GLOSSARY: A DICTIONARY FOR LINEAR ALGEBRA

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ for an undirected graph.

Affine transformation $T(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{v}_0$ = linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC.

Augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the column space of A; then $[\,A \;\; \boldsymbol{b}\,]$ has the same rank as A. Elimination on $[\,A \;\; \boldsymbol{b}\,]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1.

Basis for $\boldsymbol{V}$. Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ whose linear combinations give every $\boldsymbol{v}$ in $\boldsymbol{V}$. A vector space has many bases!

Big formula for n by n determinants. $\mathrm{Det}(A)$ is a sum of $n!$ terms, one term for each permutation P of the columns. That term is the product $a_{1\alpha} \cdots a_{n\omega}$ down the diagonal of the reordered matrix, times $\det(P) = \pm 1$.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit (the columns of A and rows of B must be in matching blocks).

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix*.

Change of basis matrix M. The old basis vectors $\boldsymbol{v}_j$ are combinations $\sum m_{ij}\boldsymbol{w}_i$ of the new basis vectors. The coordinates of $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = d_1\boldsymbol{w}_1 + \cdots + d_n\boldsymbol{w}_n$ are related by $\boldsymbol{d} = M\boldsymbol{c}$. (For $n = 2$ set $\boldsymbol{v}_1 = m_{11}\boldsymbol{w}_1 + m_{21}\boldsymbol{w}_2$, $\boldsymbol{v}_2 = m_{12}\boldsymbol{w}_1 + m_{22}\boldsymbol{w}_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A.

Cholesky factorization $A = CC^{\mathrm{T}} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C. Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1}S^{n-1}$. $C\boldsymbol{x} =$ **convolution** $\boldsymbol{c} * \boldsymbol{x}$. Eigenvectors in F.

Cofactor C_{ij}. Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ becomes a combination of the columns of A. The system is solvable only when $\boldsymbol{b}$ is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ = space of all combinations of the columns of A.

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put $c_1, \ldots, c_n$ in row n and put $n - 1$ 1's along diagonal 1. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots)$.

Complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ to $A\boldsymbol{x} = \boldsymbol{b}$. (Particular $\boldsymbol{x}_p$) + ($\boldsymbol{x}_n$ in nullspace).

Complex conjugate $\overline{z} = a - ib$ for any complex number $z = a + ib$. Then $z\overline{z} = |z|^2$.

Condition number $cond(A) = \kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $A\boldsymbol{x} = \boldsymbol{b}$, the relative change $\|\delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is less than $cond(A)$ times the relative change $\|\delta\boldsymbol{b}\|/\|\boldsymbol{b}\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $A\boldsymbol{x} = \boldsymbol{b}$ by minimizing $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{b}$ over growing Krylov subspaces.

Covariance matrix Σ. When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\overline{x}_i$, the matrix Σ = mean of $(\boldsymbol{x} - \overline{\boldsymbol{x}})(\boldsymbol{x} - \overline{\boldsymbol{x}})^{\mathrm{T}}$ is positive (semi)definite; it is diagonal if the x_i are independent.

Cramer's Rule for $A\boldsymbol{x} = \boldsymbol{b}$. B_j has $\boldsymbol{b}$ replacing column j of A, and $x_j = |B_j|/|A|$.

Cross product $\boldsymbol{u} \times \boldsymbol{v}$ in $\mathbf{R}^3$. Vector perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$, length $\|\boldsymbol{u}\|\|\boldsymbol{v}\||\sin\theta|$ = parallelogram area, computed as the "determinant" of $[\,\boldsymbol{i}\ \ \boldsymbol{j}\ \ \boldsymbol{k}\,;\ u_1\ \ u_2\ \ u_3\,;\ v_1\ \ v_2\ \ v_3\,]$.

Cyclic shift S. Permutation with $s_{21} = 1, s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n-1$, volume of box $= |\det(A)|$.

Diagonal matrix D. $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A. Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. Λ = eigenvalue matrix and S = eigenvector matrix. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(\boldsymbol{V})$ = number of vectors in any basis for $\boldsymbol{V}$.

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n$. Complex dot product is $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$. Perpendicular vectors have zero dot product. $(AB)_{ij}$ = (row i of A)·(column j of B).

