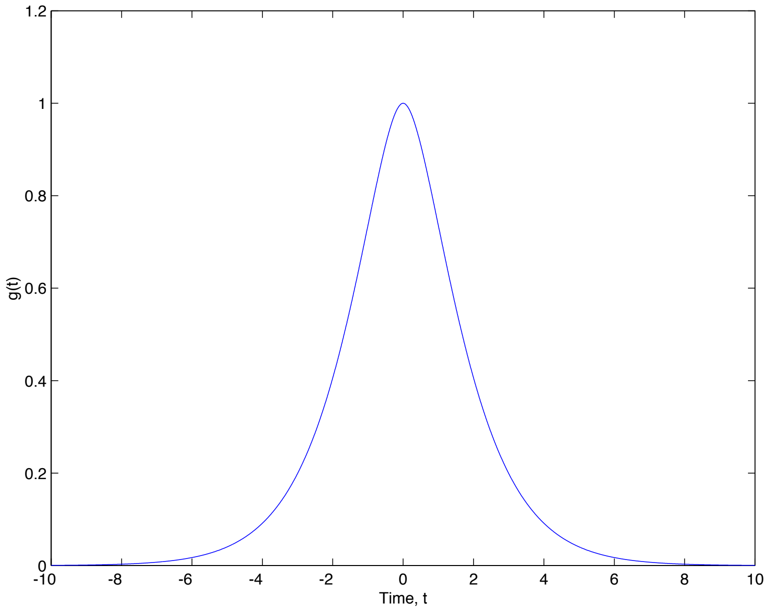
Problem S21 (Signals and Systems)

Solution:

1. The signal is plotted below:



The signal is very smooth, almost like a Gaussian. Therefore, I expect that the duration bandwidth product will be close to the theoretical lower bound.

2.
$$\left(\frac{\Delta t}{2}\right)^2 = \frac{\int t^2 g^2(t)\,dt}{\int g^2(t)\,dt}$$
The two integrals are easily evaluated for the given $g(t)$. The result is
$$\int t^2 g^2(t)\,dt = \frac{7}{2}$$
$$\int g^2(t)\,dt = \frac{5}{2}$$
Therefore,
$$\Delta t = 2\sqrt{\frac{7}{5}}$$

3. The time domain formula for the bandwidth is
$$\left(\frac{\Delta \omega}{2}\right)^2 = \frac{\int \dot{g}^2(t)\,dt}{\int g^2(t)\,dt}$$
The numerator integral is
$$\int \dot{g}^2(t)\,dt = \frac{1}{2}$$

Therefore,

$$\Delta\omega = \frac{2}{\sqrt{5}}$$

4. The duration-bandwidth product is

$$\Delta t\, \Delta\omega = \frac{4\sqrt{7}}{5} \approx 2.1166$$

which is very close to the theoretical lower limit of 2. This is not surprising, since the shape of $g(t)$ is close to a gaussian.

Problem S22 (Signals and Systems)

Solution:

I used Mathematica to find some of the integrals, although you could use tables or integrate by parts.

(a)

$$\bar{t} = \int t g^2(t)\, dt = \int_0^\infty t^7 e^{-2t/tau}\, dt = \frac{315}{16}\tau^8$$

$$\bar{t} = \int g^2(t)\, dt = \int_0^\infty t^6 e^{-2t/tau}\, dt = \frac{45}{8}\tau^7$$

Therefore,

$$\bar{t} = \frac{7}{2}\tau$$

(b)

$$\int (t-\bar{t})^2 g^2(t)\, dt = \frac{315}{32}\tau^9$$

Therefore,

$$\Delta t = \sqrt{7}\,\tau$$

(c)

$$\int \dot{g}^2(t)\, dt = \frac{9}{8}\tau^5$$

Therefore,

$$\Delta\omega = \frac{2}{\sqrt{5}\,\tau}$$

(d) The duration-bandwidth product is

$$\Delta t\, \Delta\omega = 2\sqrt{\frac{7}{5}} \approx 2.366$$

which compares favotably with the theoretical lower bound

$$\Delta t\, \Delta\omega \geq 2$$

PAL
5/9/07

Unified Engineering Problem Set
Week 14 Spring, 2007
SOLUTIONS

M14.1 Spring modeled as a rod in uniaxial tension

(a) Total energy of "rod" (spring) in terms of stress and strain

$$u = \int_0^\epsilon \sigma \, d\epsilon \qquad \text{per } \underline{\text{unit}} \ \underline{\text{volume}}$$

stress prior to yielding (and assuming linear behavior):

$$\sigma = E\epsilon$$

Combine these two equations to get:

$$u = \int_0^\epsilon E\epsilon \, d\epsilon$$

$$\Rightarrow u = \tfrac{1}{2} E\epsilon^2 \Big]_0^\epsilon = \tfrac{1}{2} E\epsilon^2$$

