Recitation 10

The matrix of a linear transformation

Definition 0.1. A *vector space* is a set V together with operations $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$ called "addition" and "scalar multiplication," respectively, satisfying various axioms, which should be intuitive by now.

Definition 0.2. We say that vectors $v_1, \ldots, v_n \in V$ are *linearly dependent* if there exists $c_1, \ldots, c_n \in \mathbb{R}$ not all zero, with $c_1 v_1 + \ldots + c_n v_n = 0$. If $v_1, \ldots, v_n$ are not linearly dependent then we say they are *linearly independent*.

Definition 0.3. $v_1, \ldots, v_n \in V$ are said to be a *basis* for V if $v_1, \ldots, v_n$ are linearly independent and any $v \in V$ can be written as a linear combination of $v_1, \ldots, v_n$: that is $v = c_1 v_1 + \ldots + c_n v_n$ for come $c_1, \ldots, c_n \in \mathbb{R}$.

Definition 0.4. If V and W are vector spaces, a *linear transformation* $T : V \longrightarrow W$ is a function such that $T(v + cw) = T(v) + cT(w)$ for any $v, w \in V$ and $c \in \mathbb{R}$.

Definition 0.5. Suppose V and W are vector spaces with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, respectively. Then for each $j \in \{1, \ldots, n\}$, $T(v_j)$ is a linear combination of the $w_1, \ldots, w_m$. So there exist

$$a_{1,j}, \ldots, a_{m,j} \text{ with } Tv_j = a_{1,j} w_1 + a_{2,j} w_2 + \ldots + a_{m,j} w_m.$$

The *matrix of* T with respect to the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ is the $m \times n$ matrix A, with entries $a_{i,j}$.

The most basic example of the matrix of a linear transformation

1. $\mathbb{R}^n$ is a vector space:
   $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ and $c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)$.

2. Let $e_i \in \mathbb{R}^n$ be the vector with 1 in the i^{th} entry and 0 in the other entries. Then $e_1, \ldots, e_n$ are linearly independent in $\mathbb{R}^n$.

3. $e_1, \ldots, e_n$ is a basis for $\mathbb{R}^n$.

4. An $m \times n$ matirx A defines a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T(x) = Ax$.

5. $\mathbb{R}^n$ has basis $e_1, \ldots, e_n$ and $\mathbb{R}^m$ has basis $e'_1, \ldots, e'_m$, where we use the prime to highlight the vectors are in $\mathbb{R}^m$ as opposed to $\mathbb{R}^n$, but they have an identical definition. We have

   $$Ae_j = a_{1,j} e'_1 + a_{2,j} e'_2 + \ldots + a_{m,j} e'_m$$

   and so the matrix of the linear transformation $T(x) = Ax$ with respect to the bases, $e_1, \ldots, e_n$ and $e'_1 \ldots, e'_m$ is A.

Changing basis

Definition 0.6. If V is a vector space and $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are two bases for V. Then there exists an invertible $n \times n$ matrix B with entries $b_{i,j}$ such that

$$v'_j = b_{1,j}v_1 + b_{2,j}v_2 + \ldots + b_{n,j}v_n.$$

B is said to be the *basis change matrix* from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$.

Suppose V and W are vector spaces. Suppose that V has bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$, that W has bases $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_m$, and that $T : V \longrightarrow W$ is a linear transformation.

If T has matrix A with respect to the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, then T has matrix $C^{-1}AB$ with respect to the bases $v'_1, \ldots, v'_n$ and $w'_1, \ldots, w'_m$, where B and C denote the basis change matrices from the unprimed bases to the primed bases of V and W, respectively.

The most basic example of changing basis

Suppose A is an $m \times n$ matrix. Then we have seen that the linear transformation $T(v) = Av$ has matrix A with respect to the standard bases $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_m$. The matrix of T with respect to another pair of bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ is given by $C^{-1}AB$ where

$$B = \left(v_1 \middle| \cdots \middle| v_n \right) \text{ and } C = \left(w_1 \middle| \cdots \middle| w_m \right).$$

SVD

Here is an algorithm that will always work for the SVD.
Suppose you are given an $m \times n$ matrix A.