Echelon matrix U. The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ **and eigenvector** $\boldsymbol{x}$. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\boldsymbol{x} \neq \boldsymbol{0}$ so $\det(A - \lambda I) = 0$.

Eigshow. Graphical 2 by 2 eigenvalues and singular values (MATLAB or Java).

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = **Elementary matrix** E_{ij}. The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|\boldsymbol{x}\| = 1$ the vectors $\boldsymbol{y} = A\boldsymbol{x}$ lie on the ellipse $\|A^{-1}\boldsymbol{y}\|^2 = \boldsymbol{y}^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}\boldsymbol{y} = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}\boldsymbol{u}(0)$ solves $\boldsymbol{u}' = A\boldsymbol{u}$.

Factorization $A = LU$. If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n\boldsymbol{x}$ and $F_n^{-1}\boldsymbol{c}$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$ satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1 + \sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Four fundamental subspaces of $A = \boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A^{\mathrm{T}})$.

Fourier matrix F. Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}} F = nI$. Then $\boldsymbol{y} = F\boldsymbol{c}$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A. Columns without pivots; combinations of earlier columns.

Free variable x_i. Column i has no pivot in elimination. We can give the $n - r$ free variables any values, then $A\boldsymbol{x} = \boldsymbol{b}$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $A\boldsymbol{x} = \boldsymbol{b}$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements (perpendicular subspaces of $\mathbf{R}^n$ with dimensions r and $n - r$) from $A\boldsymbol{x} = \boldsymbol{0}$. Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$.

Gauss-Jordan method. Invert A by row operations on $[\,A \;\; I\,]$ to reach $[\,I \;\; A^{-1}\,]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A, orthonormal columns in Q. Each column $\boldsymbol{q}_j$ of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\mathsf{diag}(R) > \boldsymbol{0}$.

Graph G. Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n - 1$ edges and no closed loops. A **directed graph** has a direction arrow specified on each edge.

Hankel matrix H. Constant along each antidiagonal; h_{ij} depends on $i + j$.

Hermitian matrix $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} = A$. Complex analog of a symmetric matrix: $\overline{a_{ji}} = a_{ij}$.

Hessenberg matrix H. Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix $\mathsf{hilb}(n)$. Entries $H_{ij} = 1/(i + j - 1) = \int_0^1 x^{i-1} x^{j-1} dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number.

Hypercube matrix P_L^2. Row $n + 1$ counts corners, edges, faces, $\ldots$ of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**)**. Diagonal entries $= 1$, off-diagonal entries $= 0$.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix. A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. No combination $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k =$ zero vector unless all $c_i = 0$. If the $\boldsymbol{v}$'s are the columns of A, the only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$.

Inverse matrix A^{-1}. Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\operatorname{rank}(A) < n$ and $A\boldsymbol{x} = \boldsymbol{0}$ for a nonzero vector $\boldsymbol{x}$. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1's on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector $(1, 0, \ldots, 0)$.

Kirchhoff's Laws. *Current law*: net current (in minus out) is zero at each node. *Voltage law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \bigotimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, \boldsymbol{b})$. The subspace spanned by $\boldsymbol{b}, A\boldsymbol{b}, \ldots, A^{j-1}\boldsymbol{b}$. Numerical methods approximate $A^{-1}\boldsymbol{b}$ by $\boldsymbol{x}_j$ with residual $\boldsymbol{b} - A\boldsymbol{x}_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\widehat{\boldsymbol{x}}$. The vector $\widehat{\boldsymbol{x}}$ that minimizes the error $\|\boldsymbol{e}\|^2$ solves $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Then $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is orthogonal to all columns of A.

Left inverse A^+. If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$. Nullspace of A^{T} = "left nullspace" of A because $\boldsymbol{y}^{\mathrm{T}}A = \boldsymbol{0}^{\mathrm{T}}$.

Length $\|\boldsymbol{x}\|$. Square root of $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ (Pythagoras in n dimensions).

Linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ or $\sum c_j \boldsymbol{v}_j$. Vector addition and scalar multiplication.

