Question from Professor Ian Christie, West Virginia University

Find unit vectors $h(t)$ and $m(t)$ in the direction of the hour and minute hands of a clock, where t denotes the elapsed time in hours. If $t = 0$ represents noon then $m(0) = h(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. At what time will the hands of the clock first be perpendicular? At what time after noon will the hands first form a straight line? Remember that in the dot product $m(t) \cdot h(t)$, $\sin x \sin y + \cos x \cos y = \cos(x - y)$.

Solution:

The minute hand completes a full circle at time $t = 1$, and the hour hand moves one-twelfth as fast:

$$m(t) = \begin{bmatrix} \sin 2\pi t \\ \cos 2\pi t \end{bmatrix} \quad \text{and} \quad h(t) = \begin{bmatrix} \sin \frac{\pi t}{6} \\ \cos \frac{\pi t}{6} \end{bmatrix}.$$

The dot product is $m \cdot h = \sin 2\pi t \, \sin \frac{\pi t}{6} + \cos 2\pi t \, \cos \frac{\pi t}{6} = \cos\left(2\pi t - \frac{\pi t}{6}\right) = \cos\left(\frac{11\pi t}{6}\right)$. The hands are perpendicular when this cosine is zero: $\frac{11\pi t}{6} = \frac{\pi}{2}$ and $t = \frac{3}{11}$ hours $= 16\,\frac{4}{11}$ minutes. The hands will be in a straight line (opposite directions!) when $\frac{11\pi t}{6} = \pi$ and the cosine is -1. This happens after $16\,\frac{4}{11}$ more minutes, or at $t = \frac{6}{11}$ hours $= 32\,\frac{8}{11}$ minutes past noon.

When do the hands first point in the same direction? It must be after 1:00, so the minute hand can come around to meet the hour hand.