1. Let $\lambda_1, \ldots, \lambda_i$ be the non-zero eigenvalues of A^TA and $\lambda_{i+1}, ..., \lambda_n$ be the zero eigenvalues of A^TA. Choose corresponding ORTHONORMAL eigenvectors $v_1, \ldots, v_n$ for A^TA.
2. Let $\sigma_j = \sqrt{\lambda_j}$. Let $u_1, \ldots, u_i$ be given by $u_j = Av_j/\sigma_j$.
3. Let $u_{i+1}, \ldots, u_m$ be an ORTHONORMAL basis of $N(AA^T)$.
4. The SVD is
$$\left(u_1 \middle| \cdots \middle| u_m \right) \Sigma \left(v_1 \middle| \cdots \middle| v_n \right)^T$$
where Σ is an $m \times n$ matrix with (j, j)-entry given by σ_j and all other entries 0.

If $m < n$ it is a little quicker to do the following.

1. Let $\lambda_1, \ldots, \lambda_i$ be the non-zero eigenvalues of AA^T and $\lambda_{i+1}, ..., \lambda_m$ be the zero eigenvalues of AA^T. Choose corresponding ORTHONORMAL eigenvectors $u_1, \ldots, u_m$ for AA^T.
2. Let $\sigma_j = \sqrt{\lambda_j}$. Let $v_1, \ldots, v_i$ be given by $v_j = A^Tu_j/\sigma_j$.
3. Let $v_{i+1}, \ldots, v_n$ be an ORTHONORMAL basis of $N(A^TA)$.

Recitation 10 questions

Question 1

Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$.

(a) Compute $A(1,1,1)^T$, $A(1,1,0)^T$, $A(1,-1,0)^T$. Write each as a linear combination of the vectors $(1,1)^T$ and $(0,1)^T$.

(b) Find the matrix of the linear transformation $T(x) = Ax$ with respect to the bases $(1,1,1)^T$, $(1,1,0)^T$, $(1,-1,0)^T$ and $(1,1)^T$, $(0,1)^T$ by using the definition of the matrix of a linear transformation.

(c) Find the matrix of the linear transformation $T(x) = Ax$ with respect to the bases $(1,1,1)^T$, $(1,1,0)^T$, $(1,-1,0)^T$ and $(1,1)^T$, $(0,1)^T$ by using basis change matrices.

Question 2

Suppose A is 3×2 matrix with the property that

$$A(4,5)^T = (1,2,3)^T \text{ and } A(3,4)^T = (3,2,1)^T.$$

(a) What is the matrix of $T(x) = Ax$ with respect to the bases

$$(4,5)^T,\ (3,4)^T \text{ and } (1,2,3)^T,\ (3,2,1)^T,\ (0,1,0)^T.$$

(b) Use the basis change formula to write the matrix you just calculated as $C^{-1}AB$.

(c) What is A?

Question 3

Let V be the set of cubic polynomials $V = \{f(x) = ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\}$.

(a) Recall why V is a vector space; what happens to the coefficients under addition and scalar multiplication?

(b) Define $T_1(f(x)) = f'(x)$. Is this a linear transformation $V \longrightarrow V$? Why?

(c) Define $T_2(f(x)) = f(x+1)$. Is this a linear transformation $V \longrightarrow V$? Why?

(d) Define $T_3(f(x)) = x^3 f(1/x)$. Is this a linear transformation $V \longrightarrow V$? Why?

(e) Recall that $1, x, x^2, x^3$ is a basis for V. Does this make sense to you?

(f) What is $T_1(1), T_1(x), T_1(x^2), T_1(x^3)$? What is the matrix of T_1 with respect to the basis $1, x, x^2, x^3$ (using this as a basis for the domain and codomain)?

(g) What is the matrix of T_2 with respect to the basis $1, x, x^2, x^3$?

(h) What is the matrix of T_3 with respect to the basis $1, x, x^2, x^3$?

(i) What is the matrix of T_1 with respect to the basis 1, $x-1$, $(x-1)(x-2)$, $x(x^2 - \frac{9}{2}x + 6)$?

Question 4

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$.

1. Find the eigenvalues $\lambda_1 \neq 0$, λ_2 and unit eigenvectors v_1, v_2 of $A^T A$.
2. Let $\sigma_1 = \sqrt{\lambda_1}$, $u_1 = Av_1/\sigma_1$. Verify that u_1 is a unit eigenvector for AA^T with eigenvalue λ_1.
3. Extend u_1 to an orthonormal basis u_1, u_2.
4. Check that

$$A = (u_1|u_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} (v_1|v_2)^T.$$