Linear transformation T. Each vector $\boldsymbol{v}$ in the input space transforms to $T(\boldsymbol{v})$ in the output space, and linearity requires $T(c\boldsymbol{v} + d\boldsymbol{w}) = c\,T(\boldsymbol{v}) + d\,T(\boldsymbol{w})$. Examples: Matrix multiplication $A\boldsymbol{v}$, differentiation in function space.

Linearly dependent $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. A combination other than all $c_i = 0$ gives $\sum c_i \boldsymbol{v}_i = \boldsymbol{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \ldots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with eigenvalues $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ of the Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Compare $L_0 = 2$ with Fibonacci.

Markov matrix M. All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $M\boldsymbol{s} = \boldsymbol{s} > \boldsymbol{0}$.

Matrix multiplication AB. The i, j entry of AB is (row i of A)·(column j of B) $= \sum a_{ik}b_{kj}$. By columns: Column j of AB = A times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times $\boldsymbol{x}$ equals A times $B\boldsymbol{x}$.

Minimal polynomial of A. The lowest degree polynomial with $m(A)$ = zero matrix. The roots of m are eigenvalues, and $m(\lambda)$ divides $\det(A - \lambda I)$.

Multiplication $A\boldsymbol{x} = x_1$(column 1) $+ \cdots + x_n$(column n) = combination of columns.

Multiplicities AM **and** GM. The algebraic multiplicity AM of an eigenvalue λ is the number of times λ appears as a root of $\det(A - \lambda I) = 0$. The geometric multiplicity GM is the number of independent eigenvectors (= dimension of the eigenspace for λ).

Multiplier ℓ_{ij}. The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: $\ell_{ij} = (\text{entry to eliminate})/(j\text{th pivot})$.

Network. A directed graph that has constants $c_1, \ldots, c_m$ associated with the edges.

Nilpotent matrix N. Some power of N is the zero matrix, $N^k = 0$. The only eigenvalue is $\lambda = 0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$ **of a matrix**. The "ℓ^2 norm" is the maximum ratio $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = \sigma_{\max}$. Then $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$ and $\|AB\| \le \|A\|\|B\|$ and $\|A+B\| \le \|A\| + \|B\|$. **Frobenius norm** $\|A\|_F^2 = \sum\sum a_{ij}^2$; ℓ^1 and ℓ^∞ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Gives the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ if A has full rank n. The equation says that (columns of A)$\cdot(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = 0$.

Normal matrix N. $NN^{\mathrm{T}} = N^{\mathrm{T}}N$, leads to orthonormal (complex) eigenvectors.

Nullspace $\boldsymbol{N}(A)$ = Solutions to $A\boldsymbol{x} = \mathbf{0}$. Dimension $n - r = (\#\text{ columns}) - \text{rank}$.

Nullspace matrix N. The columns of N are the $n - r$ special solutions to $A\boldsymbol{s} = \mathbf{0}$. **Orthogonal matrix** Q. Square matrix with orthonormal columns, so $Q^{\mathrm{T}}Q = I$ implies $Q^{\mathrm{T}} = Q^{-1}$. Preserves length and angles, $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ and $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$. All $|\lambda| = 1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces. Every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$.

Orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. Dot products are $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ if $i \ne j$ and $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = 1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q = I$. If $m = n$ then $Q^{\mathrm{T}} = Q^{-1}$ and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $\boldsymbol{v} = \sum(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{q}_j)\boldsymbol{q}_j$.

Outer product $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row = rank one matrix.

Partial pivoting. In elimination, the jth pivot is chosen as the largest available entry (in absolute value) in column j. Then all multipliers have $|\ell_{ij}| \le 1$. Roundoff error is controlled (depending on the *condition number* of A).

Particular solution $\boldsymbol{x}_p$. Any solution to $A\boldsymbol{x} = \boldsymbol{b}$; often $\boldsymbol{x}_p$ has free variables $= 0$.

Pascal matrix $P_S = \mathsf{pascal}(n)$. The symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S = P_L P_U$ all contain Pascal's triangle with $\det = 1$ (see index for more properties).

Permutation matrix P. There are $n!$ orders of $1, \ldots, n$; the $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij}; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A. Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d. The diagonal entry (*first nonzero*) when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$. Solutions to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = 0$ give the plane (dimension $n - 1$) perpendicular to $\boldsymbol{a} \ne \mathbf{0}$.