Place this in terms of σ and E using the stress-strain relation in the form:

$$\epsilon = \frac{\sigma}{E}$$

This gives:

$$u = \frac{1}{2} E \left(\frac{\sigma}{E}\right)^2$$

$$\Rightarrow \boxed{u = \frac{1}{2} \frac{\sigma^2}{E}}$$ per unit volume

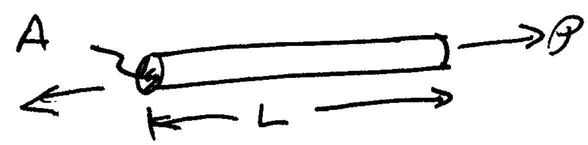
(b) Need to place the total energy in terms of the pertinent parameters

(i) For a given volume, from (a) we see it depends on σ and E. We maximize σ to the point of yielding (σ_y)

$$u = \frac{1}{2} \frac{\sigma_Y^2}{E}$$ per unit volume

For a rod, we know that:

$$\sigma = \frac{P}{A}$$



and Volume = AL = constant

So for a given volume, $\boxed{\text{maximize } \frac{\sigma_Y^2}{E}}$

Note that ½ is just a constant

(ii) for a given mass of material....
this means that the Volume will change.
In (i) noted that Volume = LA.
Knowing density ρ, then:

$$\text{Mass} = \rho LA = \text{Constant} \qquad (1)$$

To determine total energy, need:

$$(\text{Energy per unit volume})(\text{Volume}) = \text{Energy}$$

Use equation (1) to get an expression for the volume, LA:

$$\text{Volume} = LA = \frac{\text{Constant}}{\rho}$$

Use with the expression for U per unit volume:

$$\Rightarrow \text{Energy} = \frac{1}{2}\frac{\sigma_Y^2}{E}\frac{\text{Constant}}{\rho}$$

disregarding the constants

$$\Rightarrow \boxed{\text{maximize } \frac{\sigma_Y^2}{E\rho}}$$

(iii) for a given cost of material...
This (again) means that the Volume will change.
Operate with the material parameter of cost per mass, c. So:

$$\text{total cost} = (\text{cost per mass})(\text{mass})$$

with the ~~total~~ cost as a constant. We must also use an expression for mass involving material parameters as we did in (ii)

$$\text{Mass} = \rho L A$$

So:

$$\text{Cost} = (c)(\rho L A) = \text{constant}$$

we can thus, once again, get an expression for the volume in terms of pertinent material parameters:

$$\text{Volume} = LA = \frac{\text{constant}}{\rho c}$$

Use this in the expression for total energy:

$$\text{Energy} = \frac{1}{2}\frac{\sigma_Y^2}{E}\frac{\text{constant}}{\rho c}$$

Again, disregarding constants.

$$\Rightarrow \boxed{\text{maximize } \frac{\sigma_Y^2}{E\rho c}}$$

(c) Looking at these six materials, calculate the pertinent combinations of parameters (i.e. the criterion for each case) and compare:

Material	Given Volume: maximize (σ_Y^2/E) $[10^6\ Pa]$	Given Mass: maximize $(\sigma_Y^2/E\rho)$ $[Pa/(g/m^3)]$	Given Cost: maximize $(\sigma_Y^2/E\rho c)$ $\left[\frac{10^3\ N\cdot m}{\$}\right]$
Al alloy	3.57	1.32	0.66
Spring Steel	27.43	3.43	1.14
Rubber	18.0	20.0	15.4
Titanium	16.90	3.76	0.37
Nickel	18.69	2.10	0.49
Graphite/Epoxy	4.23	2.82	0.014

Note: Being clear and consistent on units is important. Look for each case.

Volume: σ_Y^2/E : $\dfrac{[10^6\ Pa]^2}{[10^9\ Pa]} = [10^3\ Pa]$ extra 10^3 comes from numbers

Mass: $\sigma_Y^2/E\rho$: Start from $\dfrac{\sigma_Y^2}{E}$ and add $\dfrac{1}{\rho}$

$$[10^6\ Pa]\cdot\frac{1}{[10^6\ g/m^3]} = [Pa/(g/m^3)]$$

could also go to: $[N/m^2/(g/m^3)] = \left[\dfrac{N\cdot m}{g}\right]$

Cost: $\sigma_Y^2/E\rho c$: Start from $\sigma_Y^2/E\rho$ and add $\dfrac{1}{c}$

$$\left[Pa/(g/m^3)\right] \cdot \frac{1}{[\$/10^3 g]} = \left[\frac{10^3\ Pa \cdot m^3}{\$}\right]$$

could also go to: $\left[\frac{10^3\ N \cdot m}{\$}\right]$

Comments:

- For the volume criterion, spring steel is the best material by at least 50%
- For the mass criterion, rubber is the best material by almost an order of magnitude
- For the cost criterion, rubber is the best material by over an order of magnitude
- Overall, rubber is easily the best in two of the three criteria and is second (close) in the other, so it is the most likely choice overall.

