9:00 - 10:00
35-225

MIT ID# (last four digits) SOLUTIONS

Unified Quiz 4M

April 6, 2007

- Put the last four digits of your MIT ID # on each page of the exam.
- Read all questions carefully.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit cannot be given without intermediate results.
- Show the logical path of your work. Explain clearly your reasoning and what you are doing. *In some cases, the reasoning is worth as much (or more) than the actual answers.*
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units. Intermediate answers and final answers are not correct without the units.
- Report significant digits only.
- Box your final answers.
- ***Calculators and handwritten "crib sheets" are allowed.***
- ***Unified Handout entitled "Summary of Simple Beam Theory" allowed.***

EXAM SCORING

#1M (1/3)	
#2M (1/3)	
#3M (1/3)	
FINAL SCORE	

Unified Quiz 4 - Materials and Structures (Q4M)
April 6, 2007

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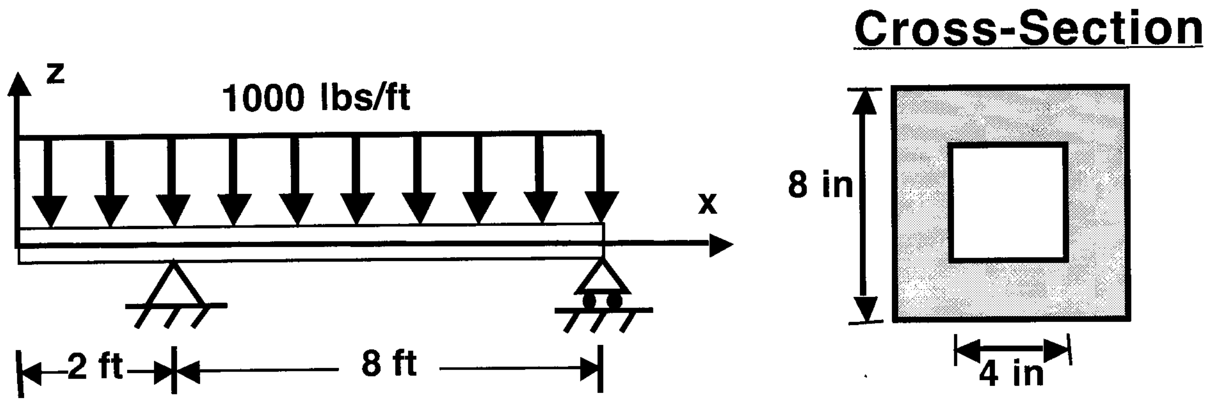
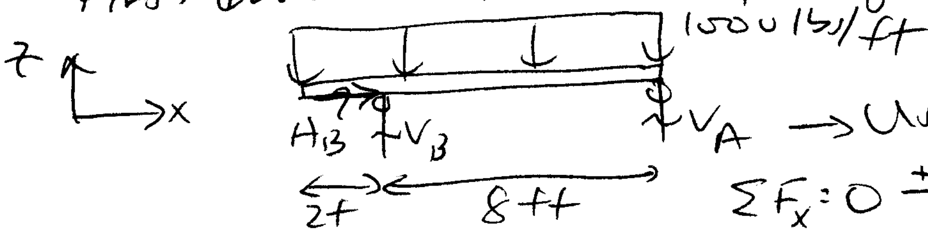
PROBLEM #1M (1/3)

An aluminum beam (E = 10 Msi, ν = 0.3) is supported by a roller-pin configuration with a roller at one end and a pin 2 feet inboard of its tip, as shown in the accompanying figure. The beam is a total of 10 feet long and has a square box cross-section with outer dimensions of 8 inches and inner dimensions of 4 inches. The beam has a distributed downward loading of 1000 pounds per foot.



(a) Sketch the shear force and bending moment resultant distributions as a function of position along the beam. Be sure to note the key values of each and their locations.