Polar decomposition $A = QH$. Orthogonal Q, positive (semi)definite H.

Positive definite matrix A. Symmetric matrix with positive eigenvalues and positive pivots. Definition: $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > 0$ unless $\boldsymbol{x} = \mathbf{0}$.

Projection $\boldsymbol{p} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$ onto the line through $\boldsymbol{a}$. $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ has rank 1.

Projection matrix P onto subspace $\boldsymbol{S}$. Projection $\boldsymbol{p} = P\boldsymbol{b}$ is the closest point to $\boldsymbol{b}$ in $\boldsymbol{S}$, error $\boldsymbol{e} = \boldsymbol{b} - P\boldsymbol{b}$ is perpendicular to $\boldsymbol{S}$. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in $\boldsymbol{S}$ or $\boldsymbol{S}^{\perp}$. If columns of A = basis for $\boldsymbol{S}$ then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Pseudoinverse A^+ (Moore-Penrose inverse). The n by m matrix that "inverts" A from column space back to row space, with $\boldsymbol{N}(A^+) = \boldsymbol{N}(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. $\mathrm{Rank}(A^+) = \mathrm{rank}(A)$.

Random matrix $\mathsf{rand}(n)$ or $\mathsf{randn}(n)$. MATLAB creates a matrix with random entries, uniformly distributed on $[\,0 \;\; 1\,]$ for rand and standard normal distribution for randn.

Rank one matrix $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \neq 0$. Column and row spaces = lines $c\boldsymbol{u}$ and $c\boldsymbol{v}$.

Rank $r(A)$ = number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ for symmetric A: $\lambda_{\min} \leq q(\boldsymbol{x}) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors $\boldsymbol{x}$ for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \mathsf{rref}(A)$. Pivots = 1; zeros above and below pivots; r nonzero rows of R give a basis for the row space of A.

Reflection matrix $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. The unit vector $\boldsymbol{u}$ is reflected to $Q\boldsymbol{u} = -\boldsymbol{u}$. All vectors $\boldsymbol{x}$ in the plane mirror $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{x} = 0$ are unchanged because $Q\boldsymbol{x} = \boldsymbol{x}$. The "Householder matrix" has $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+. If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the plane by θ and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Orthogonal matrix, eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors $(1, \pm i)$.

Row picture of $A\boldsymbol{x} = \boldsymbol{b}$. Each equation gives a plane in $\mathbf{R}^n$; planes intersect at $\boldsymbol{x}$.

Row space $\boldsymbol{C}(A^{\mathrm{T}})$ = all combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j$ = **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$.

Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$. Then $|\boldsymbol{v}^{\mathrm{T}}A\boldsymbol{w}|^2 \leq (\boldsymbol{v}^{\mathrm{T}}A\boldsymbol{v})(\boldsymbol{w}^{\mathrm{T}}A\boldsymbol{w})$ if $A = C^{\mathrm{T}}C$.

Semidefinite matrix A. (Positive) semidefinite means symmetric with $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} \geq 0$ for all vectors $\boldsymbol{x}$. Then all eigenvalues $\lambda \geq 0$; no negative pivots.

Similar matrices A and B. Every $B = M^{-1}AM$ has the same eigenvalues as A.

Simplex method for linear programming. The minimum cost vector $\boldsymbol{x}^*$ is found by moving from corner to lower cost corner along the edges of the feasible set (where the constraints $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ are satisfied). Minimum cost at a corner!

Singular matrix A. A square matrix that has no inverse: $\det(A) = 0$.

Singular Value Decomposition (SVD) $A = U\Sigma V^{\mathrm{T}}$ = **(orthogonal U) times (diagonal Σ) times (orthogonal V^{T}).** First r columns of U and V are orthonormal bases of $\boldsymbol{C}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$ with $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i$ and singular value $\sigma_i > 0$. Last columns of U and V are orthonormal bases of the nullspaces of A^{T} and A.

Skew-symmetric matrix K. The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $A\boldsymbol{x} = \boldsymbol{b}$. The right side $\boldsymbol{b}$ is in the column space of A.