M14.2 Condition A: $\sigma_{11} = -p \quad \sigma_{12} = 0$
$\sigma_{22} = -p \quad \sigma_{13} = 0$
$\sigma_{33} = -p \quad \sigma_{23} = 0$

Condition B: $\sigma_{11} = 0.5p \quad \sigma_{12} = 0$
$\sigma_{22} = p \quad \sigma_{13} = 0$
$\sigma_{33} = 2p \quad \sigma_{23} = 0$

Condition C: $\sigma_{11} = 2p \quad \sigma_{12} = 0$
$\sigma_{22} = -p \quad \sigma_{13} = 0$
$\sigma_{33} = 0.5p \quad \sigma_{23} = 0$

Condition D: $\sigma_{11} = p \quad \sigma_{12} = 0$
$\sigma_{22} = 4p \quad \sigma_{13} = 0$
$\sigma_{33} = 0.5p \quad \sigma_{23} = 0$

(a) Application of the Tresca condition requires knowledge of the principal stresses.

For conditions A, B, and C, there are no applied shear stresses, so the applied normal stresses are the principal stresses.

Put these in appropriate order based on magnitude:

Condition A	Condition B	Condition C
$\sigma_{I} = \sigma_{11} = -p$	$\sigma_{I} = \sigma_{33} = 2p$	$\sigma_{I} = \sigma_{11} = 2p$
$\sigma_{II} = \sigma_{22} = -p$	$\sigma_{II} = \sigma_{22} = p$	$\sigma_{II} = \sigma_{22} = -p$
$\sigma_{III} = \sigma_{33} = -p$	$\sigma_{III} = \sigma_{11} = 0.5p$	$\sigma_{III} = \sigma_{33} = 0.5p$

For condition D, there is no applied stress in the 3-axis since $\sigma_{13} = \sigma_{23} = 0$ so σ_{33} is a principal stress. However, σ_{12} is non zero, so the principal stresses in the 1-2 plane need to be determined.
We'll call $\sigma_{33} = \sigma_{III}$ and label the two in the 1-2 plane as σ_{I} and σ_{II} (we'll get them in order afterwards)
From last term for planar stress:

principal stresses are roots of equation:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

For Condition D $\Rightarrow$

$$\tau^2 - \tau(p + 4p) + ([p](4p) - [2p]^2) = 0$$

$$\Rightarrow \tau^2 - 5p\tau + (4p^2 - 4p^2) = 0$$

$$\Rightarrow \tau(\tau - 5p) = 0$$

$$\Rightarrow \tau = \sigma_{I} = 5p$$

$$\tau = \sigma_{II} = 0$$

Finally in order for

Condition D
$\sigma_{I} = 5p$
$\sigma_{II} = 0.5p$
$\sigma_{III} = 0$

Now apply the Tresca criterion where yield occurs if:

$$|\sigma_I - \sigma_{II}| = \sigma_Y$$

or

$$|\sigma_{II} - \sigma_{III}| = \sigma_Y$$

or

$$|\sigma_{III} - \sigma_I| = \sigma_Y$$

in addition, the directionality associated with this is that yielding occurs via shear on the plane of maximum shear stress corresponding to the difference in those two principal stresses

Here: $\sigma_Y = 1500$ MPa

Apply each condition...

Condition A: hydrostatic stress

$\Rightarrow$ All differences $= 0$ $\Rightarrow$ $\boxed{\text{No yielding}}$ A

Condition B: $|\sigma_I - \sigma_{III}| = |2p - p| = p = \sigma_Y$

$\Rightarrow p = 1500$ MPa

$|\sigma_{II} - \sigma_{III}| = |p - 0.5p| = 0.5p = \sigma_Y$

$\Rightarrow p = 3000$ MPa

$|\sigma_{III} - \sigma_I| = |0.5p - 2p| = 1.5p = \sigma_Y$

$\Rightarrow p = 1000$ MPa

critical case is last one.