First draw the Free Body Diagram:



→ Use equilibrium:

$\Sigma F_x = 0 \xrightarrow{+} \Rightarrow H_B = 0$

$\Sigma F_z = 0 \uparrow + \Rightarrow V_A + V_B - (1000\ lbs/ft)(10\ ft) = 0$ (*)

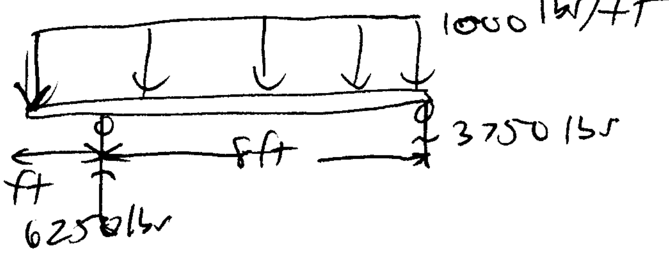
$\Sigma M_A = 0 \circlearrowleft + \Rightarrow -V_B(8ft) + \int_0^{10ft} (1000\ lbs/ft)\, x' dx' = 0$

so: $V_B(8ft) = (1000\ lbs/ft)\ (x')^2/2 \Big]_0^{10ft}$

$\Rightarrow$ $\boxed{\begin{array}{l} V_B = 6250\ lbs \\ V_A = 3750\ lbs \end{array}}$

using (*):

Draw this:



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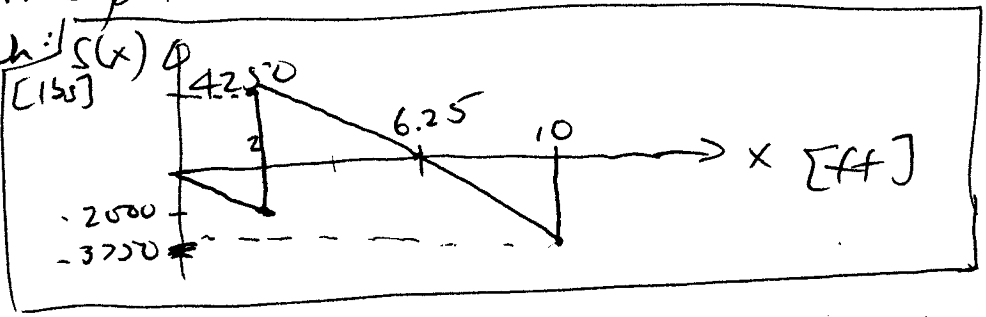
PROBLEM #1M (continued)

→ Now use $\frac{dS}{dx} = q(x) = -1000$ lbs/ft

- At $x=0$, $S=0$ (it is a free end)
- It hits $S = -2000$ lbs (-1000 lbs/ft × 2ft) and jumps by $V_A = 6250$ lbs to $S = +4250$ lbs
- At $x = 10$ft, $S = -V_B = -3750$ lbs
- Find point where $S = 0$: $V_A + q(x)x = 0 \Rightarrow x = 6.25$ft

Sketch:



→ Now use $\frac{dM}{dx} = S(x)$

- At $x = 0$ and $x = L$, M must be zero (no moment reaction)
- Note key points of slope:
  - discontinuous at $x = 2$ft
  - zero (maximum point) at $x = 6.25$ft
- Calculate values at 2 key points:

$x = 2$ ft: 1000 lbs/ft, x', $M(2\text{ft})$ $\quad \Sigma M = 0$ ↺+

$\Rightarrow M(x) + \int_0^{2ft} (1000 \text{ lbs/ft}) x' dx'$

$\Rightarrow M(2ft) = -1000 \text{ lbs/ft} \left. \frac{(x')^2}{2} \right]_0^{2ft}$

