fracture mechanics equation:

\[ \sigma_f = \frac{K_c}{\sqrt{\pi a}} \]

Here \( 2a = 0.25 \text{ in} \Rightarrow a = 0.125 \text{ in} \)

\[ K_c = 31 \text{ ksi} / \sqrt{\text{in}} \text{ for the 2024 aluminum} \]

\[ \Rightarrow \sigma_f = \frac{31 \text{ ksi} / \sqrt{\text{in}}}{\sqrt{\pi (0.125 \text{ in})}} \]

\[ \Rightarrow \sigma_f = 49.5 \text{ ksi} \]

Thus, if the stress perpendicular to the crack exceeds 49.5 ksi, there is failure. However, the crack could be oriented in many directions, so we must find the principal stresses (i.e., the maximum extensional stresses) and then the related direction for the worst case. Consider the possible cases of the loads that are applied:

1. Joint pressure and those resultant stresses.

   Then \( \sigma_{11} = 5143 \text{ psi} \)
   \( \sigma_{22} = 10,286 \text{ psi} \) (as previously)

Both are principal, since no shear stress is applied and both are well below the critical value of 49.5 ksi.
2) Consider pressure stresses with stress due to longitudinal load. From before (1):

\[ \sigma_1 = 5743 \text{ psi} + \sigma_{11} \text{ A.C.} \]

\[ \sigma_{22} = 10,256 \text{ psi} \]

we see that \( \sigma_{22} \) is well below the critical stress, so the criterion here is:

\[ \sigma_1 = 5,143 \text{ ksi} + \sigma_{11} \text{ A.C.} < 49.5 \text{ ksi} \]

\[ \Rightarrow \sigma_{11} \text{ A.C.} < 48.4 \text{ ksi} \] (4)

3) All three loads: pressure, longitudinal, and torsional. From before (1), (2), (3) and then the previous work to find the principal stresses for the full case:

\[ \sigma_I = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) + \sqrt{\frac{\sigma_{11}^2 + \left( \sigma_{11} - \sigma_{22} \right)^2}{2}} < 49.5 \text{ ksi} \] (7)

\[ \sigma_{II} = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) - \sqrt{\frac{\sigma_{11}^2 + \left( \sigma_{11} - \sigma_{22} \right)^2}{2}} < 49.5 \text{ ksi} \] (8)

All stresses in [ksi]

Note that \( \sigma_I \) always is larger than \( \sigma_{II} \), so only (7) needs to be considered. Further note that this is only valid for tensile stresses.

As before, assemble data for (7) and then plot these with the other condition (a quadrant).
\[
\begin{array}{|c|c|c|}
\hline
\sigma_{11} & \sigma_{12} & \sigma_{22} \\
\hline
0 & \pm 41.7 & \pm 46.2 \\
10~10 & \pm 39.3 & \pm 56.2 \\
20~20 & \pm 30.9 & \pm 59.1 \\
30~30 & \pm 23.7 & \pm 54.0 \\
40~40 & \pm 13.1 & \pm 57.5 \\
50~50 & \pm 60.8 & 0 \\
\hline
\end{array}
\]

Plot:

"Operating stress envelope" for buckle material via damage tolerant approach with limit pressure already accounted for

\[
\sigma_{11} \text{ [ksi]} \quad \gamma_{\text{A.T.}} \text{ [ksi]}
\]

\(\gamma\) = operating
(c) Each of these approaches are different criteria and thus plots do look substantially different as may well be expected. The Tresca condition gives the yield point while the damage tolerant approach gives the stress at which a crack will critically propagate. This latter case occurs only for tensile stresses whereas the Tresca condition considers compressive stress as well. The damage tolerant approach gives the fracture possible operating area.

M14.4 1. D
2. G
3. A
4. H
5. B
6. F
7. E
8. C