B

$\Rightarrow$ $\boxed{\text{yielding at } p = 1000 \text{ MPa on plane at } 45° \text{ between } \sigma_{11} \text{ and } \sigma_{33}}$

<u>Condition C</u>: $|\sigma_I - \sigma_{II}| = |2p - (-p)| = \sigma_Y$

$\Rightarrow 3p = 1500 \text{ MPa}$

$\Rightarrow p = 500 \text{ MPa}$

$|\sigma_{II} - \sigma_{III}| = |-p - 0.5p| = \sigma_Y$

$\Rightarrow 1.5p = 1500 \text{ MPa}$

$\Rightarrow p = 1000 \text{ MPa}$

$|\sigma_{III} - \sigma_I| = |0.5p - 2p| = \sigma_Y$

$\Rightarrow 1.5p = 1500 \text{ MPa}$

$\Rightarrow p = 1000 \text{ MPa}$

critical case is first (c)

$\Rightarrow$ $\boxed{\text{yielding at } p = 500 \text{ MPa on plane at } 45° \text{ between } \sigma_{11} \text{ and } \sigma_{22}}$

<u>Condition D</u>: $|\sigma_I - \sigma_{II}| = |5p - 0| = \sigma_Y$

$\Rightarrow 5p = 1500 \text{ MPa}$

$\Rightarrow p = 300 \text{ MPa}$

$|\sigma_{II} - \sigma_{III}| = |0 - 0.5p| = \sigma_Y$

$\Rightarrow 0.5p = 1500 \text{ MPa}$

$\Rightarrow p = 3000 \text{ MPa}$

$|\sigma_{III} - \sigma_I| = |0.5p - 5p| = \sigma_Y$

$\Rightarrow 4.5p = 1500 \text{ MPa}$

$\Rightarrow p = 333 \text{ MPa}$

critical case is first D

$$\Rightarrow \boxed{\text{yielding at } p = 300 \text{ MPa on plane at } 45^\circ \text{ to direction of principal stress}^* \text{ in 1-2 plane and } \sigma_{33}}$$

* find this angle in 1-2 plane by using:

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4p}{p-(-4p)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta_p = \frac{1}{2}(38.7^\circ)$$

$$\Rightarrow \theta_p = 19.3^\circ$$

(b) the von Mises criterion is:

$$(\sigma_{I}-\sigma_{II})^2 + (\sigma_{II}-\sigma_{III})^2 + (\sigma_{III}-\sigma_{I})^2 = 2\sigma_Y^2$$

look at each condition again.

Condition A: hydrostatic stress
All ~~differ~~ differences are zero, so again A

$$\boxed{\text{NO yielding}}$$

Condition B:

$$(2p-p)^2 + (p-0.5p)^2 + (0.5p-2p)^2 = 2\sigma_Y^2$$

$$\Rightarrow p^2 + 0.25p^2 + 2.25p^2 = 2\sigma_Y^2$$

$$\Rightarrow 3.5p^2 = (1500\ MPa)^2 \times 2$$

$$\Rightarrow p = \sqrt{\frac{2}{3.5}}\,(1500\ MPa) = 1134\ MPa$$

for (B): $\boxed{p = 1134\ MPa}$

<u>Condition C</u>:

$$(2p-(-p))^2 + (-p-0.5p)^2 + (0.5p-2p)^2 = 2\sigma_Y^2$$

$$\Rightarrow 9p^2 + 2.25p^2 + 2.25p^2 = 2\sigma_Y^2$$

$$\Rightarrow p = \sqrt{\frac{2}{13.5}}\,\sigma_Y$$

$\Rightarrow$ for (C): $\boxed{p = 577\ MPa}$

<u>Condition D</u>:

$$(5p-0.5p)^2 + (0.5p-0)^2 + (0-5p)^2 = 2\sigma_Y^2$$

$$\Rightarrow 20.25p^2 + 0.25p^2 + 25p^2 = 2\sigma_Y^2$$

$$\Rightarrow 45.5p^2 = 2\sigma_Y^2$$

$$\Rightarrow p = \sqrt{\frac{2}{45.5}}\,\sigma_Y^2$$

$\Rightarrow$ for (D): $\boxed{p = 314\ MPa}$

(c) For each of the conditions, the Tresca Criterion gives a more conservative estimate of the yielding load characteristic, p (see summary table that follows). The one case where this is not true is Condition A which is a hydrostatic case and both criteria predict no yielding as this is a fundamental basis for each.

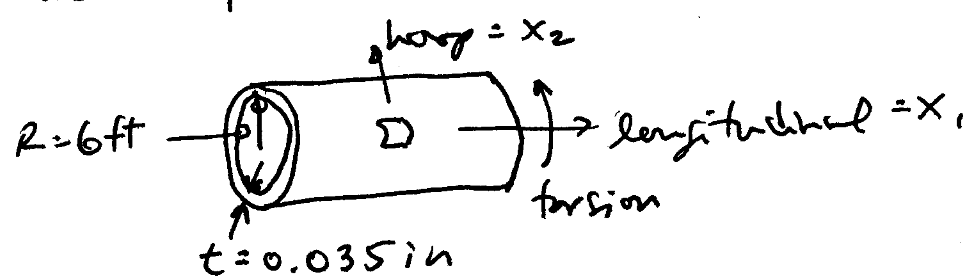
Summary:

Critical p [MPa]

Condition	Tresca	von Mises
A	—	—
B	1000	1134
C	500	577
D	300	314

The Tresca criterion considers yielding on a single plane and thus the two principal stresses acting on that plane. In contrast, the von Mises criterion involves and interacts all the applied stresses and thus slightly higher values.