$M(2ft) = -2000$ ft·lbs

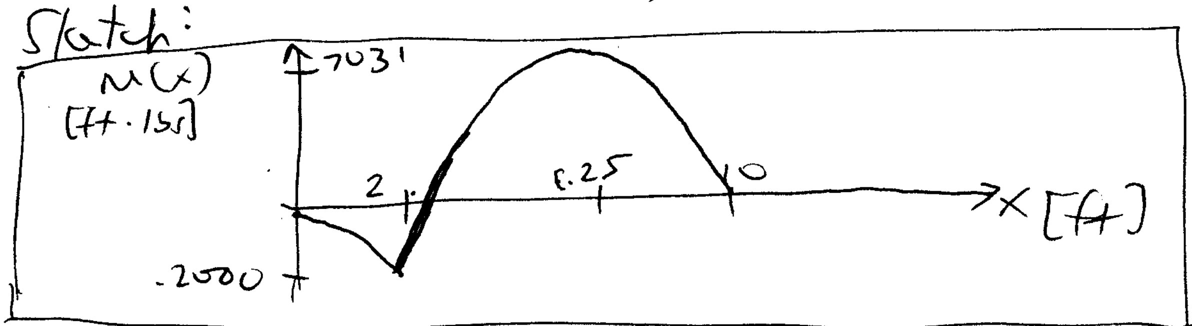
$x = 6.25$ft: $M(6.25ft)$, x', 1000 lbs/ft, 3750 lbs $\quad \Sigma M = 0$ ↻+

$\Rightarrow M(6.25ft) - (3750 \text{ lbs})(3.75ft) + \int_0^{3.75ft} (1000 \text{ lbs/ft}) x' dx'$

$M = 14063 \text{ ft·lbs} - 1000 \text{ lbs/ft} \left. \frac{(x')^2}{2} \right]_0^{3.75ft}$

$M(6.25ft) = 7031$ ft·lbs

Sketch:



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PROBLEM #1M (continued)

(b) Determine the x-location of the maximum axial stress (i.e. σ_{xx}).

The pertinent equation is: $\sigma_{xx} = -\frac{Mz}{I}$

x-location of maximum axial stress is therefore at x-location of maximum moment resultant, $M(x)$.

This was shown to be at $x = 6.25$ ft with an $M(x)$ value of 703 ft·lbs.

maximum axial stress, σ_{xx} at $\boxed{x = 6.25 \text{ ft}}$

(c) Determine the x-location of the maximum shear stress (i.e. σ_{xz}).

The pertinent equation is:

$$\sigma_{xz} = \frac{-SQ}{Ib}$$

The x-location of the maximum shear stress is therefore at the x-location of the maximum shear resultant, $S(x)$.

This was shown to be at $x = 2$ ft with a value of $S(x)$ of 4250 lbs.

maximum shear stress, σ_{xz} at $\boxed{x = 2 \text{ ft}}$

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PROBLEM #1M (continued)

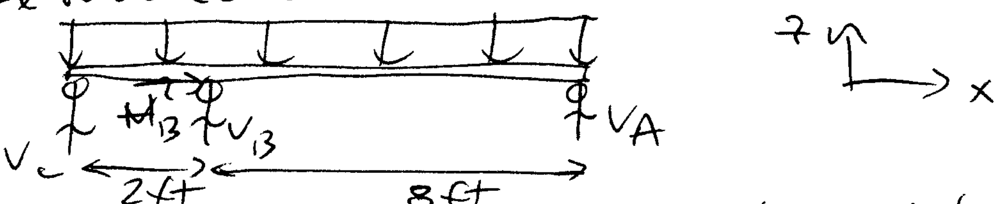
(d) How do the answers to parts (a), (b), and (c) change if wood (E = 3.5 Msi, ν = 0.3) is used rather than aluminum?

This is a statically determinate system. Thus, all values of force resultants and stresses are determined through equilibrium and the modulus thus does not affect these results.

$\boxed{\text{Not affected}}$

(e) A third support (a second roller) is added at the unsupported tip of the beam (z = 0). Would the procedure for determining the answers to part (a) change? Be sure to explain ***clearly***. Use figures, ratios, etc. as appropriate. *Obtaining final values or operative quantified equations is **not necessary***.

