A Factorization Review

A major theme in the linear algebra course has been the interpretation and computation of matrix factorizations ($A = LU$, $\texttt{rref}(A)$, $A = LDL^{\mathrm{T}}$, etc.). One way to summarize the course is to review aspects of those factorizations and their uses. A list of tasks is given below. Use your understanding of the course material and the text, and fill in the details as one way of preparing for the final examination.

1. The LU factorization is written as $A = LU$ and A may be rectangular. Describe the main properties of L and U . What are the sizes of L and U in terms of the size of A? See section 2.6.

2. Describe two legitimate uses of the LU decomposition in applied problems. See sections 2.6 and 5.1.

3. Evaluate $\det(A^2)$ when

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 & 8 & 7 \\ 0 & 3/2 & 4 & 3 \\ 0 & 0 & 4/3 & 2 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

4. Find a basis for the nullspace $N(A)$ when $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 8 & 2 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

5. The LDL^{T} factorization is a refinement of the LU factorization and we generally write $A = LDL^{\mathrm{T}}$. Describe the main properties of A, L, and D. See section 2.7.

6. Describe two uses of the LDL^{T} factorization in linear algebra problems. See sections 2.7, 6.5 and problem 25 on page 341.

7. Is there an LDL^{T} factorization for $A = \begin{bmatrix} 16 & 2 & 9 & 13 \\ 2 & 11 & 10 & 13 \\ 9 & 8 & 6 & 5 \\ 13 & 14 & 5 & 1 \end{bmatrix}$? Give reasons.

8. Find the LDL^{T} factorization of A when

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Is A a positive definite matrix? Give reasons.

9. The Gram-Schmidt algorithm generates orthogonal or orthonormal vectors from linearly independent column vectors in a matrix A. The result is a factorization $A = QR$. Describe the matrix R and properties of the columns of Q. See section 4.4.

10. If Q is 4 by 3 and has orthonormal columns, find x that minimizes $\|b - Qx\|^2$ in terms of Q and b. What is the size of x? See sections 4.1 and 4.3.

11. One version of the Gram-Schmidt algorithm finishes with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

What is the size of A? Find the QR factorization $A = QR$ when Q has orthonormal columns and no steps of the Gram-Schmidt algorithm are repeated. Write out the factors Q and R. See sections 4.1 and 4.4.

12. The implicit factorization $AS = S\Sigma$ occurs in eigenvalue problems. If Σ is a diagonal matrix, describe the significance of the individual columns of S in relation to the diagonal entries in Σ. See section 6.2.

13. Describe two uses of the eigenvector factorization $A = S\Sigma S^{-1}$. See sections 6.2, 6.3, 6.4, and 6.5.

14. If $AS = S \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$ and S is invertible, what are the eigenvalues of A? What is the determinant of A? What are the eigenvalues of $A - I$, of $A - A^{-1}$, and of A^{-1}? Find a factored form of e^{tA} in terms of S and the diagonal matrix above. Calculate the trace of A and then the trace of e^{tA}. All tasks and questions can be answered without knowing A explicitly. See sections 6.2, 6.3, and 6.6.

15. The quadratic form $5x^2 + 8xy + 5y^2 = 1$ describes a tilted ellipse in the xy-plane and eigenvalue methods can be used to analyze it.

(a) Find the length of the major axis and minor axis.

(b) Find the directions of the axes of this tilted ellipse in the xy-plane.

See section 6.5.

16. A singular value decomposition (SVD) is a powerful factorization that is used extensively in applied problems. It can be presented as $AV = U\Sigma$ for a matrix A of any size. Describe the properties of the factors U, V, Σ and their sizes in relation to the size of A. See section 6.7.

17. One important use of the SVD is in finding orthonormal bases for some or all of the four fundamental subspaces. Other uses include finding the rank of a matrix and solving least squares problems. As discussed in the text, you can extract subspace information by selecting appropriate columns from U or V. The code below shows MATLAB output for a specific "magic matrix." Use the SVD output and find an orthonormal basis for the column space $C(A)$ and nullspace $N(A)$. What is the rank of A?

```
>>A=magic(4) % matrix in question
A =
        16         2         3        13
         5        11        10         8
         9         7         6        12
         4        14        15         1
>>help svd
  SVD   Singular value decomposition.
     [U,S,V] = SVD(X) produces a diagonal matrix S, of the same
     dimension as X and with nonnegative diagonal elements in
     decreasing order, and unitary matrices U and V so that
     X = U*S*V'.

>>[U,S,V] = svd(A); % MATLAB calculates the parts
>>disp(U) % show U
        0.5000   -0.6708    0.5000    0.2236
        0.5000    0.2236   -0.5000    0.6708
        0.5000   -0.2236   -0.5000   -0.6708
        0.5000    0.6708    0.5000   -0.2236
```

```
>>disp(S)
      34.0000          0          0          0
            0    17.8885          0          0
            0          0     4.4721          0
            0          0          0     0.0000
>>disp(V)
       0.5000    -0.5000     0.6708    -0.2236
       0.5000     0.5000    -0.2236    -0.6708
       0.5000     0.5000     0.2236     0.6708
       0.5000    -0.5000    -0.6708     0.2236
```

18. Suppose that A is 5 by 3 and $A = U\Sigma V^{\mathrm{T}}$ is its SVD, where

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^{\mathrm{T}}A$ using $A = U\Sigma V^{\mathrm{T}}$.

(b) Remember properties of orthogonal matrices and evaluate $\det(AA^{\mathrm{T}})$ and $\det(A^{\mathrm{T}}A)$. See sections 6.7 and 5.1.

19. Write $(x_1 + x_2 + x_3 + x_4)^2 + (x_1 - x_2 - x_3)^2 = x^{\mathrm{T}}Ax$ in terms of a symmetric matrix A and vector $x^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$. What is the rank of A?

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