M14.3 Airplane fuselage



At limit $p = 10$ psi (pressure differential)

$$\sigma_{hoop} = \sigma_{22} = \frac{pR}{t}$$

$$\sigma_{long} = \sigma_{11} = \frac{pR}{2t}$$

(a) stress of material is sum of stress due to pressure and stress from empennage loads accounting for 50% load carrying factor of skin

so:

$$\sigma_{11} = \frac{1}{2}\left(\sigma_{11\,(\text{due to } p)} + \sigma_{11\,(\text{applied wing})}\right) \qquad (1)$$

$$\sigma_{22} = \frac{1}{2}\left(\sigma_{22\,(\text{due to } p)}\right) \qquad (2)$$

$$\sigma_{12} = \frac{1}{2}\left(\sigma_{12\,(\text{applied torsion})}\right) \qquad (3)$$

using the pressure equations. At this limit condition:

$$\sigma_{11\,(\text{due to } p)} = \frac{10\text{ psi}\,(6\text{ ft})(12\text{ in}/\text{ft})}{2\,(0.035\text{ in})} = 10{,}286\text{ psi}$$

$$\sigma_{22\,(\text{due to } p)} = \frac{10\text{ psi}\,(6\text{ ft})(12\text{ in}/\text{ft})}{0.035\text{ in}} = 20{,}572\text{ psi}$$

using in the above:

$$\sigma_{11} = 5143 + \sigma_{11\,(\text{applied wing})} \;[\text{psi}] \qquad (1')$$

$$\sigma_{22} = 10{,}286\text{ psi} \qquad (2')$$

$$\sigma_{12} = \left(\sigma_{12\,(\text{applied torsion})}\right) \;[\text{psi}] \qquad (3')$$

recognizing that $\sigma_{11\,A.L.}$ and $\sigma_{12\,A.T.}$ are half of the applied load condition. Now use the Tresca condition. As before:

$$|\sigma_{I} - \sigma_{II}| = \sigma_Y \;\underline{\text{or}}\; |\sigma_{II} - \sigma_{III}| = \sigma_Y \;\underline{\text{or}}\; |\sigma_{III} - \sigma_{I}| = \sigma_Y$$

Here we have plane stress with $\sigma_{III} = 0$, so this becomes:

$$|\sigma_I - \sigma_{II}| = \sigma_Y$$

$$|\sigma_{II}| = \sigma_Y \qquad |\sigma_{III}| = \sigma_Y$$

with $\sigma_Y = 50$ ksi

It is necessary to find the principal stresses for the planar stress case. Again (as in first problem) use:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

and find roots. Do so in this form. Use quadratic solution for:

$$\tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \sigma_I, \sigma_{II} = \frac{1}{2}\left\{(\sigma_{11} + \sigma_{22}) \pm \left[(\sigma_{11} + \sigma_{22})^2 - 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)\right]^{1/2}\right\}$$

write out:

$$\Rightarrow = \frac{1}{2}\left\{(\sigma_{11} + \sigma_{22}) \pm \left[\sigma_{11}^2 + 2\sigma_{11}\sigma_{22} + \sigma_{22}^2 - 4\sigma_{11}\sigma_{22} + 4\sigma_{12}^2\right]^{1/2}\right\}$$

$$= \frac{1}{2}\left\{(\sigma_{11} + \sigma_{22}) \pm \left[\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2 + 4\sigma_{12}^2\right]^{1/2}\right\}$$

So:

$$\sigma_I = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) + \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

$$\sigma_{II} = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) - \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

The Tresca conditions can be rewritten using these expressions:

$$50\,ksi = |\sigma_I - \sigma_{II}| = \left| 2\sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2} \right| \quad (4)$$

$$50\,ksi = |\sigma_I| = \left| \left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) + \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2} \right| \quad (5)$$

$$50\,ksi = |\sigma_{II}| = \left| \left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) - \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2} \right| \quad (6)$$

Now rewrite these in terms of the total stresses from equations (1'), (2') and (3'):

from (4)

$$50\,ksi = 2\sqrt{\sigma_{12_{A.T.}}^2 + \left(\frac{5143 + \sigma_{11_{A.L.}} - 10{,}286}{2}\right)^2}$$

$$\Rightarrow \boxed{50\,ksi = 2\sqrt{\sigma_{12_{A.T.}}^2 + \left(\frac{\sigma_{11_{A.L.}} - 5143}{2}\right)^2}} \quad (4)'$$

from (5)

$$50\,ksi = \left| \left(\frac{5143 + \sigma_{11_{A.L.}} + 10{,}286}{2}\right) + \sqrt{\sigma_{12_{A.T.}}^2 + \left(\frac{\sigma_{11_{A.L.}} - 5143}{2}\right)^2} \right|$$