There would now be four reaction loads:



There are only 3 degrees of freedom in the 2-D plane.
So.... # of reactions > # of d.o.f.

$\boxed{\text{YES}}$ ⟹ $\boxed{\text{Statically Indeterminate}}$

Approach changes
All the pertinent equations must be solved simultaneously to get resultants, stresses and deflections as their behavior is coupled (material modulus matters in all cases). Need equations for:

- reactions (via equilibrium)
- resultants (via general relations)
- deflection w(x) in terms of moment M(x)
- associated Boundary Conditions

$\boxed{\text{Solve all simultaneously}}$

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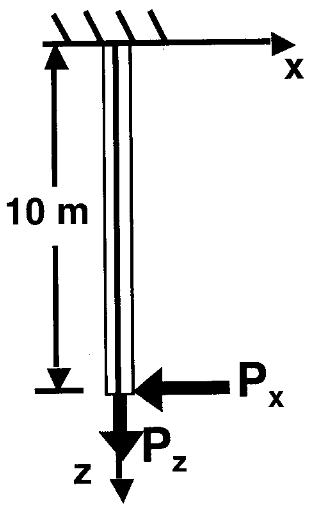
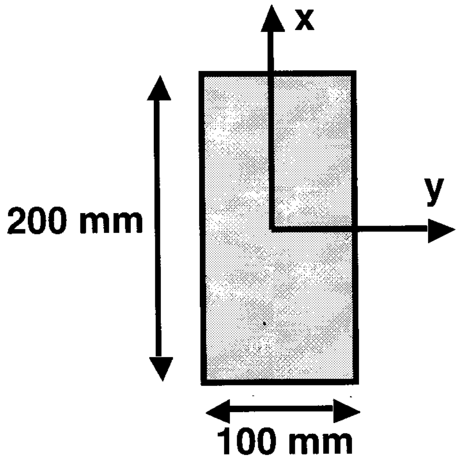
PROBLEM #2M (1/3)

A 10-meter long steel beam (E = 200 GPa, ν = 0.3) is suspended overhead in a clamped support. The beam has a rectangular cross-section 100 mm across and 200 mm deep. The structural configuration, as illustrated in the accompanying figure, is loaded by a horizontal tip load of magnitude P_x. This results in a maximum axial stress, σ_{zz}, of 50 MPa.



Cross-Section



The structural configuration is *subsequently* ***also*** subjected to a vertical tip load, P_z. of 400,000 N.

(a) Determine how the vertical tip load affects the maximum axial stress, σ_{zz}. Quantify as best as you can. **Clearly** explain any modeling assumptions and associated limitations.

To first order, the two separate models (beam and rod) can be superposed (linear modeling). Thus, determine the stress due to the axial load via the rod model:

$$\sigma_{zz(P_z)} = \frac{P_z}{A} \qquad P_z = 400{,}000\ N$$

$$A = (0.1m)(0.2m) = 0.02m^2$$

$$\Rightarrow \sigma_{zz(P_z)} = \frac{400{,}000\ N}{0.02m^2} = 20\ MPa$$

This changes the maximum axial stress due to P_x of $\sigma_{xx(P_x)} = 50$ MPa to $\boxed{70\ MPa = \sigma_{zz\ total}}$

$\Rightarrow \boxed{40\%\ \text{increase}}$

PROBLEM #2M (continued)

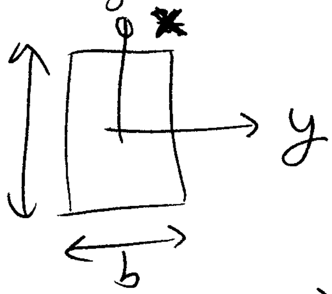
NOTE: In application of models, axes x and z are switched from what is normally used. So σ_{zz} stress here is normally σ_{xx} in models. This is only notation and does not change the basics of the model or the results in any way.

(b) The width (y-direction) of the cross-section is doubled to 200 mm. How is the total axial stress, σ_{zz}, affected for this two-load configuration?