Spanning set $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ **for** $\boldsymbol{V}$. Every vector in $\boldsymbol{V}$ is a combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$.

Special solutions to $A\boldsymbol{s} = \boldsymbol{0}$. One free variable is $s_i = 1$, other free variables $= 0$.

Spectral theorem $A = Q\Lambda Q^{\mathrm{T}}$. Real symmetric A has real λ_i and orthonormal $\boldsymbol{q}_i$ with $A\boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i$. In mechanics the $\boldsymbol{q}_i$ give the *principal axes*.

Spectrum of A = the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. **Spectral radius** $= |\lambda_{\max}|$.

Standard basis for $\mathbf{R}^n$. Columns of n by n identity matrix (written $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in $\mathbf{R}^3$).

Stiffness matrix K. If $\boldsymbol{x}$ gives the movements of the nodes in a discrete structure, $K\boldsymbol{x}$ gives the internal forces. Often $K = A^{\mathrm{T}}CA$ where C contains spring constants from Hooke's Law and $A\boldsymbol{x}$ = stretching (strains) from the movements $\boldsymbol{x}$.

Subspace $\boldsymbol{S}$ **of** $\boldsymbol{V}$. Any vector space inside $\boldsymbol{V}$, including $\boldsymbol{V}$ and $\boldsymbol{Z} = \{\text{zero vector}\}$.

Sum $\boldsymbol{V} + \boldsymbol{W}$ **of subspaces**. Space of all ($\boldsymbol{v}$ in $\boldsymbol{V}$) + ($\boldsymbol{w}$ in $\boldsymbol{W}$). **Direct sum**: $\dim(\boldsymbol{V} + \boldsymbol{W}) = \dim \boldsymbol{V} + \dim \boldsymbol{W}$ when $\boldsymbol{V}$ and $\boldsymbol{W}$ share only the zero vector.

Symmetric factorizations $A = LDL^{\mathrm{T}}$ **and** $A = Q\Lambda Q^{\mathrm{T}}$. The number of positive pivots in D and positive eigenvalues in Λ is the same.

Symmetric matrix A. The transpose is $A^{\mathrm{T}} = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric. All matrices of the form $R^{\mathrm{T}}R$ and LDL^{T} and $Q\Lambda Q^{\mathrm{T}}$ are symmetric. Symmetric matrices have real eigenvalues in Λ and orthonormal eigenvectors in Q.

Toeplitz matrix T. Constant-diagonal matrix, so t_{ij} depends only on $j - i$. Toeplitz matrices represent linear time-invariant filters in signal processing.

Trace of A = sum of diagonal entries = sum of eigenvalues of A. $\operatorname{Tr} AB = \operatorname{Tr} BA$.

Transpose matrix A^{T}. Entries $A^{\mathrm{T}}_{ij} = A_{ji}$. A^{T} is n by m, $A^{\mathrm{T}}A$ is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are $B^{\mathrm{T}}A^{\mathrm{T}}$ and $(A^{\mathrm{T}})^{-1}$.

Triangle inequality $\|\boldsymbol{u} + \boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|v\|$. For matrix norms $\|A + B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T: $t_{ij} = 0$ if $|i - j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}} = U^{-1}$. Orthonormal columns (complex analog of Q).

Vandermonde matrix V. $V\boldsymbol{c} = \boldsymbol{b}$ gives the polynomial $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$ at n points. $V_{ij} = (x_i)^{j-1}$ and $\det V$ = product of $(x_k - x_i)$ for $k > i$.

Vector $\boldsymbol{v}$ **in** $\mathbf{R}^n$. Sequence of n real numbers $\boldsymbol{v} = (v_1, \ldots, v_n)$ = point in $\mathbf{R}^n$.

Vector addition. $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, \ldots, v_n + w_n)$ = diagonal of parallelogram.

Vector space $\boldsymbol{V}$. Set of vectors such that all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ remain in $\boldsymbol{V}$. Eight required rules are given in Section 3.1 for $c\boldsymbol{v} + d\boldsymbol{w}$.

Volume of box. The rows (or columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$ **or vectors** $\boldsymbol{w}_{jk}$. Stretch and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$. Vectors from $\boldsymbol{w}_{00} = (1, 1, -1, -1)$ would be $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$.