$$\Rightarrow \boxed{50\,ksi = \left| \left(\frac{15{,}429 + \sigma_{11_{A.L.}}}{2}\right) + \sqrt{\sigma_{12_{A.T.}}^2 + \left(\frac{\sigma_{11_{A.L.}} - 5143}{2}\right)^2} \right|} \quad (5')$$

and from (6)

$$\boxed{50\,ksi = \left| \left(\frac{15{,}429 + \sigma_{11_{A.L.}}}{2}\right) - \sqrt{\sigma_{12_{A.T.}}^2 + \left(\frac{\sigma_{11_{A.L.}} - 5143}{2}\right)^2} \right|} \quad (6')$$

Now apply different values of σ_{11} for each case and determine the value of σ_{12} that causes failure, then plot them.

all in [ksi]

from equation (4'):

$\sigma_{11\,A.C.}$	$\sigma_{12\,A.T.} = \sqrt{(50)^2 - \left(\frac{\sigma_{11}}{2} - 5.14\right)^2}$
0	±24.9
+10, −10	±24.9, ±23.8
+20, −20	±23.9, ±21.6
+30, −30	±21.7, ±17.8
+40, −40	±17.9, ±10.7
+50.1, −44.8	0

from equation (5'):

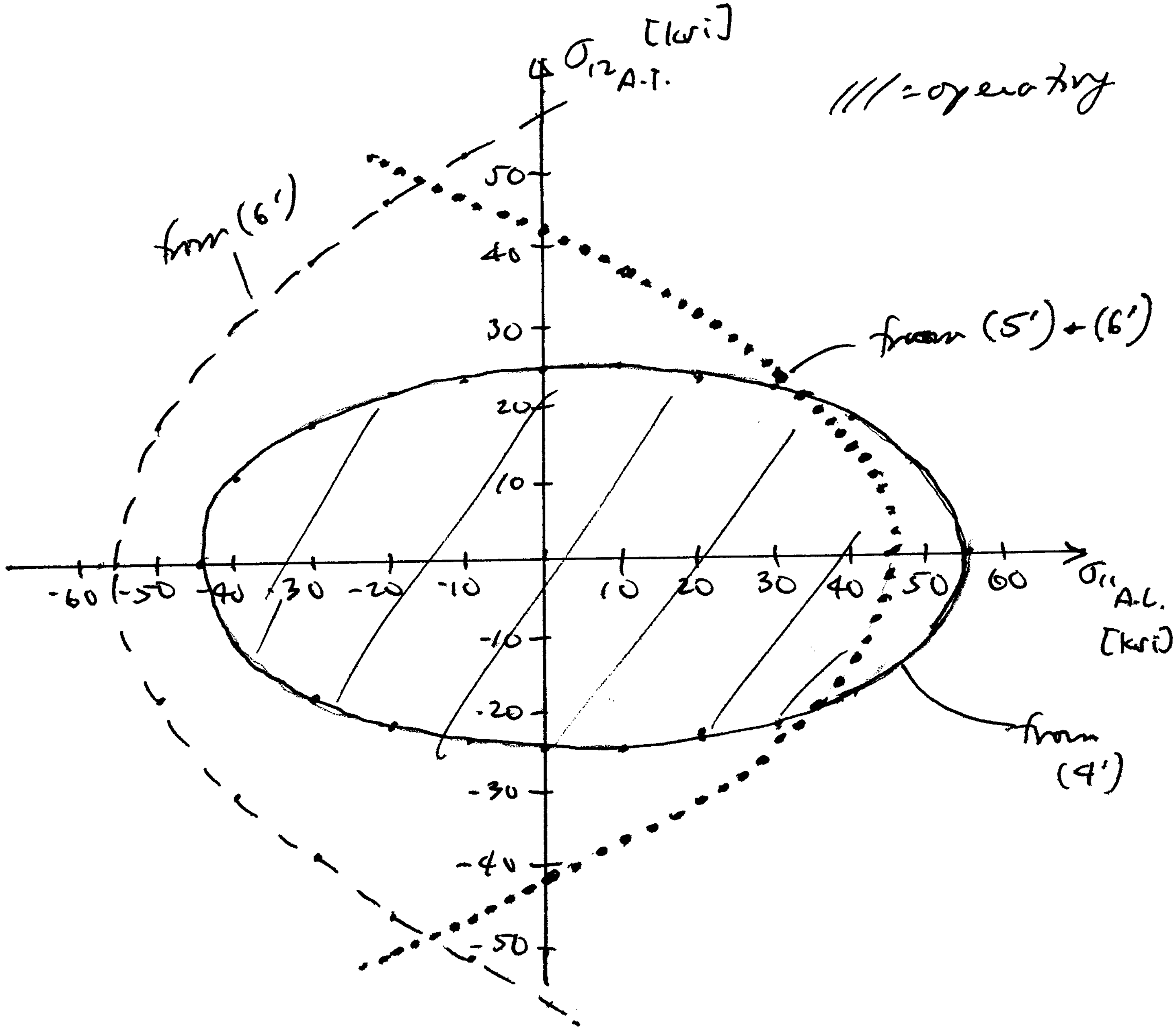
$\sigma_{11\,A.C.}$	$\sigma_{12\,A.T.} = \sqrt{\left(\frac{84.6 - \sigma_{11}}{2}\right)^2 - \left(\frac{\sigma_{11} - 5.14}{2}\right)^2}$	$\sigma_{12\,A.T.} = \sqrt{\left(\frac{\sigma_{11} - 84.6}{2}\right)^2 - \left(\frac{\sigma_{11} - 5.14}{2}\right)^2}$
0	±42.2	±42.2
+10, −10	±37.2, ±46.7	±37.2
+20, −20	±31.4, ±50.8	±31.4
+30, −30	±24.3, ±54.5	same as
+40, −40	±13.9, ±58.1	←
+50, −50	—, ±61.9	
+44.8	0	0