"Switching" axes as discussed in (a), note the pertinent equations for the stresses:

$$\sigma_{zz(P_z)} = \frac{P_z}{A} \qquad \sigma_{zz(P_x)} = -\frac{Mx}{I}$$

changing the structure cross-section does not change the loading only the geometry:



$$A = bh$$
$$I = \frac{bh^3}{12}$$

If $b' = 2b \Rightarrow A' = 2bh = 2A \Rightarrow \sigma'_{zz(P_z)} = \frac{P_z}{A'} = \frac{P_z}{2A} = \frac{1}{2}\sigma_{zz(P_z)}$

$(') =$ new cross-section

$\Rightarrow I' = \frac{2bh^3}{12} = 2I \Rightarrow \sigma'_{zz(P_x)} = -\frac{Mx}{I'} = -\frac{Mx}{2I} = \frac{1}{2}\sigma'_{zz(P_x)}$

$\Rightarrow$ total axial stress, σ_{zz}

$\boxed{\text{reduced by factor of 2}}$

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PROBLEM #2M (continued)

(c) The height (x-direction) of the cross-section is doubled to 400 mm. How is the total axial stress, σ_{zz}, affected for this two-load configuration?

Proceeding as in (b)

$$\text{If } h' = 2h \Rightarrow A' = 2bh = 2A \Rightarrow \sigma'_{zz}(P_z) = \frac{P_z}{A'} = \frac{P_z}{2A} = \frac{1}{2}\sigma_{zz}(P_z)$$

$(') =$ new cross-section

$$\Rightarrow I' = \frac{b(2h)^3}{12} = \frac{8bh^3}{12} = 8I \Rightarrow \sigma'_{zz}(P_x) = -\frac{Mx'}{I'} = -\frac{M(2x)}{8I}$$

$$x' = 2x$$

$$\Rightarrow \sigma'_{zz}(P_x) = \frac{1}{4}\sigma_{zz}(P_x)$$

Using values from (a):

$\max \sigma_{zz}(P_x) = 50\text{ MPa} \Rightarrow \sigma'_{zz}(P_x) = 12.5\text{ MPa}$

$\max \sigma_{zz}(P_z) = 20\text{ MPa} \Rightarrow \sigma'_{zz}(P_z) = 10\text{ MPa}$

total $\underset{\max}{\sigma'_{zz}} = 22.5\text{ MPa} \Rightarrow$ $\boxed{\text{Reduced by } 47.5\text{ MPa}}$

(d) Describe how you would check results to determine whether your modeling is applicable.

The main thing is to check for $\boxed{\text{consistency}}$. This means checking to see that results are "consistent" with the assumptions and limitations.
For example, the full stress and strain states could be calculated and compared to the baseline assumptions. The judgement would then be based on "how good" these are (i.e. "how consistent") -- it is not a clear Yes/No.

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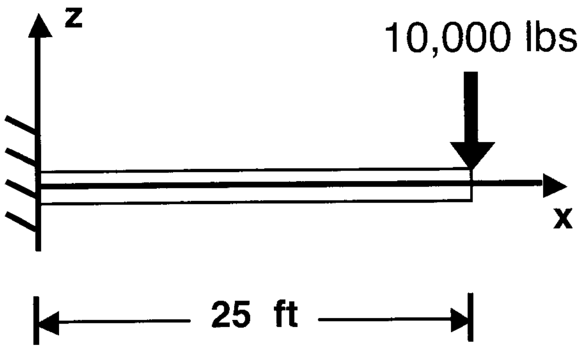
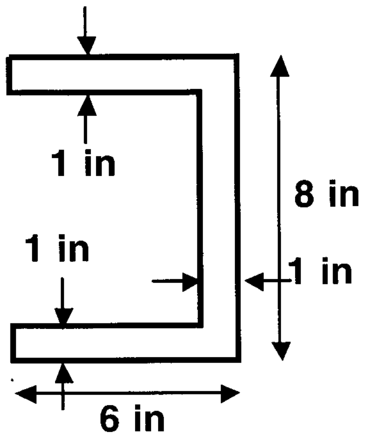
PROBLEM #3M (1/3)

A cantilevered beam has a C-shape cross-section with the dimensions as in the accompanying figure. The beam is 25 feet long and is made of steel with a modulus of 30 Msi. The beam is subjected to a tip load of magnitude 10,000 pounds.