from equation (6'):

$\sigma_{11\,A.C.}$	$\sigma_{12\,A.T.} = \sqrt{\left(\frac{\sigma_{11} - 84.6}{2}\right)^2 - \left(\frac{\sigma_{11} - 5.14}{2}\right)^2}$	$\sigma_{12\,A.T.} = \sqrt{\left(\frac{115.4 + \sigma_{11}}{2}\right)^2 - \left(\frac{\sigma_{11} - 5.14}{2}\right)^2}$
0	±42.2	±57.7
+10, −10	±37.2	±62.7, ±52.2
+20, −20	±31.4	±67.3, ±46.0
+30, −30	±24.3	±71.6, ±38.9
+40, −40	±13.9	±75.7, ±30.2
+50, −50	outside of operating limits	±79.6, ±17.6
+60, −60		±83.3, —
−55.1		0

Plot the limiting lines

"Operating stress envelope" for fuselage material via Tresca condition with limit pressure already accounted for



(b) With the "damage tolerant" approach, use the basic fracture mechanics equation

$$\sigma_f = \frac{K_c}{\sqrt{\pi a}}$$

Here $2a = 0.25\text{ in} \Rightarrow a = 0.125\text{ in}$

$K_{Ic} = 31\ \text{ksi}/\sqrt{\text{in}}$ for the 2024 aluminum

$$\Rightarrow \sigma_f = \frac{31\ \text{ksi}/\sqrt{\text{in}}}{\sqrt{\pi(0.125\text{ in})}}$$

$$\Rightarrow \sigma_f = 49.5\ \text{ksi}$$

Thus, if the stress perpendicular to the crack exceeds 49.5 ksi, there is failure. However, the crack could be oriented many directions, so we must find the principal stresses (i.e. the maximum extensional stresses) and then the related direction for the worst case. Consider the possible cases of the loads that are applied

① Just pressure and those resultant stresses.

There: $\sigma_{11} = 5,143$ psi

$\sigma_{22} = 10,286$ psi (as per earlier)

Both are principal, since no shear stress is applied and both are well below the critical value of 49.5 ksi

(2) Consider pressure stresses with stresses due to longitudinal load. From before (1'):

$$\sigma_{11} = 5743\,psi + \sigma_{11_{A.C.}}$$

$$\sigma_{22} = 10{,}286\,psi$$

we see that σ_{22} is well below the critical stress, so the criterion here is:

$$\sigma_{11} = 5.143\,ksi + \sigma_{11_{A.C.}} < 49.5\,ksi$$

$$\Rightarrow \sigma_{11_{A.C.}} < 44.4\,ksi \quad (*)$$

(3) All three loads: pressure, longitudinal, and torsional. From before (1'), (2'), (3') and then the previous work to find the principal stresses for the full case:

$$\sigma_{I} = \left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) + \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2} \quad < 49.5\,ksi \ (7)$$

$$\sigma_{II} = \left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) - \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2} \quad < 49.5\,ksi \ (8)$$

All stresses in [ksi]

Note that σ_I always is larger than σ_{II}, so only (7) needs to be considered. Further note that this is only valid for tensile stresses.