Cross-Section

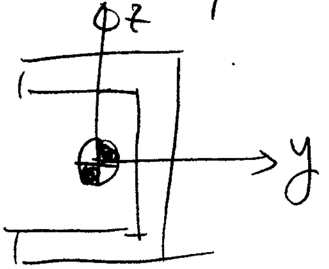


(a) Determine the ratio of the maximum tensile axial stress to the maximum compressive axial stress (i.e. σ_{xx}).

The pertinent equation for the axial stress is:

$$\sigma_{xx} = \frac{-Mz}{I}$$

I is a cross-section property and M is a constant value of any location and this will be maximized where M is maximized. The variation as to tension or compression depends on the value z as measured from the centroid. This cross-section is symmetric about the y-axis at its center:



So the maximum positive z value is equal in magnitude to the maximum negative z value.

$$|+z|_{max} = |-z|_{max}$$

This gives that the maximum tensile σ_{xx} is equal in magnitude to the maximum compressive σ_{xx}

$\boxed{\text{Ratio: } 1:1}$

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PROBLEM #3M (continued)

(b) Determine the location of the maximum value of the transverse shear stress, σ_{xz}, in the cross-sectional plane.

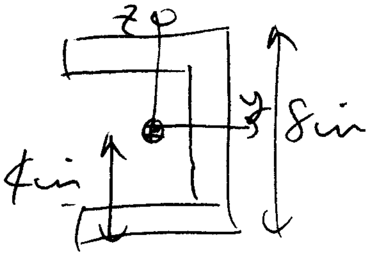
The pertinent equation for transverse shear stress is:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

At any x-location, S and I are constant values, so one must consider how Q and b vary in the cross-sectional plane:

$$\sigma_{xz}(z) = -\frac{S}{I}\frac{Q(z)}{b(z)}$$

To maximize the value of σ_{xz}, one wishes to maximize Q(z) and minimize b(z). The maximum Q(z) occurs at the centroid and the width is also minimum there. So maximum value of σ_{xz} occurs at center point:



$\boxed{\text{At center point in plane}}$

(c) You can double one cross-sectional dimension in order to reduce the maximum deflection. Which dimension would you change to make this the most effective? **Clearly** explain your reasoning. Also indicate how this change would affect the location of the maximum deflection and the reasoning associated with this.

The pertinent equation is

$$M = EI\frac{d^2w}{dx^2}$$

In being able to change cross-sectional geometry, the key parameter in the equation is the moment of inertia, I. This is defined as:

$$I = \int z^2 b(z)\, dz$$

Thus once to get as much material as far away from the centroid (= z) as possible.

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PROBLEM #3M (continued)

Furthermore, this is directly affected by distance from centroid as expressed in the Parallel Axis Theorem:

$$I_{total} = I_0 + Az^2$$

$$I_{rectangle} = \frac{bh^3}{12}$$

So doubling the dimension in the z-direction would be far more effective in reducing the maximum deflection than doubling the dimension in the y-direction.

$$\boxed{\text{Double dimension in z-direction}}$$

(i.e. 8 in → 16 in)

(d) How do the answers to parts (a), (b), and (c) change if the beam is made of titanium (modulus of 15 Msi)? Explain carefully.

All stress values depend only directly on beam geometrical properties. They can depend on material modulus in a statically indeterminate case, but only indirectly through values of the resultants. However this does not change location of maximum values, ratios of stresses in that plane, or dimensional change to reduce deflection as this also is only directly attributable to the geometry change itself. Thus

$$\boxed{\text{the answers do not change}}$$