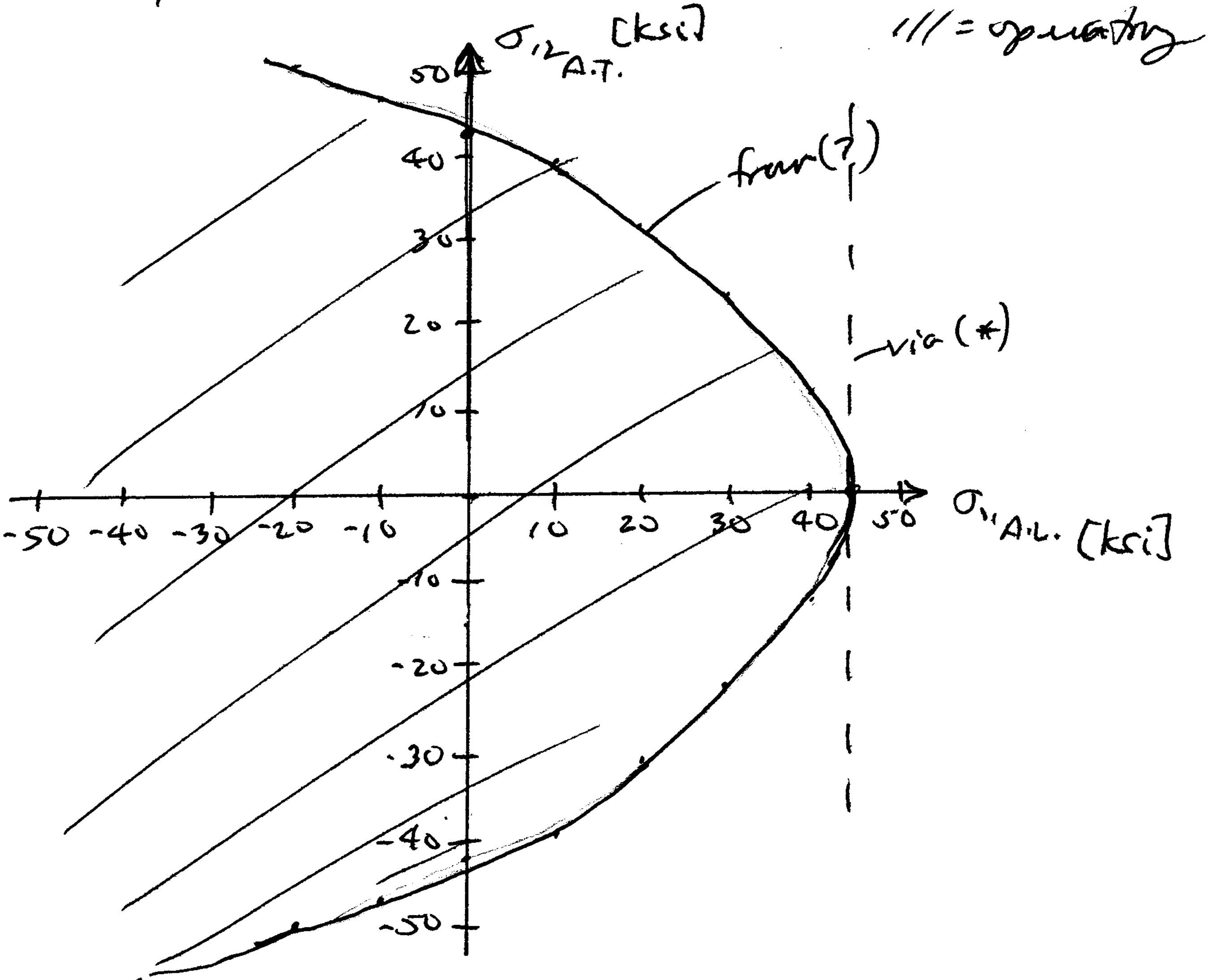
As before, assemble data for (7), and then plot these with the other condition (equation (*))

via equation (7) all in [ksi]

$\sigma_{11_{A.L.}}$	$\sigma_{12_{A.T.}} \lessgtr \sqrt{\left(\frac{83.6-\sigma_{11}}{2}\right)^2 - \left(\frac{\sigma_{11}-5.14}{2}\right)^2}$	$\sigma_{12_{A.T.}} \lessgtr \sqrt{\left(\frac{\sigma_{11}-83.6}{2}\right)^2 - \left(\frac{\sigma_{11}-5.14}{2}\right)^2}$
0	± 41.7	
+10, -10	±39.3, ± 46.2	same as ←
+20, -20	± 30.9, ± 50.2	
+30, -30	± 23.7, ± 54.0	
+40, -40	±13.1, ± 57.5	
+50, -50	—, ± 60.8	
+44.4	0	

Plot:

"Operating stress envelope" for fuselage material via damage-tolerant approach with limit pressure already accounted for



(c) Each of these approaches are different criteria and the plots do look substantially different as may well be expected. The Tresca condition gives the yield point while the damage tolerant approach gives the stress at which a crack will critically propagate. This latter case occurs only for tensile stresses whereas the Tresca condition considers compressive stresses as well. The damage tolerant approach gives the greatest possible operating area.

M14.4 1. D
2. G
3. A
4. H
5. B
6. F
7. E
8. C